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$SU(3)$ revisited

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Abstract. The 'D' matrices for all states of the two fundamental representations and octet are given in the Euler angle parametrization of $SU(3)$. The raising and lowering operators are given in terms of linear combinations of the left-invariant vector fields of the group manifold in this parametrization. Using these differential operators the highest weight state of an arbitrary irreducible representation is found and a description of the calculation of Clebsch–Gordon coefficients is given.

1. Introduction

In our understanding of particle physics, studying the group $SU(3)$ has helped tremendously. It has given us an organization to the plethora of 'elementary' particles through the Eightfold way [1] and then led to the quark description of hadrons§. This, in turn, led to the fundamental theory of the strong nuclear interactions known as the colour $SU(3)$ of the now widely accepted standard model‖. It has also had numerous successes in phenomenological models such as the nuclear $SU(3)$ model of Elliot [4], and the Skyrme–Witten model [5]. Its algebra has been utilized extensively for these applications but its manifold has not. In most cases, due to the intimate relationship between the algebra of a Lie group and the group itself (subalgebras correspond to subgroups, etc), this description has been enough. Also, since the group manifold of $SU(3)$ is eight dimensional, it is not prone to 'visual' analysis. Recently, however, the manifold has been used for the study of quantum three-level systems and geometric phases [6–9]. The subgroups and coset spaces of $SU(3)$ are listed in [8] along with a discussion of the geometry of the group manifold which is relevant to the understanding of the geometric phase. It should therefore be no surprise if the group and group manifold lead to further understanding of physical phenomena beyond what the algebra has already accomplished. Further study of its structure may very well lead to an even greater understanding of nature and the way its symmetries are manifest.

Here, the raising and lowering operators of the group are given in terms of differential operators. The states of the fundamental representations are given in terms of the Euler angle parametrization. A highest weight state is given for all irreps (irreducible representations) that will enable the calculation of any state within any irrep. A determination of the ranges of the angles in the Euler angle parametrization is made. Finally, the states within the octet are given and a description of the direct calculation of the Wigner Clebsch–Gordon coefficients is given that uses the invariant volume element.

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§ A bibliography on the developement of the quark model is given in [2].
‖ An excellent review of this material is contained in [3].

2. The ladder operations

The so-called ladder, or raising and lowering operators, take one state to another within an irrep. Their representation may be in terms of matrices or differential operators. The differential operators have been constructed here from linear combinations of the left-invariant vector fields in [9]. This enables one to analyse the states within a group representation. Most of this analysis has been performed using only the properties of the commutation relations which the differential operators can be shown to satisfy. These analyses will not be repeated here since they are well explained in various texts (see, for example, [10, 11]). What is important here is that the differential operators given can be shown to satisfy the commutation relations on the D matrices of the next section and therefore represent the Lie algebra as claimed.

First the left differential operators, that is, those that are constructed from the left-invariant vector fields of [9] are given. These change the labels on the left of the brackets used to represent the elements of the D matrices which indicates the change from one state within an irrep to another. The explicit forms of these are given appendix A. These operations are given explicitly by example below. One may take note that the right 'raising' operations are given by the subtraction of two elements of the corresponding Λs. This is due to the commutation relations that are obeyed by the right operators. They satisfy (see, for instance, [12])

$$[\Lambda_i^r, \Lambda_j^r] = -2\mathrm{i}\epsilon_{ijk}\Lambda_k^r$$

whereas the left operators satisfy

$$[\Lambda_i, \Lambda_j] = 2\mathrm{i}\epsilon_{ijk}\Lambda_k.$$

3. The fundamental representations

Here the states of the fundamental representations are exhibited explicitly and one may check through a straightforward calculation that they are related through the raising and lowering operations defined above. First the **3** representation.

$$D(\alpha,\beta,\gamma,\theta,a,b,c,\phi) = \mathrm{e}^{(-\mathrm{i}\lambda_3\alpha)}\mathrm{e}^{(\mathrm{i}\lambda_2\beta)}\mathrm{e}^{(-\mathrm{i}\lambda_3\gamma)}\mathrm{e}^{(\mathrm{i}\lambda_5\theta)}\mathrm{e}^{(-\mathrm{i}\lambda_3 a)}\mathrm{e}^{(\mathrm{i}\lambda_2 b)}\mathrm{e}^{(-\mathrm{i}\lambda_3 c)}\mathrm{e}^{(-\mathrm{i}\lambda_8\phi)}. \tag{1}$$

This matrix actually corresponds to the complex conjugate of the matrix D in [9] as is common. The particular signs of the exponents correspond to a choice of phase that is a generalization of the Condon and Shortley phase convention (see [13]). This makes the root operators positive or zero. Matrix elements can be labelled by their eigenvalues as below, where the following definition is used

$$\left\langle t_3', y' \,\middle|\, t_3, y\right\rangle \equiv D^{(1,0)}_{t_3,y;t_3',y'}$$

$$D^{(1,0)}_{t_3,y;t_3',y'} = \begin{pmatrix} \left\langle \frac12,\frac13 \middle| \frac12,\frac13\right\rangle & \left\langle \frac12,\frac13 \middle| -\frac12,\frac13\right\rangle & \left\langle \frac12,\frac13 \middle| 0,-\frac23\right\rangle \\ \left\langle -\frac12,\frac13 \middle| \frac12,\frac13\right\rangle & \left\langle -\frac12,\frac13 \middle| -\frac12,\frac13\right\rangle & \left\langle -\frac12,\frac13 \middle| 0,-\frac23\right\rangle \\ \left\langle 0,-\frac23 \middle| \frac12,\frac13\right\rangle & \left\langle 0,-\frac23 \middle| -\frac12,\frac13\right\rangle & \left\langle 0,-\frac23 \middle| 0,-\frac23\right\rangle \end{pmatrix}. \tag{2}$$

These matrix elements correspond to the functions:

$$\left\langle \tfrac12,\tfrac13 \middle| \tfrac12,\tfrac13\right\rangle = \mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{-\mathrm{i}c}\mathrm{e}^{-\mathrm{i}\eta}(\mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}a}\cos\beta\cos b\cos\theta - \mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}a}\sin\beta\sin b)$$
$$\left\langle \tfrac12,\tfrac13 \middle| -\tfrac12,\tfrac13\right\rangle = \mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}c}\mathrm{e}^{-\mathrm{i}\eta}(\mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}a}\cos\beta\sin b\cos\theta + \mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}a}\sin\beta\cos b)$$
$$\left\langle \tfrac12,\tfrac13 \middle| 0,-\tfrac23\right\rangle = \mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{2\mathrm{i}\eta}\cos\beta\sin\theta$$

$$\left\langle -\tfrac{1}{2}, \tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3} \right\rangle = -\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{-\mathrm{i}c}\mathrm{e}^{-\mathrm{i}\eta}(\mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}a}\sin\beta\cos b\cos\theta + \mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}a}\cos\beta\sin b)$$
$$\left\langle -\tfrac{1}{2}, \tfrac{1}{3} \,\middle|\, -\tfrac{1}{2}, \tfrac{1}{3} \right\rangle = -\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}c}\mathrm{e}^{-\mathrm{i}\eta}(\mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}a}\sin\beta\sin b\cos\theta - \mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}a}\cos\beta\cos b)$$
$$\left\langle -\tfrac{1}{2}, \tfrac{1}{3} \,\middle|\, 0, -\tfrac{2}{3} \right\rangle = -\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{2\mathrm{i}\eta}\sin\beta\sin\theta$$
$$\left\langle 0, -\tfrac{2}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3} \right\rangle = -\mathrm{e}^{-\mathrm{i}a}\mathrm{e}^{-\mathrm{i}c}\mathrm{e}^{-\mathrm{i}\eta}\sin\theta\cos b$$
$$\left\langle 0, -\tfrac{2}{3} \,\middle|\, -\tfrac{1}{2}, \tfrac{1}{3} \right\rangle = -\mathrm{e}^{-\mathrm{i}a}\mathrm{e}^{\mathrm{i}c}\mathrm{e}^{-\mathrm{i}\eta}\sin b\sin\theta$$
$$\left\langle 0, -\tfrac{2}{3} \,\middle|\, 0, -\tfrac{2}{3} \right\rangle = \mathrm{e}^{2\mathrm{i}\eta}\cos\theta.$$

