CONTEXTUAL BACKGROUND

E.C.G. Sudarshan:
PROBABILITY AND QUANTUM DYNAMICS

Johann von Neumann, the father of Hilbert space quantum mechanics, wrote to Birkhoff in a letter dated November 13, presumably 1935: "I would like to make a confession that may seem immoral: I do not believe absolutely in Hilbert space anymore" (Birkhoff 1961). With this statement, von Neumann indicated a shift in emphasis from states in a Hilbert space to the lattice of all linear closed subspaces of this Hilbert space. In the mid 1930s he and Birkhoff introduced lattice theoretical ideas into quantum theory, thus laying the foundations for the field of quantum logic (Birkhoff and von Neumann 1936).

There is another way, though, to look at von Neumann's statement, even if this might not have been his original intention. Since quite a number of basic problems is encountered when one tries to enforce a rigorous mathematical formulation of quantum mechanics in the usual Hilbert space representation, it is an interesting option to explore the possibility of representing states of a system not by rays in Hilbert space but by distributions in generalized spaces.

Dirac's formulation of quantum theory in terms of bras and kets (Dirac 1930) leads to one such difficulty: the δ-functions required in Dirac's approach are undefined in Hilbert space. A first step to meet this problem was done when Schwartz (1950) developed his theory of distributions with δ-functions as singular limiting cases. A special class of distribution spaces was introduced somewhat later by the Russian mathematician Gel'fand and his collaborators (Gel'fand and Vilenkin 1964): the so-called rigged Hilbert spaces. At about the same time, both J.E. Roberts and A. Bohm utilized rigged Hilbert spaces to make Dirac's formalism mathematically rigorous. By the end of the 1970s, it turned out that some basic physical problems of Hilbert space quantum mechanics, notably in the context of decaying states or resonances, could be clarified in terms of rigged Hilbert spaces. For more details see Bohm and Gadella (1989).

More recently, the rigged Hilbert space approach has been used quite extensively by the Brussels-Austin-group of Prigogine and collaborators. Their special interest in rigged Hilbert spaces was due to the fact that this formalism provides a natural way to derive two semigroups from the basic unitary time evolution of both classical and quantum systems under certain conditions (Antoniou and Prigogine 1993). One of these semigroups describes an evolution toward the future, the other toward the past. Hence it is clear that the rigged Hilbert space formulation does not dispense us from the need of (ad hoc) selecting the proper semigroup. In addition to the rigged Hilbert space approach, other extensions are possible such as, e.g., the Liouville extension (Petrosky and Prigogine 1997).

Sudarshan's contribution describes yet another way to generalize quantum theory in order to cover situations for which the usual Hilbert space approach is insufficient. In his formulation, the states of dynamical systems are identified with distributions which assign numerical values to all dynamical variables. The notion of dynamics is then generalized to mappings (which could include irreversible mappings) of the convex set of states onto themselves. Stochastic quantum dynamics is studied with regard to both the convex set of dynamical maps of density matrices and their generic construction as contractions of extended systems. The analytic continuation of this dynamics to dual analytic spaces is carried out and the dominant metastable modes are identified. The various generalized spaces, often used indiscriminately in the literature, are defined and distinguished. The concept of the age of a decaying system obeying a semigroup is introduced and illustrated.

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PROBABILITY AND QUANTUM DYNAMICS

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1. Introduction

The simplest dynamical system is the point particle characterized by its mass. Its state is specified by its position and momentum. The dynamical law is the description of how these quantities change in time. For a free particle, the momentum remains constant while the position increases in the direction of the momentum. The increase is directly proportional to the elapsed time and inversely proportional to the mass. When such a particle is subjected to a force, the momentum also changes proportional to the force. The instantaneous state may be represented as a point in phase space which moves along a trajectory. For complicated interactions, the trajectories may form an intricate pattern but they do not intersect, and the generic dynamical evolution may be viewed as a mapping of the phase space onto itself.

What happens when the forces acting on the particle are randomly fluctuating? The evolved phase point would also fluctuate; and the only way to specify the final state is to describe the probability for various points in phase space (Bachelier 1900, Einstein 1905). In any particular realization of the motion there would be a final specific phase point but there is no way to predict it; rather the predictable quantity is the probability distribution. This is the quantity that evolves according to a definite law. Since now the final state is a probability distribution in phase space, can we consider a probability distribution over phase space as the generic state (Segal 1947, Haag and Kastler 1964)? The canonical transformations that bring about Hamiltonian time evolution would now be viewed as the evolution of the probability distribution. For a canonical evolution, the probability distribution over the phase space variables behaves as a scalar field:

$$\omega \to \omega'; \quad p \to p'; \quad p'(\omega') = p(\omega). \tag{1.1}$$

The case of a fluctuating force acting on the system is more complicated; it depends on the statistical distribution of the interactions. In particular, a "pure" initial state which is concentrated on a phase point can become a smooth distribution corresponding to a "mixed" statistical state. For a Hamiltonian evolution, the mappings of phase point to phase point as well as of the density distribution are invertible – there is no irreversibility. This way of viewing statistical states of a classical dynamical system (Segal 1947, Haag and Kastler 1964) is in contrast to the popular view that the probability distribution reflects our ignorance of the phase point and that any particle will have a definite position and a definite momentum. While every realization can be a pure state, these pure states do not have a definite evolution. Realizations are pure but the distributions are the entities which evolve according to a well-defined law.

We must also recognize that even the best measurements provide small patches of phase space, and even with a Hamiltonian evolution such a phase patch can spread into a long-tentacled octopus shape. So "almost pure" states can become pretty much mixed up. On the other hand, the probability distributions have a well-defined evolution law, no matter whether the evolution is Hamiltonian or not. Such a description can incorporate irreversible evolutions (Prigogine 1997).

Probability distributions are non-negative measures on phase space and form a convex set (Segal 1947, Haag and Kastler 1964). The distribution

$$p(\omega) = p_1(\omega)\cos^2\Theta + p_2(\omega)\sin^2\Theta \tag{1.2}$$

is an admissible distribution provided $p_1(\omega)$ and $p_2(\omega)$ are admissible distributions. The extremal states are the states whose distributions are concentrated on individual phase points.

Recall that phase space has a symplectic geometry and that, therefore, there is no intrinsic notion of "distance" between two distinct phase space points (Sudarshan et al. 1961, Sudarshan and Mukunda 1974). But there are no guarantees that "nearby" states evolve into "nearby" states, however the notion of "nearness" is defined. The octopus-like phase distribution evolving from a phase patch is evidence of this behavior. But this is not an irreversible evolution, since retracing the final distribution to the initial regular patch is a Hamiltonian evolution, and if we go back further the regular patch would be seen to have evolved from an octopus-like irregular patch. Of course, this does only apply for a reversible Hamiltonian evolution and not for stochastic evolutions.

