PURIFICATION OF IMPURE DENSITY OPERATORS AND THE RECOVERY OF ENTANGLEMENTS

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Abstract

The need to retain the relative phases in quantum mechanics implies an addition law parametrized by a phase of two density operators required for the purification of a density matrix. This is shown with quantum tomography and the Wigner function. Entanglement is determined in terms of phase dependent multiplication.

Quantum mechanics as traditionally formulated[1] involves three principles. The states of the system are normalized vectors in a Hilbert space, (selfadjoint) linear operators correspond to (real) dynamical variables; and the expectation value for any dynamical variable is bilinear (sesquilinear) in the state vectors. But the overall phase of the state vector is irrelevant in those computations

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involving only that one state. The state corresponds to a "ray" in Hilbert space $\{\psi : e^{i\alpha}\psi_0\}$.

Alternatively the expectation value can be expressed by means of the density matrix

$$\rho = \psi\psi^\dagger = \psi_0\psi_0^\dagger,$$

which is defined on the ray.

These density operators are "pure" and of rank one:

$$\rho^\dagger = \rho\,; \qquad \rho > 0\,; \qquad \mathrm{tr}\,\rho = 1\,; \qquad \rho^2 = \rho\,.$$

There is a probability addition of the density operators. If ρ_1 and ρ_2 are two pure density matrices

$$\rho = \cos^2\Theta\rho_1 + \sin^2\Theta\rho_2$$

also is a density matrix; but it is not pure, but mixed. These are the interior points of the convex set of density operators. Hence

So

$$\rho^2 \neq \rho\,, \qquad \rho - \rho^2 > 0\,.$$

Can we have an additional operation of constructing a **pure** density operator from the **sum** of the two density operators $\rho_1,\ \rho_2$? We must have such a construction since the superposition principle of quantum mechanics[1] tells us how to add two state vectors ψ_1 and ψ_2 to form a superposed state vector

$$\psi = \left\{\cos\Theta\ \psi_1 + e^{i\varphi^1}\sin\Theta\ \psi_2\right\}$$

which is normalized if $\psi_1,\ \psi_2$ are orthonormal. For nonorthogonal states, we have to use the normalization factor

$$(1 + \sin 2\Theta\ \ \cos\varphi \mid \langle\psi_1 \mid \psi_2\rangle \mid)^{-1/2}\,.$$

where φ includes the phase of $\langle\psi_1 \mid \psi_2\rangle$ along with φ^1. In terms of density operators, the φ-addition law which we introduce is

$$\rho(x,y,\varphi) = \{\cos\Theta\psi_1(x) + e^{i\varphi}\sin\Theta\psi_2(x)\}\{\cos\Theta\psi_1^\dagger(y) + e^{i\varphi}\sin\Theta\psi_2^\dagger(y)\}$$

$$= \cos^2\Theta\rho_1(x,y) + \sin^2\Theta\rho_2(x,y) + \sin 2\Theta\ cos\varphi\rho_{12}(x,y,\varphi).$$

In terms of the Wigner function[2], we get a φ-addition law which has an interference term proportional to $\sin 2\Theta\cos\varphi$., i.e.

$$W\left(q,p\right) = \cos^2\Theta\ W_1\left(q,p\right) + \sin^2\Theta\ W_2\left(q,p\right) + \sin 2\Theta\ \ \cos\varphi\ I_{12}\left(q,p,\varphi\right)$$

where I_{12} is the (generalized) Wigner function corresponding to the operator having the structure of the root square of a convolution of the product of the two density matrices, which we symbolically denote as

$$I_{12} \to \sqrt{\rho_1\rho_2}\,.$$

There is thus a one-parameter addition law of density operators and of Wigner functions with probabilities $\cos^2\Theta$ and $\sin^2\Theta$ and with extra intereference term. Note that this $W\left(q,p\right)$ is pure and satisfies the purity criterion

$$\int\int dp\ dq\left\{W\left(q,p\right)\right\}^2 = \frac{1}{2\pi}\,.$$

For quantum tomography[3], also we can construct superposition of tomograms using a one parameter addition law:

$$\Phi\left(\lambda,\mu;x\right) = \cos^2\Theta\ \Phi_1\left(\lambda,\mu;x\right) + \sin^2\Theta\ \Phi_2\left(\lambda,\mu;x\right) + \sin 2\Theta\ \ \cos\varphi\ \Phi_{12}\left(\lambda,\mu;x\right).$$

Here tomograms $\Phi_1.\Phi_2$ determine the probability density of quadrature $x = \lambda q + \mu p$ in the pure states ψ_1 and ψ_2. The tomogram Φ_{12} corresponds to the interference term I_{12}.

The passage from the impure density operator $\cos^2\Theta\rho_1 + \sin^2\Theta\rho_2$ to the pure φ-dependent addition may be called purification. Note that the purification introduces the relative phase φ which was not in ρ_1 or ρ_2.

The density operator of a composite system AB with subsystems A and B may be chosen pure or impure. For a pure density operator ρ_{AB}, one can get the density operator ρ_A and ρ_B by the partial trace operation

$$\rho_A = \mathrm{tr}_B\,(\rho_{AB}); \qquad \rho_B = \mathrm{tr}_A\,(\rho_{AB})\,.$$

It is not necessary that A and B have the same dimensionality. Unless ρ_{AB} is a direct product of pure states of A and B, a pure ρ_{AB} yelds impure ρ_A and ρ_B. But they will have the same rank R and the same nonnegative eigenvalues which sum up to unity. The density operator

$$\rho'_{AB} = \rho_A \otimes \rho_B \neq \rho_{AB}$$

is impure. Thus, the whole is greater than the parts: there is additional information in ρ_{AB}. These are the "entanglement" terms[4].

We purify the product ρ'_{AB} in the same way that we used before for the mixture of two density operators ρ_1 and ρ_2. Here we have n such pure states mixed together and need $(n-1)$ phase angles $\varphi_1 = 0, \varphi_2, \ldots \varphi_n$. The diagonal form of ρ'_{AB} has only n nonzero diagonal elements. We need to introduce the offdiagonal elements

$$\sqrt{\lambda_j\lambda_k}\, e^{i(\varphi_j - \varphi_k)}$$

in the (j,k) location. Note that while we have $n(n-1)/2$ offdiagonal terms, there are only $(n-1)$ phases φ_j.

The purification of the density matrix ρ'_{AB} we call as the φ- multiplication law of the density matrices ρ_A and ρ_B.

While purification of an impure density addition is dependent on one phase angle, the form of the entanglement is constructed depending on $R-1$ phase angles. These have to be obtained from other considerations.

The same kind of φ-addition law and φ-multiplication law holds for other representatives like the quantum tomograms, the diagonal coherent state distribution function in quantum optics[5] and the Husimi - Kano[6] density of coherent state projection operators. We expect to return to this discussion elsewhere.

Acknowledgements

V.I.M. thanks Dipartimento di Scienze Fisiche Universitá di Napoli "Federico II" for kind hospitality and Russian Foundation for Basic Research for partial support.

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