This is actually formed from D^* and the $\mathbf{3}^*$ representation is formed by the following replacements: $\{\lambda_1 \to \lambda_1,\ \lambda_2 \to -\lambda_2,\ \lambda_3 \to -\lambda_3,\ \lambda_4 \to \lambda_4,\ \lambda_5 \to -\lambda_5,\ \lambda_6 \to \lambda_6,\ \lambda_7 \to -\lambda_7,\ \lambda_8 \to -\lambda_8\}$ for the corresponding matrices in the $\mathbf{3}$ representation. The two fundamental representations are inequivalent so there exists no inner automorphism between them. This is the outer automorphism that preserves the ladder operations and the previous phase convention. The $\mathbf{3}^*$ representation is then found to be as follows:

$$D(\alpha, \beta, \gamma, \theta, a, b, c, \phi) = \mathrm{e}^{(\mathrm{i}\lambda_3\alpha)}\mathrm{e}^{(-\mathrm{i}\lambda_2\beta)}\mathrm{e}^{(\mathrm{i}\lambda_3\gamma)}\mathrm{e}^{(-\mathrm{i}\lambda_5\theta)}\mathrm{e}^{(\mathrm{i}\lambda_3 a)}\mathrm{e}^{(-\mathrm{i}\lambda_2 b)}\mathrm{e}^{(\mathrm{i}\lambda_3 c)}\mathrm{e}^{(\mathrm{i}\lambda_8\phi)}. \tag{3}$$

Its matrix elements can be labelled by their eigenvalues as follows:

$$D^{(0,1)}_{t_3,y;t_3',y'} = \begin{pmatrix} \left\langle -\tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, -\tfrac{1}{2}, -\tfrac{1}{3} \right\rangle & \left\langle -\tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, -\tfrac{1}{3} \right\rangle & \left\langle -\tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3} \right\rangle \\ \left\langle \tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, -\tfrac{1}{2}, -\tfrac{1}{3} \right\rangle & \left\langle \tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, -\tfrac{1}{3} \right\rangle & \left\langle \tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3} \right\rangle \\ \left\langle 0, \tfrac{2}{3} \,\middle|\, -\tfrac{1}{2}, -\tfrac{1}{3} \right\rangle & \left\langle 0, \tfrac{2}{3} \,\middle|\, \tfrac{1}{2}, -\tfrac{1}{3} \right\rangle & \left\langle 0, \tfrac{2}{3} \,\middle|\, 0, \tfrac{2}{3} \right\rangle \end{pmatrix} \tag{4}$$

$$\left\langle -\tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, -\tfrac{1}{2}, -\tfrac{1}{3} \right\rangle = \mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}c}\mathrm{e}^{\mathrm{i}\eta}(\mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}a}\cos\beta\cos b\cos\theta - \mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}a}\sin\beta\sin b)$$
$$\left\langle -\tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, -\tfrac{1}{3} \right\rangle = -\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{-\mathrm{i}c}\mathrm{e}^{\mathrm{i}\eta}(\mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}a}\cos\beta\sin b\cos\theta + \mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}a}\sin\beta\cos b)$$
$$\left\langle -\tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3} \right\rangle = -\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{-2\mathrm{i}\eta}\cos\beta\sin\theta$$
$$\left\langle \tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, -\tfrac{1}{2}, -\tfrac{1}{3} \right\rangle = \mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}c}\mathrm{e}^{\mathrm{i}\eta}(\mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}a}\sin\beta\cos b\cos\theta + \mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}a}\cos\beta\sin b)$$
$$\left\langle \tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, -\tfrac{1}{3} \right\rangle = -\mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{-\mathrm{i}c}\mathrm{e}^{\mathrm{i}\eta}(\mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}a}\sin\beta\sin b\cos\theta - \mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}a}\cos\beta\cos b)$$
$$\left\langle \tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3} \right\rangle = -\mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{-2\mathrm{i}\eta}\sin\beta\sin\theta$$
$$\left\langle 0, \tfrac{2}{3} \,\middle|\, -\tfrac{1}{2}, -\tfrac{1}{3} \right\rangle = \mathrm{e}^{\mathrm{i}a}\mathrm{e}^{\mathrm{i}c}\mathrm{e}^{\mathrm{i}\eta}\sin\theta\cos b$$
$$\left\langle 0, \tfrac{2}{3} \,\middle|\, \tfrac{1}{2}, -\tfrac{1}{3} \right\rangle = -\mathrm{e}^{\mathrm{i}a}\mathrm{e}^{-\mathrm{i}c}\mathrm{e}^{\mathrm{i}\eta}\sin\theta\sin b$$
$$\left\langle 0, \tfrac{2}{3} \,\middle|\, 0, \tfrac{2}{3} \right\rangle = \mathrm{e}^{-2\mathrm{i}\eta}\cos\theta.$$

Note that the D matrices are labelled properly in the following form (the t label was not necessary in the fundamental representations nor would it be on any triangular representation, $D^{(p,0)}$ or $D^{(0,q)}$):

$$D^{(p,q)}_{(t,t_3,y;t',t_3',y')}.$$

4. General irreducible representations

In the Euler angle coordinates, the states within an irrep may be obtained in two ways. One is to exponentiate the algebra and multiply the matrices in the decompostion given in [9], or (1). Another way is to find the maximum weight state of the irrep and use the raising and lowering operations to derive the other states within that irrep. This maximum weight state can be found as follows.

For each irrep there exists a unique maximum weight state, $D_m^{(p,q)}$, that can be defined by the following equations:

$$V_+ D_m^{(p,q)} = 0 \qquad V_+^r D_m^{(p,q)} = 0$$
$$U_+ D_m^{(p,q)} = 0 \qquad U_+^r D_m^{(p,q)} = 0$$
$$T_+ D_m^{(p,q)} = 0 \qquad T_+^r D_m^{(p,q)} = 0.$$

When one solves these equations and satisfies the conditions for the first two or three reps, one finds that in this parametrization

$$D_m^{(p,q)} = \mathrm{e}^{-\mathrm{i}(2q+p)\eta}\mathrm{e}^{-\mathrm{i}p\alpha}\mathrm{e}^{-\mathrm{i}pc}\sum_{n=0}^{p}(-1)^{n+1}\binom{p}{n} \times(\mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}a}\cos\beta\cos b\cos\theta)^n(\mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}a}\sin\beta\sin b)^{p-n}\cos^q\theta. \tag{5}$$

Note 1. This is not the maximum state defined in [10, 11].