Though we have commented on Hamiltonian classical systems so far, the concept of distributions applies equally for dynamical systems in a generic sense. For example, if we have a finite set of phase points, only the generic state is a probability vector whose elements sum up to unity. The

evolution maps are now stochastic matrices (Ramakrishnan 1959), and the only reversible evolutions are permutations. Hence, there are no continuous reversible evolutions in time for such systems. The generic stochastic evolution is a contraction map with limit states toward which the map converges. Even "pure" states which are concentrated on a specific phase point converge to "mixed" states. Given a generic state of such a system, we could retrace its evolution for a finite time. At that time the inverse image is a statistical state which cannot be traced back any further unless the non-negativity of the elements of the probability vector is violated.

The dynamical evolutions form a convex set. If the map is

$$p(\omega) \to A\{p(\omega)\} \ ,$$

then

$$A = \cos^2 \Theta A_1 + \sin^2 \Theta A_2 \ ,$$

that is,

$$A\{p(\omega)\} = \cos^2 \Theta A_1\{p(w)\} + \sin^2 \Theta A_2\{p(\omega)\}$$

is also an admissible stochastic evolution. For stochastic matrices this implies that the conditions

$$A_{j,k} \gg 0; \quad \sum A_{j,k} = 1$$

are preserved by such convex combinations. The possibility of a measurement of all dynamical variables to arbitrary accuracy enables us to view the extremal states as probability distributions concentrated on a point in phase space. If this is not true, i.e., if the algebra of dynamical variables is non-Abelian, then the states can no longer be such concentrations (Segal 1947, Haag and Kastler 1964), nor need the distributions be pointwise positive over the phase space. For a canonical quantum system, the Wigner-Moyal distribution (Wigner 1932, Moyal 1949) and the diagonal coherent state distribution (Sudarshan 1963, Mehta and Sudarshan 1965) are not non-negative, nor are the extremal distributions always concentrated at phase points.

For quantum systems with a discrete set of states, the kinematic characterization of a state is given by a non-negative matrix of unit trace. These characterizations form a convex set, the extremal states being projections of rank one. The dynamical evolution must map the corresponding matrix into a matrix of the same set. We can now have reversible non-trivial evolutions corresponding to unitary transformations of this matrix. In addition, there are also the stochastic evolutions, which are irreversible.

2. Quantum Kinematics, Convex Sets of States

The generic state of a quantum system is specified by its density distribution which may be viewed as a linear non-negative normalized number-valued linear functional (Segal 1947, Haag and Kastler 1964; singular linear functionals are used in Sudarshan (1963) and in Mehta and Sudarshan (1965)) on the operators. In a more restrictive form, in which a state is identified as a trace class operator (von Neumann 1955) in the Hilbert space, it has the canonical decomposition

$$\rho = \sum_{1}^{\infty} c_n \psi_n \tilde{\psi}_n; \quad c_n > 0; \quad \sum_{1}^{\infty} c_n = 1 \tag{2.1}$$

with

$$\tilde{\psi} = \psi^{\dagger} . \tag{2.2}$$

The set of density distributions can be enlarged provided the dynamical variables for which expectation values are sought are restricted. Conversely, if the set of density distributions is restricted a wider set of dynamical variables may be constructed.

The conditions of positivity and normalization still allow us to form normalized convex combinations:

$$\rho = \rho_1 \cos^2 \theta + \rho_2 \sin^2 \theta . \tag{2.3}$$

They *do not* form a vector space. For several purposes including that of the stochastic dynamics of density distributions, it is advantageous to consider the vector space generated by the density distributions. The additional distributions so obtained may not satisfy either positivity or normalization, or both. We will see that metastable (decaying) "states" are realized by such pseudodensity distributions.

Given the convex set of density distributions, we could seek the boundary elements and, more specifically, the generating extremal elements. The latter are those density distributions whose (normalized) convex combinations generate all density distributions but which themselves have no nontrivial decomposition. Trace class operators in Hilbert space are a compact set under the Hilbert-Schmidt norm and the extremals are one-dimensional projections. For more general definitions of the density distributions, it is necessary to investigate the situation case by case.

If the dynamical variables undergo a unitary transformation, their expectation values change. This is equivalent to a linear transformation on the density distributions:

$$\rho_U(A) = \rho(UAU) . \tag{2.4}$$

If the unitary transformations concerned form a group, then the linear transformations

$$\rho \to \rho U \tag{2.5}$$

furnish a realization of the same group. On the other hand, if we consider the linear transformations on the density distribution we have new possibilities. This is particularly the case for time evolutions. For simple Hamiltonian systems there is a one-parameter group of transformations on the density distributions, but we have the more general possibility of non-invertible dissipative transformations. Whenever such a dissipation is involved the inverse transformations cannot act on all density distributions. Instead of a time translation group the best we can obtain is a semigroup of dissipative evolutions. In either case we refer to the generator of the group or the semigroup as the Liouvillean.

The generic time evolution is a subject of stochastic dynamics (Sudarshan et al. 1961) and has been systematically studied for decades (Davies 1969, 1970, 1971). The special case of the relaxation of spin systems in an external magnetic field was studied in terms of the Bloch equations half a century ago (Lindblad 1975, Gorini et al. 1976, Gorini et al. 1978). But the convex set of dynamical maps has an intricate structure even for 2×2 density matrices.

One way of arriving at stochastic dynamics is by considering the system as being embedded in a larger system with a time translation group and then contracting out the extraneous degrees of freedom. When one does this one arrives at a subclass of stochastic dynamical maps, namely those of the completely positive type. Conversely, given a completely positive dynamical map we can realize it constructively in terms of the contraction of an extended Hamiltonian time translation (Bloch 1946).

Stochastic dynamics thus involves a *sense of time* and thus a *breaking of time symmetry*. In the contraction procedure the time symmetry breaking is explicit. It has been of continuing interest for more than a century whether the breaking can occur spontaneously, without any explicitly asymmetric procedure and without any restriction on the initial states. On the other hand, if the dynamical laws are time symmetric it is to be expected that the time reversed sequence of states corresponds to a time reversed semigroup. What, then, selects the forward semigroup for time evolution? In other words: what is the ingredient implicit in the dynamics or involved in the choice of physical states that assures the time symmetry breaking second law of thermodynamics? With the increasing recognition of the role of deterministic chaos and the relevance of the Poincaré catastrophe for large Poincaré systems the simple objections of Loschmidt and Zermelo to Boltzmann's H-theorem appear to be less compelling.

This paper is an attempt to clarify this issue by a careful characterization of the various families of states, of the varieties of dual pairs of states and dynamical variables. It appears from this analysis that states exhibiting dissipation and hence breaking time symmetry are to be selected from an extended set of states. There are the time reversed states which exhibit negative dissipation and, hence, are not acceptable as physical states obeying the second law of thermodynamics.