The maximum state could also be labelled with t_{3m} and y_m, which denote the value of t_3 and y for this maximum state. In terms of p and q these are

$$y_m = \tfrac{1}{3}(2q+p) \qquad t_{3m} = \tfrac{1}{2}p.$$

5. The octet

The octet is the smallest nontrivial example within which there exists two different states with the same t_3 and y. These will have different total isospin since they belong to different isospin representations. Thus it may be used as an example of how to find the Clebsch–Gordon coefficients using the explicit D matrices.

The octet is an irrep with eight states (hence the name). It can be obtained from the product of $D^{(1,0)}$ and $D^{(0,1)}$ from which a scalar $D^{(0,0)}$ is removed. Thus it is denoted by $D^{(1,1)}$. For it, the maximum weight state is given by the equation in the last section by substitution of the explicit p and q,

$$D_m^{(1,1)} = \mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{-\mathrm{i}c}\mathrm{e}^{-3\mathrm{i}\eta}\cos(\theta)[\mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}a}\cos(\beta)\cos(b)\cos(\theta) - \mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}a}\sin(\beta)\sin(b)].$$

For calculational purposes it is more convenient to notice that this may be written as

$$D_m^{(1,1)} = \left\langle \tfrac{1}{2}, \tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3}\right\rangle\left\langle 0, \tfrac{2}{3} \,\middle|\, 0, \tfrac{1}{3}\right\rangle.$$

From this state, operation by V_- will give one of the two different centre states, each having (t_3, y) given by $(0, 0)$. The first is given by

$$V_- D_m^{(1,1)} = \left\langle -\tfrac{1}{2}, \tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3}\right\rangle\left\langle \tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3}\right\rangle$$

and the second by

$$T_- U_- D_m^{(1,1)} = \left\langle -\tfrac{1}{2}, \tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3}\right\rangle\left\langle \tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3}\right\rangle + \left\langle \tfrac{1}{2}, \tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3}\right\rangle\left\langle -\tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3}\right\rangle.$$

The other states are as follows, listed counterclockwise around the hexagon starting from the one after the maximum weight state.

$$U_- D_m^{(1,1)} = \left\langle \tfrac{1}{2}, \tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3}\right\rangle\left\langle \tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3}\right\rangle$$
$$V_- U_- D_m^{(1,1)} = \left\langle 0, -\tfrac{2}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3}\right\rangle\left\langle \tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3}\right\rangle$$
$$T_- V_- U_- D_m^{(1,1)} = \left\langle 0, -\tfrac{2}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3}\right\rangle\left\langle -\tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3}\right\rangle$$
$$U_+ T_- V_- U_- D_m^{(1,1)} = \left\langle -\tfrac{1}{2}, \tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3}\right\rangle\left\langle -\tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3}\right\rangle$$
$$V_+ U_+ T_- V_- U_- D_m^{(1,1)} = \left\langle -\tfrac{1}{2}, \tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3}\right\rangle\left\langle 0, \tfrac{2}{3} \,\middle|\, 0, \tfrac{2}{3}\right\rangle.$$

The two of concern here are the two centre states. From these, the linear combinations that give states that are members of $SU(2)$ isospin states will be used. This is easy to do. Simply take the arbitrary linear combination of the two and demand that T_+ and T_- on this state give zero. This linear combination is then a member of an isospin singlet. The other linear combination gives the centre state in an isospin triplet. These linear combinations are found to be

$$D^{(1,1)}_{(2,0,0;2,0,0)} = \left\langle -\tfrac{1}{2}, \tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3} \right\rangle \left\langle \tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3} \right\rangle$$

which is the member of the isospin triplet, and

$$D^{(1,1)}_{(0,0,0;0,0,0)} = \left\langle -\tfrac{1}{2}, \tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3} \right\rangle \left\langle \tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3} \right\rangle - \left\langle \tfrac{1}{2}, \tfrac{1}{3} \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3} \right\rangle \left\langle -\tfrac{1}{2}, -\tfrac{1}{3} \,\middle|\, 0, \tfrac{2}{3} \right\rangle \tag{6}$$

which is an isospin singlet. Thus the Clebsch–Gordon coefficients have been determined. This can be used as a general method for calculating them. One can simply demand that the states form complete horizontal isospin irreps in the t_3–y plane. These are not $SU(3)$ WCG (Wigner–Clebsch–Gordon) coefficients, but rather the coefficients of the linear combinations of $SU(2)$ irreps within $SU(3)$. The method of calculating the $SU(3)$ WCG coefficients is now straightforward and will be discussed next.

6. WCG coefficients for $SU(3)$

The WCG coefficients may now be calculated with the orthogonality relations between different states using the following group-invariant volume element. This may be found by using the (wedge) product of the left- (or right-) invariant 1-forms calculated in [9]. The result is the following:

$$\mathrm{d}V = \sin 2\beta \sin 2b \sin 2\theta \sin^2\theta \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma \, \mathrm{d}\theta \, \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}c \, \mathrm{d}\phi$$

where the ranges of integration are

$$0 \leqslant \alpha, \gamma, a, c < \pi$$
$$0 \leqslant \beta, b, \theta \leqslant \tfrac{1}{2}\pi \qquad \text{and} \qquad 0 \leqslant \phi < \sqrt{3}\pi.$$

These are not trivial to determine [14] since their determination is equivalent to determining the invariant volume of the group. With the D matrices given for the fundamental representations, one may infer these minimum values for the ranges of the angles by enforcing the orthogonality relations that these representation functions must satisfy. These orthogonality relations are given by

$$\int D^{(p_1,q_1)*}_{t_1,(t_3)_1,y_1;t'_1,(t'_3)_1,y'_1} D^{(p_2,q_2)}_{t_2,(t_3)_2,y_2;t'_2,(t'_3)_2,y'_2} \, \mathrm{d}V = \frac{V_0}{d} \delta_{p_1,p_2} \delta_{q_1,q_2} \delta_{t_1,t_2} \delta_{(t_3)_1,(t_3)_2} \delta_{y_1,y_2} \delta_{t'_1,t'_2} \delta_{(t'_3)_1,(t'_3)_2} \delta_{y'_1,y'_2} \tag{7}$$

where V_0 is the invariant volume of the group and d is the dimension of the representation, $d = \frac{1}{2}(p+1)(q+1)(p+q+2)$. Thus knowing that the integral of the product of an element of a D matrix with its complex conjugate is a constant that depends only on the dimensionality of the representation, and that the integral of its product with anything else is zero, provides equations that may be solved to find the ranges of the angles. The result for V_0 $(= \sqrt{3}\pi^5/4)$ agrees with what Marinov found ($V_0 = 3\sqrt{3}\pi^5/4$) to within a factor of 3 [14]. This may be explained by considering the structure of the group manifold. In [12] the group-invariant volume element for $SU(2)$ is derived. The normalization factor π^2 can be viewed as arising from the angles α, and β in the ordinary Euler angle parametrization of $SU(2)$;

$$U = e^{i\alpha J_3} e^{i\beta J_2} e^{i\gamma J_3}.$$

The factor of 2 comes from the covering of the northern and southern poles, or hemispheres. In the case of $SU(3)$, one may consider the possibility of three 'poles'. Thus we may consider the ranges

$$0 \leqslant \phi/\sqrt{3} < \pi \qquad 2\pi \leqslant \phi/\sqrt{3} < 3\pi \qquad \text{and} \qquad 4\pi \leqslant \phi/\sqrt{3} < 5\pi$$

for ϕ to cover the three poles.