3. Stochastic Quantum Dynamics

For a finite dimensional system, the density distributions are non-negative density matrices of unit trace:

$$\rho \neq 0; \quad \mathrm{Tr}\,\rho = 1; \quad \dim \rho = N\,. \tag{3.1}$$

The extremal generating elements of the convex set of density matrices are projections of rank one:

$$\rho = \Pi_\psi = \psi\tilde{\psi}; \quad \tilde{\psi} = \psi^\dagger\,. \tag{3.2}$$

There is an infinite number of such extremal elements. For a time-independent (Hermitian) Hamiltonian H, the time evolutions are given by a unitary one-parameter group:

$$\rho(t) = \exp\,(-\mathrm{i}\mathcal{L}t)\rho \equiv \mathrm{e}^{-\mathrm{i}tH}\,\rho\,\mathrm{e}^{\mathrm{i}tH}\,. \tag{3.3}$$

Here, $\mathcal{L}$ is the Liouville superoperator

$$\mathcal{L}\rho = H\rho - \rho H\,. \tag{3.4}$$

A much more general evolution is given by a parametrized map:

$$\rho \to A(t)\rho; \quad \rho_{rs}(t) = \sum_{r's'} A_{rs,r's'}(t)\,\rho_{r's'}\,. \tag{3.5}$$

Since the properties of a density matrix must be preserved by the mapping, we have

$$\sum_r A_{rr,r's'} = \delta_{r's'}; \quad A_{sr,s'r'} = A^*_{rs,r's'}\,. \tag{3.6}$$

If we define the new $N^2 \times N^2$ matrix B with elements

$$B_{rr',ss'} = A_{rs,r's'}\,, \tag{3.7}$$

then the properties of A may be used to deduce

$$B_{ss',rr'} = B^*_{rr',ss'}; \quad \sum_r B_{rs',rs'} = \delta_{r's'}\,. \tag{3.8}$$

If the matrix B is non-negative, we say that the dynamical map $\rho \to A\rho$ is strictly positive. Not all maps need be strictly positive (Sudarshan 1985); the simplest not strictly positive map is the map

$$\rho \to \rho^* \,. \tag{3.9}$$

In all cases, since B is Hermitian and finite-dimensional it can be diagonalized. For strictly positive maps all eigenvalues are non-negative. In this case we obtain

$$\rho \to \sum_\mu V(\mu)\rho V^\dagger(\mu); \quad \sum_\mu V^\dagger(\mu)V(\mu) = 1 \,, \tag{3.10}$$

where the sum over μ in general runs from 1 to N^2. The dynamical maps themselves form a convex set since

$$B = B_1 \cos^2\phi + B_2 \sin^2\phi \tag{3.11}$$

is an acceptable map if B_1 and B_2 are. If both B_1 and B_2 are strictly positive, so is B. The question naturally arises how to find all generating extremal elements of (i) all dynamical maps and of (ii) all strictly positive dynamical maps. The first problem is very complicated and has been done completely only for $N = 2$ (Choi 1972, 1974). For $N > 2$ we know many extremal maps like unitary, antiunitary, and "pin" maps but a complete characterization is still missing.

For strictly positive maps, such a complete characterization is available. The extremal cases may be separated into families of rank R, where $1 \leq R \leq N$. The case $R = 1$ corresponds to unitary maps while $R = N$ corresponds to the "pin" maps. Moreover, we have a simple construction algorithm for finding all the extremal maps.

The strictly positive maps are obtained by a unitary evolution of an extended system – consisting of the N-dimensional system of interest and an auxiliary R-dimensional system – which is then contracted by taking the partial trace with respect to the auxiliary system. Moreover, it can be shown that any strictly positive map may be displayed as a contraction of a rank one map of an extended system.

In this context we note that these maps can be multiplied by performing them in sequence. The result is again a dynamical map. The dynamical maps therefore form a forward semigroup. In particular, the strictly positive dynamical maps form a forward semigroup. However, these maps are in general not invertible to form a group: they take the set of density matrices into indefinite matrices (of unit trace!).

Given a unitary map we can consider it being generated by a Liouvillean derived from a Hamiltonian. We can then speak of a continuous group with

a generator $\mathcal{L}$ and ask for a continuous parameter semigroup for the generic semigroup of completely positive maps. In view of the fact that any such map can be obtained by contraction of a unitary map, we may look for clues to the structure of the generator of a dissipative semigroup (Petrosky and Prigogine 1993, 1988) in such a procedure. By expanding the unitary matrix to second order (where dissipation starts to play a role) we get

$$\rho \to \rho - \mathrm{i}t[H,\rho] + \frac{(\mathrm{i}t)^2}{2!}\,[H,[H,\rho]] + \cdots . \tag{3.12}$$

Applying it to the extended system and then taking the partial trace, we obtain

$$\rho \to \rho - \mathrm{i}t\,[H,\rho] + \mathrm{i}t\,\left[L_\alpha,\left[L_\beta^\dagger,\rho\right]\right] . \tag{3.13}$$

It is possible to show that this is the generic generator of a completely positive semigroup. It is noteworthy that for a finite-dimensional system there is no self-adjoint Hamiltonian which could lead to dissipation for an isolated system and for which the Liouvillean eigenvalues are the differences of the Hamiltonian eigenvalues. But if it is coupled to an auxiliary finite-dimensional system the contraction map can exhibit dissipation (Sudarshan 1985). In order to obtain the time evolution as a continuous one-parameter semigroup we may have to take limiting cases of weak coupling and scaling of time (Gorini et al. 1976, 1978).

When the number of dimensions of the vector space becomes denumerably infinite but the Hamiltonian has still a discrete spectrum the situation is not changed dramatically. The only essential change is that there are dynamical maps of arbitrarily high rank which cannot only be obtained from unitary but also from *isometric* operators in the extended space.