The orthogonality relation for the $SU(3)$ representation matrices, with the constants determined, gives us a vital tool for the determination of the WCG coefficients of $SU(3)$. One may simply take a direct product of any representations and use the orthogonality relation to determine which, and how many, representations are contained in that direct product. The linear combinations of the states in a given representation can then be determined (with the coefficients being WCG coefficients) either by direct ladder operations that were given earlier, or by ensuring othogonality with the appropriate integration. The important result is the orthogonality relation with appropriate constants. This eliminates the problem faced by de Swart by solving his 'Γ' problem [15]; that is, one may now find the number of irreducible representations in any given representation by using the orthogonality conditions.

7. $SU(3)$ and $SO(8)$

The generic element of the adjoint representation, since it is real and unitary, is orthogonal. Since it also has determinant 1, it is an element of $SO(8)$. It is, however, a function of only eight angles. If we call this matrix R_{ij}, then it will satisfy the equation

$$U\lambda_i U^\dagger = R_{ij}\lambda_j$$

and

$$\Lambda_i^r = R_{ij}\Lambda_j.$$

Therefore we have a mapping from the left-invariant vector fields to the right-invariant vector fields given in [9] and therefore between the left and right differential operators. This relates the so-called body-fixed and space-fixed reference frames (see, for example, [12]).

This mapping is exhibited explicitly in appendix B.

8. Summary/conclusions

It has been shown that the operators from [9] provide a means for finding the irreps of $SU(3)$ by the construction of the ladder operators. The two fundamental reps and the octet rep have been exhibited explicitly. The highest weight state for any irrep was found, thus enabling the calculation of any state within any irrep. A determination of the ranges of the angles in the Euler angle parametrization was made and the calculation of WCG coefficients was discussed. Therefore a more complete description of the group $SU(3)$, its manifold and its explicitly parametrized irreps has been given than has been done in the past.

The Clebsch–Gordon coefficients (or WCG coefficients) were calculated by de Swart in [15] using only algebraic properties. The operators given here could mimic those results as well. The Euler angle parametrization was given by Beg and Ruegg along with a calculation of the differential operators that are valid for some particular cases and no attempt was made to find the corresponding right-invariant vector fields [16]. Holland [17] and Nelson [18] originally gave an account of the irreps of $SU(3)$, but the rep matrices were presented in a somewhat less manageable form. These were also investigated by Akyeampong and Rashid [19]. It is anticipated that this more manageable account will lead to new applications. It has already proven to be useful in the description of three-state systems. This will be discussed elsewhere.

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Appendix A. Explicit forms of the invariant vector fields

Below are the explicit forms of the raising, lowering and eigenvalue operators in the Euler angle parametrization. One may consult [10] or [11] for a review of the commutation relations and the results of these actions on states in the t_3–y plane.

In what follows

$$\partial_1 \equiv \frac{\partial}{\partial\alpha} \qquad \partial_2 \equiv \frac{\partial}{\partial\beta} \qquad \partial_3 \equiv \frac{\partial}{\partial\gamma} \qquad \partial_4 \equiv \frac{\partial}{\partial\theta}$$
$$\partial_5 \equiv \frac{\partial}{\partial a} \qquad \partial_6 \equiv \frac{\partial}{\partial b} \qquad \partial_7 \equiv \frac{\partial}{\partial c} \qquad \partial_8 \equiv \frac{\partial}{\partial\phi}$$

$$T_+ = \frac{1}{2}(\Lambda_1 + \mathrm{i}\Lambda_2) = \frac{1}{2}\mathrm{e}^{-2\mathrm{i}\alpha}\left(\mathrm{i}\cot 2\beta\,\partial_1 - \partial_2 - \frac{\mathrm{i}}{\sin 2\beta}\,\partial_3\right) \tag{A1}$$
$$T_- = \frac{1}{2}(\Lambda_1 - \mathrm{i}\Lambda_2) = \frac{1}{2}\mathrm{e}^{2\mathrm{i}\alpha}\left(\mathrm{i}\cot 2\beta\,\partial_1 + \partial_2 - \frac{\mathrm{i}}{\sin 2\beta}\,\partial_3\right) \tag{A2}$$

$$\begin{aligned} V_+ &= \frac{1}{2}(\Lambda_4 + \mathrm{i}\Lambda_5) \\ &= \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)}\frac{\sin\beta}{\sin 2\beta}\cot\theta\,\partial_1 + \frac{1}{2}\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)}\sin\beta\cot\theta\,\partial_2 - \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)}\cot 2\beta\sin\beta\cot\theta\,\partial_3 \\ &\quad + \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)}\frac{(2-\sin^2\theta)}{\sin 2\theta}\cos\beta\,\partial_3 - \frac{1}{2}\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)}\cos\beta\,\partial_4 - \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)}\frac{2\cos\beta}{\sin 2\theta}\,\partial_5 \\ &\quad - \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(\alpha-\gamma-2a)}\frac{\cot 2b}{\sin\theta}\sin\beta\,\partial_5 - \frac{1}{2}\mathrm{e}^{-\mathrm{i}(\alpha-\gamma-2a)}\frac{\sin\beta}{\sin\theta}\,\partial_6 + \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(\alpha-\gamma-2a)}\frac{\sin\beta}{\sin\theta\sin 2b}\,\partial_7 \\ &\quad - \frac{3}{4}\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)}\tan\theta\cos\beta\;Y_8 \end{aligned} \tag{A3}$$

$$
\begin{aligned}
V_- &= \frac{1}{2}(\Lambda_4 - \mathrm{i}\Lambda_5) \\
&= \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\alpha+\gamma)}\frac{\sin\beta}{\sin 2\beta}\cot\theta\,\partial_1 - \frac{1}{2}\mathrm{e}^{\mathrm{i}(\alpha+\gamma)}\sin\beta\cot\theta\,\partial_2 - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\alpha+\gamma)}\cot 2\beta\sin\beta\cot\theta\,\partial_3 \\
&\quad + \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\alpha+\gamma)}\frac{(2-\sin^2\theta)}{\sin 2\theta}\cos\beta\,\partial_3 + \frac{1}{2}\mathrm{e}^{\mathrm{i}(\alpha+\gamma)}\cos\beta\,\partial_4 - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\alpha+\gamma)}\frac{2\cos\beta}{\sin 2\theta}\,\partial_5 \\
&\quad - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\alpha-\gamma-2a)}\frac{\cot 2b}{\sin\theta}\sin\beta\,\partial_5 + \frac{1}{2}\mathrm{e}^{\mathrm{i}(\alpha-\gamma-2a)}\frac{\sin\beta}{\sin\theta}\,\partial_6 + \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\alpha-\gamma-2a)}\frac{\sin\beta}{\sin\theta\sin 2b}\,\partial_7 \\
&\quad - \frac{3}{4}\mathrm{e}^{\mathrm{i}(\alpha+\gamma)}\tan\theta\cos\beta\;Y_8
\end{aligned} \tag{A4}
$$