4. Liouville Dynamics with Continuous Spectrum

The Liouvillean dynamics of a system with a continuous spectrum furnishes richer possibilities. If ν is a point in the continuous spectrum, $0 < \nu < \infty$, then the density matrix may be parametrized by ν_1, ν_2:

$$\begin{aligned} H\,\rho(\nu_1,\nu_2) &= \nu_1\,\rho(\nu_1,\nu_2)\,, \\ \rho(\nu_1,\nu_2)H &= \nu_2\,\rho(\nu_1,\nu_2)\,. \end{aligned} \tag{4.1}$$

Then

$$\mathcal{L}\,\rho(\nu_1,\nu_2) = (\nu_1-\nu_2)\,\rho\,(\nu_1,\nu_2)\,, \tag{4.2}$$

and we can relabel the density matrix in the form

$$\rho(\nu;E) \equiv \rho\left(E+\frac{1}{2}\nu,\,E-\frac{1}{2}\nu\right); \quad -2E<\nu<2E\,. \tag{4.3}$$

The unitary time evolution is

$$e^{-i\mathcal{L}t}\rho(\nu;E) = e^{-i\nu t}\rho(\nu;E)\,. \tag{4.4}$$

The trace is invariant under this evolution,

$$\mathrm{Tr}\left(e^{-i\mathcal{L}t}\rho(\nu;E)\right) = \int dE\,\rho(0;E)\,, \tag{4.5}$$

and the positivity is preserved:

$$\rho(\nu;E) \geq 0 \;\rightarrow\; e^{-i\mathcal{L}t}\,\rho(\nu;E) \geq 0\,. \tag{4.6}$$

This is equivalent to the statement

$$\int^{\infty}\int_{|v|}^{\infty} \rho\,(\nu;E)^{\cdot}\, f^{*}\left(E+\frac{1}{2}\,\nu\right) f\left(E-\frac{1}{2}\nu\right)\,dE\,d\nu \geq 0\,. \tag{4.7}$$

For any energy E or any finite range of energies $0 < E < E_0$ the time dependent density matrix is an entire function of t and always obeys a group rather than a semigroup. But under suitable conditions, the survival probability

$$\begin{aligned} P(t) &= \mathrm{Tr}\,(\rho\,e^{i\mathcal{L}t}\rho) \equiv \int\int \rho(-\nu;E)e^{-i\nu t}\,\rho(\nu;E)\,d\nu\,dE \\ &= \int\int \rho^{\dagger}(\nu;E)\,e^{-i\nu t}\,\rho(\nu;E)\,d\nu\,dE \end{aligned} \tag{4.8}$$

may exhibit appropriate exponential behavior. Clearly, $P(t)$ is real and bounded by unity.

Since the density distribution ρ may be expressed in the form

$$\rho(\nu_1,\,\nu_2) = \sum_{\alpha} r_\alpha \psi_\alpha(\nu_1)\psi_\alpha^{\dagger}(\nu_2) \tag{4.9}$$

with $0 \leq \nu_1,\,\nu_2$ and $0 < r_\alpha < 1$, it follows that

$$\begin{aligned} \left(e^{-i\mathcal{L}t}\rho\right)(\nu_1,\nu_2) &= \sum_{\alpha} r_\alpha e^{-i(\nu_1-\nu_2)t}\,\psi_\alpha(\nu_1)\psi_\alpha^{\dagger}(\nu_2) \\ &= \sum_{\alpha} r_\alpha\,\psi_\alpha(\nu_1)\,e^{-i\nu_1 t}\,\left(\psi_\alpha(\nu_2)e^{-i\nu_2 t}\right)^{\dagger}\,. \end{aligned} \tag{4.10}$$

Hence, the survival probability has the decomposition

$$P(t) = \sum \alpha\, r_\alpha^2\, P_\alpha(t); \quad P_\alpha(t) = \left|\int_0^{\infty} \psi_\alpha^{\dagger}(\nu)\,e^{-i\nu t}\,\psi_\alpha(\nu)\right|^2, \tag{4.11}$$

so that $P_\alpha(t)$ are the absolute values squared of functions of t analytic in the lower half plane. Then, we have by the Paley-Wiener theorem:

$$\left| \int \frac{\log P_\alpha(t)}{1+t^2}\, dt \right| < \infty \,. \tag{4.12}$$

This is not possible if the convex sum of those functions decreases exponentially with t for $t > 0$. This is a slight generalization of a result derived four decades ago by Khalfin (1958).

At this stage it will be useful to classify the kinds of density distributions that we may consider, their analytic continuations, and the extension of the set of density distributions. Before going into this, we note that, given any set of dynamical variables, we may consider the density distributions as their duals. If we consider too large a class of distributions we restrict the set of observables and vice versa. Finally, if we consider analytic continuations of the dynamical variables, the density distributions themselves should be analytically continued initially (Kapur and Peierls 1938, Hu 1948, Sudarshan et al. 1978, Parravicini et al. 1980). The dual correspondence should be maintained.

5. Varieties of Statistical State Spaces

Given the density distribution $\rho(\nu; E)$, the time evolution can be displayed as

$$\left(\mathrm{e}^{-\mathrm{i}\mathcal{L}t}\rho\right)(\nu; E) = \mathrm{e}^{-\mathrm{i}\nu t}\,\rho(\nu; E) \,, \tag{5.1}$$

and the survival probability can be written in the form

$$P(t) = \int_0^\infty dE \int_{-2E}^{2E} d\nu\, \mathrm{e}^{-\mathrm{i}\nu t} \rho^*(\nu; E)\, \rho(\nu; E) \,. \tag{5.2}$$

The integration over the finite segment $-2E < \nu < 2E$ may be deformed to run along some path in the complex plane provided the function $\rho(\nu; E)$ is analytic in ν in a suitable domain in which the new open contour C from $-2E$ to $2E$ lies:

$$\begin{aligned} P(t) &= \int dE\, P(t, E) \,, \\ P(t, E) &= \int_C \rho^*(z^*, E)\, \mathrm{e}^{-\mathrm{i}zt} \rho(z, E)\, dz \,. \end{aligned} \tag{5.3}$$

We can now define various spaces associated with density distributions $\rho(z, E)$. Let us start with noting that the density distributions constitute a convex set, not a vector space. We can, however, relax the positivity condition and define the vector space spanned by the density distributions.

We distinguish the following spaces:

1. the space $\mathcal{B}$ of integrable distributions where the ν integration extends over the bounded range $-2E < \nu < 2E$;
2. the space $\mathcal{C}$ of square integrable (and integrable) distributions (this is the analog of the classical Koopman phase space densities);
3. the space $\mathcal{D}$ of distributions which are boundary values of functions analytic in a domain providing the analytic continuations for complex contours in the variables ν and E (where the E integration extends from 0 to ∞);
4. the space $\mathcal{E}$ of distributions where the variable E extends from $-\infty$ to $+\infty$;
5. the space $\mathcal{F}$ of distributions analytic in a half plane for ν (except for an essential singularity at infinity) and suitably analytic in the variable E, so that we can develop a forward semigroup in time;
6. the space $\mathcal{A}$ of distributions analytic in a half plane for ν and suitably analytic in the variable E.

It is clear that these spaces are different. The space $\mathcal{C}$ contains the space $\mathcal{B}$, the space $\mathcal{A}$ is contained in the space $\mathcal{F}$, and the space $\mathcal{B}$ is contained in the space $\mathcal{E}$. As sets, $\mathcal{D}$ and $\mathcal{B}$ are dense in each other but there are elements in $\mathcal{B}$ which have no counterpart in $\mathcal{D}$.

For an isolated system the total energy E is bounded from below, but *for a system which is open to dynamical interaction with other systems this may not be an essential requirement.* It is only under this provision that the spaces $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{A}$ are relevant.

A piecewise analytic function or any general measure which belongs to the space can be arbitrarily closely approximated by boundary values of analytic functions. Similarly a distribution along a complex contour in $\mathcal{C}$ can be approximated arbitrarily closely by functions in $\mathcal{B}$ though there is no one-to-one correspondence between the vectors (Sudarshan 1992). A specially interesting case is the complex delta distribution which assigns, to a function representing a vector in the dual space which is analytic in a domain containing the particular complex point, the value of the function at the complex point. There is *no vector in $\mathcal{B}$ or in $\mathcal{C}$* which corresponds to this vector, but in $\mathcal{E}$ there is such a vector. This vector in $\mathcal{E}$ would be appropriate for describing the simplest metastable excitations (see next section).

If we take a physical state in $\mathcal{D}$ and analytically continue it, we can consider it as a function along a complex contour together with one or more isolated poles (or, more generally, branch cuts). The pole terms control the behavior of the survival probability *but they are always accompanied by a background integral.* This background integral is essential: it reproduces the correct behavior at short (Zeno) times and long (Khalfin) times. An

isolated pole by itself would not have a corresponding state in the space $\mathcal{D}$ of physical states.

In the space $\mathcal{F}$ the situation is quite different. There exist states in $\mathcal{F}$ which correspond precisely to a discrete complex point (or points). For a single complex point these are the familiar Gamow-Breit-Wigner states with a unique exponential dependence of the survival probability. These correspond to unique vectors in $\mathcal{F}$. The correspondence between $\mathcal{F}$ and its analytic continuation is a correspondence of complete spaces, not merely dense sets.

A special subset of these functions is analytic in an entire half plane. Such functions constitute the Hardy class functions with many interesting properties and are often taken to represent *nascent* metastable states. But the Hardy class property is *not* preserved by time evolution since

$$\rho(\nu; E) \to \mathrm{e}^{-\mathrm{i}\nu t}\rho(\nu; E) \tag{5.4}$$

has an *essential (exponential) singularity at infinity*. So after any finite time has elapsed, a nascent state evolves into a non-Hardy class function. These non-Hardy class functions are labeled by the index of exponential growth at infinity.

6. Need for Extended Space: Breaking of Time Symmetry

If the states having a purely exponential survival probability are to be included as natural ("physical") states, the spectrum of energies E has to be extended from $0 < E < \infty$ to $-\infty < E < \infty$. This leads to the spaces $\mathcal{E}$ and $\mathcal{F}$ of functions which are the boundary values of analytic functions, analytic in a half plane except, perhaps, for an exponential type singularity at infinity. The extension of the energy spectrum from $0 < E < \infty$ to $-\infty < E < \infty$ is equivalent to lifting the restriction

$$-2E < \nu < 2E \tag{6.1}$$

and allowing the ν integration to run from $-\infty$ to $+\infty$. So if the survival amplitude

$$P(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(\nu; E)\, \mathrm{e}^{-\mathrm{i}\nu t}\, \rho^*(\nu; E)\, d\nu\, dE \tag{6.2}$$

is calculated for positive and negative times, we will get two distinct functions. For $t > 0$, $\mathrm{e}^{-\mathrm{i}\nu t}$ is a convergence factor for the lower half plane and an exponential increase for the upper half plane. Hence,

$$P(t) = \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\nu\, \mathrm{e}^{-\mathrm{i}\nu t}\rho(\nu; E)\, \rho^*(\nu^*; E) \tag{6.3}$$

can be considered as a closed contour integration where the integration in the lower half plane is closed by an infinite semicircle whose contribution vanishes. The result is, then,

$$P(t) = \int_{-\infty}^{\infty} dE \cdot 2\pi \mathrm{i} \sum_{\substack{\text{Residues in the} \\ \text{upper half plane}}} \rho(z, E)\rho^*(z^*, E)\, \mathrm{e}^{-\mathrm{i}zt}\,, \tag{6.4}$$

while for $t < 0$:

$$P(t) = \int_{-\infty}^{\infty} dE \cdot 2\pi \mathrm{i} \sum_{\substack{\text{Residues in the} \\ \text{upper half plane}}} \rho(z, E)\, \rho^*(z^*, E)\, \mathrm{e}^{-\mathrm{i}z^*t}\,. \tag{6.5}$$

In particular, if there are no poles in the upper half plane then

$$P(t) = \sum a_j\, \mathrm{e}^{-\gamma_j t}, \qquad t \leq 0\,. \tag{6.6}$$

For the density functions in the extended space $\mathcal{F}$, with no poles in the lower half plane but poles in the upper half plane, $P(t)$ is exponentially decreasing with $|t|$ for the past ($t < 0$).

These two classes of functions are disjoint except for the constant function, but functions constant in ν lead to an unphysical survival "probability"

$$P(t) = \delta\,(t)\,. \tag{6.7}$$

In the extended space $\mathcal{F}$ there are thus two disjoint sets of states. The forward evolving states with

$$1 \geq P(t) = \mathrm{Tr}\,(\rho(t)\rho(0)) > 0, \quad t > 0\,, \tag{6.8}$$

are the states consistent with the second law of thermodynamics. The second set is a time reversed set of backward regressing states with

$$1 \geq P(t) > 0, \quad t < 0\,. \tag{6.9}$$

These states are *not* suitable for a system that obeys the second law. The choice of physical states as forward evolving *is the breaking of time symmetry.* It is not dependent upon objective information or the act of isolated measurements but it is a property of thermodynamically adapted states and is picked automatically and universally. Open systems must have this time symmetry breaking if the second law of thermodynamics is regarded to be generally valid.

We now consider in detail the correspondence between the states of open systems in the space $\mathcal{F}$ and the states of closed systems in the spaces $\mathcal{C}$ and $\mathcal{D}$. Given any element of $\mathcal{F}$, we can restrict it to the domain

$$-2E < \nu < 2E, \quad 0 < E < \infty\,, \tag{6.10}$$

and this yields an element of $\mathcal{D}$. However, given an element of $\mathcal{D}$, we cannot automatically extend it to $\mathcal{F}$ since analytic continuation to the negative real axis may not be possible.