$$
\begin{aligned}
U_+ &= \frac{1}{2}(\Lambda_6 + \mathrm{i}\Lambda_7) \\
&= \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\alpha-\gamma)}\frac{\cos\beta}{\sin 2\beta}\cot\theta\,\partial_1 + \frac{1}{2}\mathrm{e}^{\mathrm{i}(\alpha-\gamma)}\cos\beta\cot\theta\,\partial_2 - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\alpha-\gamma)}\cot 2\beta\cos\beta\cot\theta\,\partial_3 \\
&\quad - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\alpha-\gamma)}\frac{(2-\sin^2\theta)}{\sin 2\theta}\sin\beta\,\partial_3 + \frac{1}{2}\mathrm{e}^{\mathrm{i}(\alpha-\gamma)}\sin\beta\,\partial_4 + \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\alpha-\gamma)}\frac{2\sin\beta}{\sin 2\theta}\,\partial_5 \\
&\quad - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\alpha+\gamma+2a)}\frac{\cot 2b}{\sin\theta}\cos\beta\,\partial_5 - \frac{1}{2}\mathrm{e}^{\mathrm{i}(\alpha+\gamma+2a)}\frac{\cos\beta}{\sin\theta}\,\partial_6 + \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\alpha+\gamma+2a)}\frac{\cos\beta}{\sin\theta\sin 2b}\,\partial_7 \\
&\quad + \frac{3}{4}\mathrm{e}^{\mathrm{i}(\alpha-\gamma)}\tan\theta\sin\beta\;Y_8
\end{aligned} \tag{A5}
$$

$$
\begin{aligned}
U_- &= \frac{1}{2}(\Lambda_6 - \mathrm{i}\Lambda_7) \\
&= \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(\alpha-\gamma)}\frac{\cos\beta}{\sin 2\beta}\cot\theta\,\partial_1 - \frac{1}{2}\mathrm{e}^{-\mathrm{i}(\alpha-\gamma)}\cos\beta\cot\theta\,\partial_2 \\
&\quad - \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(\alpha-\gamma)}\cot 2\beta\cos\beta\cot\theta\,\partial_3 - \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(\alpha-\gamma)}\frac{(2-\sin^2\theta)}{\sin 2\theta}\sin\beta\,\partial_3 \\
&\quad - \frac{1}{2}\mathrm{e}^{-\mathrm{i}(\alpha-\gamma)}\sin\beta\,\partial_4 \\
&\quad + \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(\alpha-\gamma)}\frac{2\sin\beta}{\sin 2\theta}\,\partial_5 - \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(\alpha+\gamma+2a)}\frac{\cot 2b}{\sin\theta}\cos\beta\,\partial_5 \\
&\quad + \frac{1}{2}\mathrm{e}^{-\mathrm{i}(\alpha+\gamma+2a)}\frac{\cos\beta}{\sin\theta}\,\partial_6 + \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(\alpha+\gamma+2a)}\frac{\cos\beta}{\sin\theta\sin 2b}\,\partial_7 \\
&\quad + \frac{3}{4}\mathrm{e}^{-\mathrm{i}(\alpha-\gamma)}\tan\theta\sin\beta\;Y_8
\end{aligned} \tag{A6}
$$

$$
T_3 = \frac{\mathrm{i}}{2}\,\partial_1 \tag{A7}
$$

$$
Y = \mathrm{i}\,\partial_3 - \mathrm{i}\,\partial_5 + \mathrm{i}\frac{1}{\sqrt{3}}\,\partial_8 \tag{A8}
$$

where I have omitted a 'left' designation. The right differential operators have a superscript r. These are given by the following equations:

$$
T_-^r = \frac{1}{2}(\Lambda_1^r + \mathrm{i}\Lambda_2^r) = \frac{1}{2}\mathrm{e}^{2\mathrm{i}c}\left(-\mathrm{i}\cot 2b\,\partial_7 - \partial_6 + \frac{\mathrm{i}}{\sin 2b}\,\partial_5\right) \tag{A9}
$$

$$
T_+^r = \frac{1}{2}(\Lambda_1^r - \mathrm{i}\Lambda_2^r) = \frac{1}{2}\mathrm{e}^{-2\mathrm{i}c}\left(-\mathrm{i}\cot 2b\,\partial_7 + \partial_6 + \frac{\mathrm{i}}{\sin 2b}\,\partial_5\right) \tag{A10}
$$

$$
\begin{aligned}
V_-^r &= \frac{1}{2}(\Lambda_4^r + \mathrm{i}\Lambda_5^r) \\
&= -\frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(c+a+3\eta)}\frac{\sin b}{\sin 2b}\cot\theta\,\partial_7 + \frac{1}{2}\mathrm{e}^{\mathrm{i}(c+a+3\eta)}\sin b\cot\theta\,\partial_6 \\
&\quad + \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(c+a+3\eta)}\cot 2b\sin b\cot\theta\,\partial_5 - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(c+a+3\eta)}\frac{(2-\sin^2\theta)}{\sin 2\theta}\cos b\,\partial_5 \\
&\quad - \frac{1}{2}\mathrm{e}^{\mathrm{i}(c+a+3\eta)}\cos b\,\partial_4 + \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(c+a+3\eta)}\frac{2\cos b}{\sin 2\theta}\,\partial_3 + \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(c-a-2\gamma+3\eta)}\frac{\cot 2\beta}{\sin\theta}\sin b\,\partial_3 \\
&\quad - \frac{1}{2}\mathrm{e}^{\mathrm{i}(c-a-2\gamma+3\eta)}\frac{\sin b}{\sin\theta}\,\partial_2 - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(c-a-2\gamma+3\eta)}\frac{\sin b}{\sin\theta\sin 2\beta}\,\partial_1 + \frac{3}{4}\mathrm{e}^{\mathrm{i}(c+a+3\eta)}\tan\theta\cos b\,Y_8^r
\end{aligned}
\tag{A11}
$$

$$
\begin{aligned}
V_+^r &= \frac{1}{2}(\Lambda_4^r - \mathrm{i}\Lambda_5^r) \\
&= -\frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(c+a+3\eta)}\frac{\sin b}{\sin 2b}\cot\theta\,\partial_7 - \frac{1}{2}\mathrm{e}^{-\mathrm{i}(c+a+3\eta)}\sin b\cot\theta\,\partial_6 \\
&\quad + \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(c+a+3\eta)}\cot 2b\sin b\cot\theta\,\partial_5 - \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(c+a+3\eta)}\frac{(2-\sin^2\theta)}{\sin 2\theta}\cos b\,\partial_5 \\
&\quad + \frac{1}{2}\mathrm{e}^{-\mathrm{i}(c+a+3\eta)}\cos b\,\partial_4 + \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(c+a+3\eta)}\frac{2\cos b}{\sin 2\theta}\,\partial_3 + \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(c-a-2\gamma+3\eta)}\frac{\cot 2\beta}{\sin\theta}\sin b\,\partial_3 \\
&\quad + \frac{1}{2}\mathrm{e}^{-\mathrm{i}(c-a-2\gamma+3\eta)}\frac{\sin b}{\sin\theta}\,\partial_2 - \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(c-a-2\gamma+3\eta)}\frac{\sin b}{\sin\theta\sin 2\beta}\,\partial_1 \\
&\quad + \frac{3}{4}\mathrm{e}^{-\mathrm{i}(c+a+3\eta)}\tan\theta\cos b\,Y_8^r
\end{aligned}
\tag{A12}
$$

$$
\begin{aligned}
U_-^r &= \frac{1}{2}(\Lambda_6^r + \mathrm{i}\Lambda_7^r) \\
&= \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(c-a-3\eta)}\frac{\cos b}{\sin 2b}\cot\theta\,\partial_7 - \frac{1}{2}\mathrm{e}^{-\mathrm{i}(c-a-3\eta)}\cos b\cot\theta\,\partial_6 \\
&\quad - \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(c-a-3\eta)}\cot 2b\cos b\cot\theta\,\partial_5 - \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(c-a-3\eta)}\frac{(2-\sin^2\theta)}{\sin 2\theta}\sin b\,\partial_5 \\
&\quad - \frac{1}{2}\mathrm{e}^{-\mathrm{i}(c-a-3\eta)}\sin b\,\partial_4 + \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(c-a-3\eta)}\frac{2\sin b}{\sin 2\theta}\,\partial_3 - \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(c+a+2\gamma-3\eta)}\frac{\cot 2\beta}{\sin\theta}\cos b\,\partial_3 \\
&\quad + \frac{1}{2}\mathrm{e}^{-\mathrm{i}(c+a+2\gamma-3\eta)}\frac{\cos b}{\sin\theta}\,\partial_2 + \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(c+a+2\gamma-3\eta)}\frac{\cos b}{\sin\theta\sin 2\beta}\,\partial_1 \\
&\quad + \frac{3}{4}\mathrm{e}^{-\mathrm{i}(c-a-3\eta)}\tan\theta\sin b\,Y_8^r
\end{aligned}
\tag{A13}
$$

$$
\begin{aligned}
U_+^r &= \frac{1}{2}(\Lambda_6^r - \mathrm{i}\Lambda_7^r) \\
&= \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(c-a-3\eta)}\frac{\cos b}{\sin 2b}\cot\theta\,\partial_7 + \frac{1}{2}\mathrm{e}^{\mathrm{i}(c-a-3\eta)}\cos b\cot\theta\,\partial_6 \\
&\quad - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(c-a-3\eta)}\cot 2b\cos b\cot\theta\,\partial_5 - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(c-a-3\eta)}\frac{(2-\sin^2\theta)}{\sin 2\theta}\sin b\,\partial_5
\end{aligned}
$$

$$+\frac{1}{2}\mathrm{e}^{\mathrm{i}(c-a-3\eta)}\sin b\,\partial_4+\frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(c-a-3\eta)}\frac{2\sin b}{\sin 2\theta}\,\partial_3-\frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(c+a+2\gamma-3\eta)}\frac{\cot 2\beta}{\sin\theta}\cos b\,\partial_3$$
$$-\frac{1}{2}\mathrm{e}^{\mathrm{i}(c+a+2\gamma-3\eta)}\frac{\cos b}{\sin\theta}\,\partial_2+\frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(c+a+2\gamma-3\eta)}\frac{\cos b}{\sin\theta\sin 2\beta}\,\partial_1+\frac{3}{4}\mathrm{e}^{\mathrm{i}(c-a-3\eta)}\tan\theta\sin b\;Y_8^r \tag{A14}$$

$$T_3^r=\frac{\mathrm{i}}{2}\,\partial_7 \tag{A15}$$

$$Y^r=\frac{1}{\sqrt{3}}\mathrm{i}\,\partial_8 \tag{A16}$$

where $\eta\equiv\phi/\sqrt{3}$.

Appendix B. The adjoint representation

$$R_{11}=\cos 2\alpha\cos 2\beta\cos\theta[\cos(2a+2\gamma)\cos 2b\cos 2c-\sin(2a+2\gamma)\sin 2c]$$
$$-\sin 2\alpha\cos\theta[\sin(2a+2\gamma)\cos 2b\cos 2c+\cos(2a+2\gamma)\sin 2c]$$
$$-\cos 2\alpha\sin 2\beta\left(1-\tfrac{1}{2}\sin^2\theta\right)\sin 2b\cos 2c$$

$$R_{12}=\sin 2\alpha\cos 2\beta\cos\theta[\cos(2a+2\gamma)\cos 2b\cos 2c-\sin(2a+2\gamma)\sin 2c]$$
$$+\cos 2\alpha\cos\theta[\sin(2a+2\gamma)\cos 2b\cos 2c+\cos(2a+2\gamma)\sin 2c]$$
$$-\sin 2\alpha\sin 2\beta\left(1-\tfrac{1}{2}\sin^2\theta\right)\sin 2b\sin 2c$$

$$R_{13}=\sin 2\beta\cos(2a+2\gamma)\cos 2b\cos 2c\cos\theta-\sin 2\beta\sin(2a+2\gamma)\sin 2c\cos\theta$$
$$+\cos 2\beta\left(1-\tfrac{1}{2}\sin^2\theta\right)\sin 2b\cos 2c$$

$$R_{14}=-\tfrac{1}{2}\cos(\alpha+\gamma)\cos\beta\sin 2\theta\sin 2b\cos 2c-\cos(\alpha-\gamma-2a)\sin\beta\cos 2b\cos 2c\sin\theta$$
$$+\sin(\alpha+\gamma+2a)\sin\beta\sin 2c\sin\theta$$

$$R_{15}=\tfrac{1}{2}\sin(\alpha+\gamma)\cos\beta\sin 2\theta\sin 2b\cos 2c+\sin(\alpha-\gamma-2a)\sin\beta\cos 2b\cos 2c\sin\theta$$
$$+\cos(\alpha+\gamma+2a)\sin\beta\sin 2c\sin\theta$$

$$R_{16}=\tfrac{1}{2}\cos(\alpha-\gamma)\sin\beta\sin 2\theta\sin 2b\cos 2c-\cos(\alpha-\gamma-2a)\cos\beta\cos 2b\cos 2c\sin\theta$$
$$+\sin(\alpha+\gamma+2a)\cos\beta\sin 2c\sin\theta$$

$$R_{17}=\tfrac{1}{2}\sin(\alpha-\gamma)\sin\beta\sin 2\theta\sin 2b\cos 2c-\sin(\alpha-\gamma-2a)\cos\beta\cos 2b\cos 2c\sin\theta$$
$$-\cos(\alpha+\gamma+2a)\cos\beta\sin 2c\sin\theta$$

$$R_{18}=-\frac{\sqrt{3}}{2}\sin^2\theta\sin 2b\cos 2c$$

$$R_{21}=\cos 2\alpha\cos 2\beta\cos\theta[\sin(2a+2\gamma)\cos 2c+\cos(2a+2\gamma)\cos 2b\sin 2c]$$
$$-\sin 2\alpha\cos\theta[\sin(2a+2\gamma)\cos 2b\sin 2c-\cos(2a+2\gamma)\cos 2c]$$
$$-\cos 2\alpha\sin 2\beta\left(1-\tfrac{1}{2}\sin^2\theta\right)\sin 2b\sin 2c$$

$$R_{22}=-\sin 2\alpha\cos 2\beta\cos\theta[\sin(2a+2\gamma)\cos 2c+\cos(2a+2\gamma)\cos 2b\sin 2c]$$
$$-\cos 2\alpha\cos\theta[\sin(2a+2\gamma)\cos 2b\sin 2c-\cos(2a+2\gamma)\cos 2c]$$
$$+\sin 2\alpha\sin 2\beta\left(1-\tfrac{1}{2}\sin^2\theta\right)\sin 2b\sin 2c$$

$$R_{23} = \sin 2\beta \cos\theta[\cos(2a + 2\gamma) \cos 2b \sin 2c + \sin(2a + 2\gamma) \cos 2c] + \cos 2\beta\left(1 - \tfrac{1}{2}\sin^2\theta\right) \sin 2b \sin 2c$$