Despite this *there is a natural splitting of any vector in $\mathcal{C}$ or in $\mathcal{D}$* into two vectors in $\mathcal{F}$ with different domains of analytic continuation. Given the function $f(\nu; E)$ for a vector in $\mathcal{C}$ which vanishes outside the range $-2E < \nu < 2E$, $E > 0$, we define (Sudarshan 1992)

$$g(\nu; E) = \frac{1}{\pi \mathrm{i}} \int_{-2E}^{2E} d\nu' \, \frac{f(\nu'; E)}{\nu - \nu' + \mathrm{i}\epsilon} \,. \tag{6.11}$$

This integral, if it exists, defines a function for all values of ν and is analytic in the lower half plane. Hence, it is a suitable member of $\mathcal{F}$ appropriate for describing an open system with forward (dissipative) evolution. A companion state with backward (dissipative) evolution is given by

$$h(\nu; E) = \frac{1}{\pi \mathrm{i}} \int_{-2E}^{2E} \frac{f(\nu'; E)}{\nu' - \nu - \mathrm{i}\epsilon} \,. \tag{6.12}$$

Clearly,

$$g(\nu; E) + h(\nu; E) = f(\nu; E) \,, \tag{6.13}$$

and hence

$$h(\nu; E) = -g(\nu; E) \quad \text{for} \quad |\nu| > 2E \,. \tag{6.14}$$

The functions $g(\nu; E)$ obtained here belong to the space $\mathcal{A}$ of functions analytic in the upper half plane, more restrictive than the space $\mathcal{F}$ admitting essential singularities of exponential type at infinity. This class of functions are the Hardy class functions. They are sometimes used to describe the corresponding states (Rosenblum and Rovyak 1985, Bohm and Gadella 1989).

If $f(\nu; E)$ behaves like $\exp(-\mathrm{i}\nu t)$ at infinity for $t > 0$, the definition of $g(\nu; E)$ remains unaltered, except that it will also behave like $\exp(-\mathrm{i}\nu t)$ at infinity. These types of behavior are therefore quite appropriate to describe forward dissipative evolution. The corresponding functions are not in the Hardy space $\mathcal{A}$ but in the space $\mathcal{F}$ discussed above.

The behavior of the density distribution is automatic with temporal evolution. Given $\rho(\nu; E)$ at time $t = 0$, the density distribution function at time t becomes

$$\rho_t(\nu; E) = \rho_o(\nu; E) \, \mathrm{e}^{-\mathrm{i}\nu t} \tag{6.15}$$

which belongs to the space $\mathcal{F}$ but not to the Hardy space $\mathcal{A}$. Each such state is labeled by a parameter τ characterizing the rate of exponential growth at infinity (see next section for more details). This parameter increases linearly

with time evolution and may therefore be called the age of the state. The nascent states introduced before correspond to states of age zero.

7. Dynamical Processes and Dissipative Evolution

Our discussion has been focussed on generic systems so far. We have not yet talked about interactions, scattering, and explicit dissipative evolutions. Let us now address a generic system with a "total Hamiltonian" H which may be written

$$H = H_C + V \,, \tag{7.1}$$

where H_C is *isospectral* with H and is a simple structure, say a collection of "free Hamiltonians". If the states $|E,r\rangle$ are a set of (ideal) eigenstates of H_C with degeneracy label r, and if the states $|E,r\rangle\rangle$ are a corresponding set of (ideal) eigenstates of H, then there would be, by definition of the isospectral property of H and H_C, a one-to-one correspondence of states with the same degeneracies. Apart from normalization this correspondence may be written as:

$$\begin{aligned} Z^{-1/2}|E\rangle\rangle &= |E\rangle + (E - H_C + \mathrm{i}\epsilon)^{-1}\, V|E\rangle\rangle \\ &= \left[1 - (E - H_C + \mathrm{i}\epsilon)^{-1}\, V\right]^{-1} |E\rangle \,, \end{aligned} \tag{7.2}$$

where Z is the wavefunction renormalization constant (Sudarshan et al. 1994). We shall omit this factor in what follows. Equation (7.2) can be formally expanded in a perturbation series

$$|E\rangle\rangle = |E\rangle + \sum_1^\infty \{G_C(E)\, V\}^n \; |E\rangle; \quad G_C(E) = (E - H_C + \mathrm{i}\epsilon)^{-1} \,. \tag{7.3}$$

This corresponds to the "in" state appropriate for the initial state of a scattering process. Choosing the energy denominators with $-\mathrm{i}\epsilon$ furnishes the "out" states. Both the solution and the perturbation expansion can be extended from the Hilbert spaces to the analytically continued spaces (Sudarshan et al. 1994). No substantial change is needed if the spectrum condition is relaxed to include arbitrarily large negative energy continua.

Rather than discussing the problem of the correspondence between the (ideal) eigenstates of H and H_C we could do the same in terms of (ideal) density distributions in relation to the Liouville operators $\mathcal{L}$ and $\mathcal{L}_C$:

$$\begin{aligned} \mathcal{L}\, R(\nu;E) &= \nu\, R(\nu;E) \,, \\ \mathcal{L}_C\, R_C(\nu;E) &= \nu\, R_C(\nu;E) \,. \end{aligned} \tag{7.4}$$

Here, $R(\nu; E)$ and $R_C(\nu; E)$ are the (ideal) density distributions

$$\begin{aligned} R_{rs}(\nu; E) &= |E + \frac{1}{2}\nu, r\rangle\rangle\langle\langle E - \frac{1}{2}\nu, s| \,, \\ R_{c_{rs}}(\nu; E) &= |E + \frac{1}{2}\nu, r\rangle \, \langle E - \frac{1}{2}\nu, s| \,. \end{aligned} \tag{7.5}$$

They are related by

$$\begin{aligned} R_{\text{in}}(\nu; E) &= \left\{1 - G_C\left(E + \frac{1}{2}\nu + i\epsilon\right) V\right\}^{-1} \times \\ &\times \; R_C(\nu; E) \left\{1 - G_C^\dagger\left(E - \frac{1}{2}\nu + \mathrm{i}\epsilon\right) V\right\}^{-1} , \\ R_{\text{out}}(\nu; E) &= \left\{1 - G_C\left(E + \frac{1}{2}\nu - \mathrm{i}\epsilon\right) V\right\}^{-1} \times \\ &\times \; R_C(\nu; E) \left\{1 - G_C^\dagger\left(E - \frac{1}{2}\nu - i\epsilon\right) V\right\}^{-1} . \end{aligned} \tag{7.6}$$

The scattering probability for ideal states is

$$\begin{aligned} &P_{\text{scatt}} = \text{Tr}\left\{R^\dagger_{\text{out}_{rs}}(\nu, E)\, R_{\text{in}_{r's'}}(\nu', E')\right\} = \\ &= \langle\langle E - \frac{1}{2}\nu, s, \text{out} \,|\, E' - \frac{1}{2}\nu', s', \text{in}\rangle\rangle \, \langle\langle E' + \frac{1}{2}\nu', r', \text{in} \,|\, E + \frac{1}{2}\nu, r, \text{out}\rangle\rangle \,. \end{aligned} \tag{7.7}$$

But

$$\langle\langle E - \frac{1}{2}\nu, s, \text{out} \,|\, E' - \frac{1}{2}\nu', s', \text{in}\rangle\rangle = \delta\left(E - E' - \frac{1}{2}\nu + \frac{1}{2}\nu\right) S_{rs}\left(E - \frac{1}{2}\nu\right) \tag{7.8}$$

is the scattering matrix. So the scattering probability is given by

$$P_{\text{scatt}} = \delta(E - E')\,\delta(\nu - \nu')\, S_{rs}\left(E - \frac{1}{2}\nu\right) S^\dagger_{r's'}\left(E + \frac{1}{2}\nu\right) . \tag{7.9}$$

With the proper understanding of the adjoints and duals, these considerations do not only apply to the real spectrum representations but also to analytic continuations. Of course, as long as one deals with the real spectrum representation, no metastabilities per se occur in the scattering probabilities. Rather, the resonances manifest themselves by characteristic "resonant shapes" of the probability distributions. If we want to consider the role of metastable states and the scattering of metastable excitations we should consider the analytic continuations which would uncover the resonant states as members of complete sets of states.

The (unnormalized) states $|E\rangle\rangle$ can be normalized by suitable state sensitive multiplicative changes. When this is done we denote

$$|E, \text{in}\rangle\rangle = \Omega_{\text{in}} |E\rangle; \quad |E, r; \text{in}\rangle\rangle = \Omega_{\text{in},r,s}|E, s\rangle \tag{7.10}$$

with $\Omega_{\text{in},r,s}$ as a *unitary* operator. Then, if F_C is any invariant for the Hamiltonian H_C,

$$[F_C, H_C] = 0\,, \tag{7.11}$$

then there is an invariant of H given by

$$F = \Omega F_C \Omega^\dagger, \;\; [F, H] = 0\,, \tag{7.12}$$

by virtue of

$$H = \Omega H_C \Omega^\dagger. \tag{7.13}$$

But there is no guarantee that, if the matrix elements of F_C are smooth non-singular functions of E, the matrix elements of F are non-singular functions. In cases of non-trivial scattering the matrix elements of F will definitely be singular functions. F and F_C are unitarily equivalent and are constants of motion for H and H_C, respectively. If H and H_C share some symmetry properties, the corresponding operators are regular constants of motion for both H_C and H; this is analogous to the traditional constants of motion for the total Hamiltonian in classical dynamics. But there are additional constants of motion.

Let us now consider the time evolution. For the Hilbert space, the time evolution is the exponential of an imaginary multiple of a Hermitian Hamiltonian and, as such, it is unitary (norm preserving) no matter whether the energy spectrum is bounded from below or not. When we generalize to dual spaces, there is no longer a norm for the state. We must rather consider the invariance of the scalar product bilinear in the vector of the two dual spaces. If $\psi, \tilde{\phi}$ are such vectors, we have

$$\tilde{\phi}\,\psi \to \tilde{\phi}(t)\,\psi(t) = \tilde{\phi}\,\mathrm{e}^{\mathrm{i}Ht}\,\mathrm{e}^{-\mathrm{i}Ht}\psi = \tilde{\phi}\psi\,. \tag{7.14}$$

If H has complex eigenvalues for ψ, there are ϕ with the *same* complex eigenvalues, and therefore the product of the two remains constant (Sudarshan et al. 1978, Sudarshan and Chiu 1993). But it is no longer true that $\psi(t)$ has the same "length" as $\psi(0)$: the "length" of a vector is not defined in dual spaces. But if there were complex eigenvalues of H, then it is clear that $\psi(t)$ can be a complex multiple of $\psi(0)$. But $\tilde{\phi}(t)$ would be the inverse multiple of $\tilde{\phi}(0)$.

Similar considerations apply for the spaces spanned by density distributions. In this case, there is always an invariant state with normalized trace,

and all the other states are pseudodensities with trace zero. The evolution is "unitary", that is, it preserves scalar products between duals.

A measure of this scale change is provided by the survival amplitude

$$A(t) = \tilde{\phi}(0)\,\psi(t) \equiv \tilde{\phi}\,\mathrm{e}^{-\mathrm{i}Ht}\psi\,. \tag{7.15}$$

As ψ changes, so does $A(t)$. In particular, if the state ψ is dominated by a complex pole at z, then the survival amplitude has the dependence

$$A(t) = A(0)\,\mathrm{e}^{-\mathrm{i}zt}\,. \tag{7.16}$$

Whenever $\mathrm{Im}\,z < 0$ for $t > 0$, then we have:

$$|A(t)| \;<\; |A(0)|\,. \tag{7.17}$$

Thus, *in this sense*, the complex energy state is a *decaying* state.

From the vectors $\psi, \tilde{\phi}$ in the dual spaces we can construct pseudodensity distributions

$$\rho = \psi_1\psi_2^\dagger, \quad \sigma = \tilde{\phi}_1^\dagger\tilde{\phi}_2\,, \tag{7.18}$$

which generate dual spaces (Segal 1947, Haag and Kastler 1964). With these we can calculate the survival probability

$$\begin{aligned} P(t) &= \mathrm{Tr}(\sigma\rho(t)) = \mathrm{Tr}\left(\sigma\,\mathrm{e}^{-\mathrm{i}Ht}\rho\,\mathrm{e}^{\mathrm{i}Ht}\right) \\ &= \left(\tilde{\phi}\,\mathrm{e}^{-\mathrm{i}Ht}\psi\right)\left(\psi^\dagger\mathrm{e}^{\mathrm{i}Ht}\tilde{\phi}^\dagger\right) \\ &= |A(t)|^2\,. \end{aligned} \tag{7.19}$$

Therefore, if we know the survival amplitude, the survival probability can be computed.

When the states contain a superposition of eigenvectors of H, the behavior is given by:

$$P(t) = \left|\sum \tilde{\phi}_E\psi_E\,\mathrm{e}^{-\mathrm{i}Et}\right|^2 \longrightarrow \left|\int \tilde{\phi}(E)\psi(E)\,\mathrm{e}^{-\mathrm{i}Et}dE\right|^2\,. \tag{7.20}$$

As long as $\psi(E)$ and $\tilde{\phi}(E)$ are boundary values of functions analytic in the lower half plane except for poles (or "short" branch cuts), we can evaluate the integral over E by closing the contour with an infinite semicircle in the lower half plane (for $t > 0$). If there is only one pole in the lower half plane, the entire survival amplitude A is as if there were only one complex "energy" point eigenvalue contributing to the integral. More generally, A will be the superposition of several such "energies" and possibly an integral over them.