$$R_{24} = -\tfrac{1}{2}\cos(\alpha + \gamma) \cos\beta \sin 2\theta \sin 2b \sin 2c + \sin(\alpha - \gamma - 2a) \sin\beta \sin\theta \cos 2c - \cos(\alpha - \gamma - 2a) \sin\beta \sin\theta \cos 2b \sin 2c$$

$$R_{25} = \tfrac{1}{2}\sin(\alpha + \gamma) \cos\beta \sin 2\theta \sin 2b \sin 2c + \cos(\alpha - \gamma - 2a) \sin\beta \sin\theta \cos 2c + \sin(\alpha - \gamma - 2a) \sin\beta \sin\theta \cos 2b \sin 2c$$

$$R_{26} = \tfrac{1}{2}\cos(\alpha - \gamma) \sin\beta \sin 2\theta \sin 2b \sin 2c - \sin(\alpha + \gamma + 2a) \cos\beta \sin\theta \cos 2c - \cos(\alpha + \gamma + 2a) \cos\beta \sin\theta \cos 2b \sin 2c$$

$$R_{27} = \tfrac{1}{2}\sin(\alpha - \gamma) \sin\beta \sin 2\theta \sin 2b \sin 2c + \cos(\alpha + \gamma + 2a) \cos\beta \sin\theta \cos 2c - \sin(\alpha + \gamma + 2a) \cos\beta \sin\theta \cos 2b \sin 2c$$

$$R_{28} = -\tfrac{1}{2}\sqrt{3}\sin^2\theta \sin 2b \sin 2c$$

$$R_{31} = -\cos 2\alpha \cos 2\beta \cos\theta \sin 2b \cos(2a + 2\gamma) + \sin 2\alpha \cos\theta \sin 2b \sin(2a + 2\gamma) - \cos 2\alpha \sin 2\beta\left(1 - \tfrac{1}{2}\sin^2\theta\right) \cos 2b$$

$$R_{32} = \sin 2\alpha \cos 2\beta \cos\theta \sin 2b \cos(2a + 2\gamma) + \cos 2\alpha \cos\theta \sin 2b \sin(2a + 2\gamma) + \sin 2\alpha \sin 2\beta\left(1 - \tfrac{1}{2}\sin^2\theta\right) \cos 2b$$

$$R_{33} = -\sin 2\beta \cos\theta \sin 2b \cos(2a + 2\gamma) + \cos 2\beta\left(1 - \tfrac{1}{2}\sin^2\theta\right) \cos 2b$$
$$R_{34} = -\tfrac{1}{2}\cos(\alpha + \gamma) \cos\beta \sin 2\theta \cos 2b + \cos(\alpha - \gamma - 2a) \sin\beta \sin\theta \sin 2b$$
$$R_{35} = \tfrac{1}{2}\sin(\alpha + \gamma) \cos\beta \sin 2\theta \cos 2b - \sin(\alpha - \gamma - 2a) \sin\beta \sin\theta \sin 2b$$
$$R_{36} = \tfrac{1}{2}\cos(\alpha - \gamma) \sin\beta \sin 2\theta \cos 2b + \cos(\alpha + \gamma + 2a) \cos\beta \sin\theta \sin 2b$$
$$R_{37} = \tfrac{1}{2}\sin(\alpha - \gamma) \sin\beta \sin 2\theta \cos 2b + \sin(\alpha + \gamma + 2a) \cos\beta \sin\theta \sin 2b$$
$$R_{38} = -\tfrac{1}{2}\sqrt{3}\sin^2\theta \cos 2b$$

$$R_{41} = -\cos 2\alpha \cos 2\beta \sin\theta \sin b \cos(a - c - 2\gamma - 3\eta) - \cos 2\alpha \sin 2\beta \sin 2\theta \cos(a + c + 3\eta) \cos b + \sin 2\alpha \sin\theta \sin b \sin(a - c - 2\gamma - 3\eta)$$

$$R_{42} = \sin 2\alpha \cos 2\beta \sin\theta \sin b \cos(a - c - 2\gamma - 3\eta) + \sin 2\alpha \sin 2\beta \sin 2\theta \cos(a + c + 3\eta) \cos b - \cos 2\alpha \sin\theta \sin b \sin(a - c - 2\gamma - 3\eta)$$

$$R_{43} = \sin 2\beta \sin\theta \sin b \cos(a - c - 2\gamma - 3\eta) + \cos 2\beta \sin 2\theta \cos(a + c + 3\eta) \cos b$$

$$R_{44} = \cos(\alpha + \gamma) \cos\beta \cos 2\theta \cos(a + c + 3\eta) \cos b - \sin(\alpha + \gamma) \cos\beta \sin(a + c + 3\eta) \cos b - \sin\beta \sin\theta \sin b \cos(a + \gamma - \alpha - c - 3\eta)$$

$$\begin{aligned} R_{45} = &- \sin(\alpha+\gamma)\cos\beta\cos 2\theta\cos(a+c+3\eta)\cos b \\ &- \cos(\alpha+\gamma)\cos\beta\sin(a+c+3\eta)\cos b \\ &- \sin\beta\sin\theta\sin b\sin(a+\gamma-\alpha-c-3\eta) \end{aligned}$$

$$\begin{aligned} R_{46} = &- \cos(\alpha-\gamma)\sin\beta\cos 2\theta\cos(a+c+3\eta)\cos b \\ &- \sin(\alpha-\gamma)\sin\beta\sin(a+c+3\eta)\cos b \\ &- \cos\beta\cos\theta\sin b\cos(a+\gamma+\alpha-c-3\eta) \end{aligned}$$

$$\begin{aligned} R_{47} = &- \sin(\alpha-\gamma)\sin\beta\cos 2\theta\cos(a+c+3\eta)\cos b \\ &+ \cos(\alpha-\gamma)\sin\beta\sin(a+c+3\eta)\cos b \\ &- \cos\beta\cos\theta\sin b\sin(a+\gamma+\alpha-c-3\eta) \end{aligned}$$

$$R_{48} = \sqrt{3}\sin 2\theta\cos(a+c+3\eta)\cos b$$

$$\begin{aligned} R_{51} = &\cos 2\alpha\cos 2\beta\sin\theta\sin b\sin(a-c-2\gamma-3\eta) \\ &- \cos 2\alpha\sin 2\beta\sin 2\theta\sin(a+c+3\eta)\cos b \\ &- \sin 2\alpha\sin\theta\sin b\cos(a-c-2\gamma-3\eta) \end{aligned}$$

$$\begin{aligned} R_{52} = &- \sin 2\alpha\cos 2\beta\sin\theta\sin b\sin(a-c-2\gamma-3\eta) \\ &+ \sin 2\alpha\sin 2\beta\sin 2\theta\sin(a+c+3\eta)\cos b \\ &- \cos 2\alpha\sin\theta\sin b\cos(a-c-2\gamma-3\eta) \end{aligned}$$

$$R_{53} = \sin 2\beta\sin\theta\sin b\sin(a-c-2\gamma-3\eta) + \cos 2\beta\sin 2\theta\cos b\sin(a+c+3\eta)$$

$$\begin{aligned} R_{54} = &\cos(\alpha+\gamma)\cos\beta\cos 2\theta\sin(a+c+3\eta)\cos b \\ &+ \sin(\alpha+\gamma)\cos\beta\cos(a+c+3\eta)\cos b \\ &+ \sin\beta\cos\theta\sin b\sin(a+\gamma-\alpha-c-3\eta) \end{aligned}$$

$$\begin{aligned} R_{55} = &- \sin(\alpha+\gamma)\cos\beta\cos 2\theta\sin(a+c+3\eta)\cos b \\ &+ \cos(\alpha+\gamma)\cos\beta\cos(a+c+3\eta)\cos b \\ &- \sin\beta\cos\theta\sin b\cos(a+\gamma-\alpha-c-3\eta) \end{aligned}$$

$$\begin{aligned} R_{56} = &- \cos(\alpha-\gamma)\sin\beta\cos 2\theta\sin(a+c+3\eta)\cos b \\ &+ \sin(\alpha-\gamma)\sin\beta\cos(a+c+3\eta)\cos b \\ &+ \cos\beta\cos\theta\sin b\sin(a+\gamma+\alpha-c-3\eta) \end{aligned}$$

$$\begin{aligned} R_{57} = &- \sin(\alpha-\gamma)\sin\beta\cos 2\theta\sin(a+c+3\eta)\cos b \\ &- \cos(\alpha-\gamma)\sin\beta\cos(a+c+3\eta)\cos b \\ &- \cos\beta\cos\theta\sin b\cos(a+\gamma+\alpha-c-3\eta) \end{aligned}$$

$$R_{58} = \sqrt{3}\sin 2\theta\sin(a+c+3\eta)\cos b$$

$$\begin{aligned} R_{61} = &\cos 2\alpha\cos 2\beta\sin\theta\cos b\cos(a+c-2\gamma-3\eta) \\ &- \cos 2\alpha\sin 2\beta\sin 2\theta\cos(a-c+3\eta)\sin b \\ &+ \sin 2\alpha\sin\theta\cos b\sin(a+c-2\gamma-3\eta) \end{aligned}$$

$$R_{62} = -\sin 2\alpha \cos 2\beta \sin\theta \cos b \sin(a + c - 2\gamma - 3\eta) + \sin 2\alpha \sin 2\beta \sin 2\theta \cos(a - c + 3\eta) \sin b + \cos 2\alpha \sin\theta \cos b \sin(a + c - 2\gamma - 3\eta)$$

$$R_{63} = -\sin 2\beta \sin\theta \cos b \cos(a + c - 2\gamma - 3\eta) - \cos 2\beta \sin 2\theta \sin b \cos(a - c + 3\eta)$$

$$R_{64} = \cos(\alpha + \gamma) \cos\beta \cos 2\theta \cos(a - c + 3\eta) \sin b - \sin(\alpha + \gamma) \cos\beta \sin(a - c + 3\eta) \sin b + \sin\beta \cos\theta \cos b \cos(a + c + \gamma - \alpha - 3\eta)$$

$$R_{65} = -\sin(\alpha + \gamma) \cos\beta \cos 2\theta \cos(a - c + 3\eta) \sin b - \cos(\alpha + \gamma) \cos\beta \sin(a - c + 3\eta) \sin b + \sin\beta \cos\theta \cos b \sin(a + c + \gamma - \alpha - 3\eta)$$

$$R_{66} = -\cos(\alpha - \gamma) \sin\beta \cos 2\theta \cos(a - c + 3\eta) \sin b - \sin(\alpha - \gamma) \sin\beta \sin(a - c + 3\eta) \sin b + \cos\beta \cos\theta \cos b \cos(a + c + \gamma + \alpha - 3\eta)$$

$$R_{67} = -\sin(\alpha - \gamma) \sin\beta \cos 2\theta \cos(a - c + 3\eta) \sin b + \cos(\alpha - \gamma) \sin\beta \sin(a - c + 3\eta) \sin b + \cos\beta \cos\theta \cos b \sin(a + c + \gamma + \alpha - 3\eta)$$

$$R_{68} = -\sqrt{3} \sin 2\theta \cos(a - c + 3\eta) \sin b$$

$$R_{71} = -\cos 2\alpha \cos 2\beta \sin\theta \cos b \sin(a + c - 2\gamma - 3\eta) - \cos 2\alpha \sin 2\beta \sin 2\theta \sin(a - c + 3\eta) \sin b + \sin 2\alpha \sin\theta \cos b \cos(a + c - 2\gamma - 3\eta)$$

$$R_{72} = \sin 2\alpha \cos 2\beta \sin\theta \cos b \sin(a + c - 2\gamma - 3\eta) + \sin 2\alpha \sin 2\beta \sin 2\theta \sin(a - c + 3\eta) \sin b + \cos 2\alpha \sin\theta \cos b \cos(a + c - 2\gamma - 3\eta)$$

$$R_{73} = -\sin 2\beta \sin\theta \cos b \sin(a + c - 2\gamma - 3\eta) + \cos 2\beta \sin 2\theta \sin b \sin(a - c + 3\eta)$$

$$R_{74} = \cos(\alpha + \gamma) \cos\beta \cos 2\theta \sin(a - c + 3\eta) \sin b + \sin(\alpha + \gamma) \cos\beta \cos(a - c + 3\eta) \sin b - \sin\beta \cos\theta \cos b \sin(a + c + \gamma - \alpha - 3\eta)$$

$$R_{75} = -\sin(\alpha + \gamma) \cos\beta \cos 2\theta \sin(a - c + 3\eta) \sin b + \cos(\alpha + \gamma) \cos\beta \cos(a - c + 3\eta) \sin b + \sin\beta \cos\theta \cos b \cos(a + c + \gamma - \alpha - 3\eta)$$

$$R_{76} = -\cos(\alpha - \gamma) \sin\beta \cos 2\theta \sin(a - c + 3\eta) \sin b + \sin(\alpha - \gamma) \sin\beta \cos(a - c + 3\eta) \sin b - \cos\beta \cos\theta \cos b \sin(a + c + \gamma + \alpha - 3\eta)$$

$$
\begin{aligned}
R_{77} = &-\sin(\alpha-\gamma)\sin\beta\cos 2\theta\sin(a-c+3\eta)\sin b \\
&-\cos(\alpha-\gamma)\sin\beta\cos(a-c+3\eta)\sin b \\
&+\cos\beta\cos\theta\cos b\cos(a+c+\gamma+\alpha-3\eta)
\end{aligned}
$$

$$R_{78} = \sqrt{3}\sin 2\theta\sin(a-c+3\eta)\sin b$$

$$
\begin{aligned}
R_{81} &= \tfrac{1}{2}\sqrt{3}\cos 2\alpha\sin 2\beta\sin^2\theta \\
R_{82} &= -\tfrac{1}{2}\sqrt{3}\sin 2\alpha\sin 2\beta\sin^2\theta \\
R_{83} &= -\tfrac{1}{2}\sqrt{3}\cos 2\beta\sin^2\theta \\
R_{84} &= -\tfrac{1}{2}\sqrt{3}\cos(\alpha+\gamma)\cos\beta\sin 2\theta \\
R_{85} &= \tfrac{1}{2}\sqrt{3}\sin(\alpha+\gamma)\cos\beta\sin 2\theta \\
R_{86} &= \tfrac{1}{2}\sqrt{3}\cos(\alpha-\gamma)\sin\beta\sin 2\theta \\
R_{87} &= \tfrac{1}{2}\sqrt{3}\sin(\alpha-\gamma)\sin\beta\sin 2\theta \\
R_{88} &= 1-\tfrac{3}{2}\sin^2\theta.
\end{aligned}
$$

Recall that $\eta \equiv \phi/\sqrt{3}$.

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