Let us consider the single complex eigenvalue in detail, even though the discrete complex eigenvalue must be accompanied by associated branch cuts. The state vector

$$\psi_0(E) = N_0\,(E-z)^{-1}; \quad N_0^2 = \frac{z^*-z}{2\pi\mathrm{i}}\,, \tag{7.21}$$

and its dual

$$\tilde{\phi}_0(E) = N_0\,(E-z^*)^{-1} \tag{7.22}$$

give the survival amplitude for $t>0$:

$$\begin{aligned} A_0(t) &= \int \tilde{\phi}_0(E)\,\mathrm{e}^{\mathrm{i}Et}\phi_0(E)dE \\ &\doteq N_0^2 \int \frac{\mathrm{e}^{-\mathrm{i}Et}}{(E-z)(E-z^*)}\,dE \\ &= \frac{-2\pi\mathrm{i}}{z-z*}\,N_0^2\,\mathrm{e}^{-\mathrm{i}zt} = \mathrm{e}^{-\mathrm{i}zt}\,. \end{aligned} \tag{7.23}$$

In this case, the analytic continuation of the wave function vanishes as z^{-1} at infinity. For $t<0$ we get similarly:

$$A_0(t) = \mathrm{e}^{\mathrm{i}z^*t}\,. \tag{7.24}$$

The survival amplitude as a function of t is therefore the join of two *distinct* analytic functions, one for $t>0$ and another one for $t<0$.

Now consider the state

$$\begin{aligned} \psi_\tau(E) &= N_0(E-z)^{-1}\,\mathrm{e}^{-\mathrm{i}zt} \\ \tilde{\phi}_\tau(E) &= N_0(E-z^*)^{-1}\mathrm{e}^{-\mathrm{i}z^*t} \end{aligned} \tag{7.25}$$

obtained by a multiplicative transformation. Then the survival amplitude is

$$\begin{aligned} A_\tau(t) &= N_0^2 \int (E-Z)^{-1}\,(E-Z^*)^{-1}\,\mathrm{e}^{\mathrm{i}(z-z^*)\tau}\,\mathrm{e}^{-\mathrm{i}zt}dE \\ &= N_0^2\,\frac{-2\pi\mathrm{i}}{z-z^*}\,\mathrm{e}^{-\mathrm{i}zt} \end{aligned} \tag{7.26}$$

which may be written

$$A_\tau(t) = A_0(t+\tau)\,. \tag{7.27}$$

In other words: the state ψ_τ may be thought of as having been created at time $t=\tau$. If τ is positive, we extrapolate, *for these states*, the semigroup for negative values of t such that

$$t+\tau>0\,. \tag{7.28}$$

This quantity τ may be called the "age" of the state in the *extended* space (Ramakrishnan 1959, Sudarshan 1992).

Having defined the age and the survival amplitude for the states we can define the age and survival probability for density distributions. Analytic density distributions in the space $\mathcal{F}$ can be chosen so that we can define the forward semigroup on them. But after the time evolution for any finite time is considered, the states are no longer in $\mathcal{F}$ but are in $\mathcal{E}$. If we denote a state in $\mathcal{F}$ at $\tau = 0$ by $\rho_o(\nu; E)$, then

$$\rho_\tau(\nu; E) = \mathrm{e}^{-\mathrm{i}\nu\tau}\, \rho_o(\nu; E) \tag{7.29}$$

is not in the space $\mathcal{F}$ but remains in the space $\mathcal{E}$. For them, the forward semigroup can be extrapolated to *negative* values of t such that $t + \tau > 0$. These are the metastable states with age τ.

In the preceding discussions we have labeled the density distribution $\rho(\nu; E)$ with the labels appropriate for the total energy and total Liouvillean. In many cases, however, we have a comparison Hamiltonian H_C and an interaction V such that H_C is isospectral with H and

$$H = H_C + V\,. \tag{7.30}$$

Then we could have an alternative labeling of the states by ν_C, E_C appropriate to

$$\begin{aligned} H_C\,\rho &= \left(E_C + \frac{1}{2}\nu_C\right)\rho\,, \\ \rho\, H_C &= \left(E_C - \frac{1}{2}\,\nu_C\right)\rho\,. \end{aligned} \tag{7.31}$$

To avoid confusion we use the symbol R for the density distribution labeled by ν_C, E_C so that

$$R(\nu_C; E_C) \equiv \rho\,(\nu; E)\,, \tag{7.32}$$

with ν_C having the same range as ν and E_C the same range as E. Then

$$R(\nu; E) = \left(1 - G_C\left(E + \frac{1}{2}\nu\right)V\right)^{-1} R_0 \left(\left(1 - G_C\left(E - \frac{1}{2}\nu\right)V\right)^{\dagger}\right)^{-1}. \tag{7.33}$$

It may be that the creation of the state is most simply described in the comparison Hamiltonian representation R_C. Then the dependence on the variables ν, E is governed by the wave matrix factors preceding and following R_C. In the special case of the Dirac-Friedrichs-Lee model of a discrete (metastable) state coupled to a continuum, the resonant complex pole plus background dependence is immediately realized if the initial state

is the discrete state of the comparison Hamiltonian with the discrete energy level chosen to prevent instability, and then by continuing in the "mass" parameter of the model. The time dependence of the survival amplitude and survival probability have been studied extensively in the literature (see, e.g., Chiu et al. 1997, Chiu and Sudarshan 1990, Sudarshan et al. 1978).

More generally, the wave matrix

$$\Omega(E) = (1 - G(E)\,V)^{-1} \tag{7.34}$$

has an *analytic dependence* on E. As a consequence, if $R_C(\nu; E)$ is a simple function of E and ν, then $R(\nu; E)$ will be *analytic* in both ν and E. The singularities of the wave matrix in the complex variable of energy reappears in the survival probability. While both the scattering amplitude and wave matrix depend on both the total Hamiltonian H and the comparison Hamiltonian H_C, it is known that only the singularities of the wave matrix appear in the survival amplitude. The redundant poles of the scattering amplitude, if there are any, do not contribute. Anyway, the *survival amplitude depends on both the total Hamiltonian H and the comparison Hamiltonian* H_C.

Another point to be noted is that when we consider the time evolution, despite the fact that the $(\text{norm})^2$ of the state is not defined directly, we still can talk of affine scale; that is, whether the state gets multiplied by a number $e^{iz t}$. Such states *do not* exist in $\mathcal{C}$ but they exist in the extended space $\mathcal{E}$. They may be realized along the real axis but could equally well be identified as complex discrete energy states. As a consequence, while the product of a state and its dual is invariant under time evolution, the survival amplitude depends on time. For the special state corresponding to a discrete complex pole, the dependence is purely exponential.

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On Quanta, Mind and Matter

Hans Primas in Context

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