Inner Composition Law of Pure-Spin States

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Abstract. Superposition principle for spin degrees of freedom is described in terms of density operators only using a formulated composition law of pure-state density operators. Decoherence phenomenon and visibility of the interference pattern are discussed.

INTRODUCTION

The pure states in quantum mechanics are associated with vectors $|\,\psi\rangle$ in Hilbert space [1] which correspond to the state wave functions [2]. The pure state can be also described by the projector, which is the density operator introduced in [3]. Recently [4–6] the composition of the pure-state density operators was discussed in the connection with the interference problem in quantum mechanics.

The discrete spin degrees of freedom have no classical limit. Due to this, it is worthy to consider the composition law of the spin-state projectors which provides the pure-state projector of another spin state.

Let us consider spin state of spin $j = 1/2$.

The generic normalized spin state is described by the vector $|\,\psi\rangle$ in two-dimensional Hilbert space

$$|\,\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad a = \left\langle \frac{1}{2}, \frac{1}{2} \middle| \psi \right\rangle, \qquad b = \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \psi \right\rangle . \tag{1}$$

We used base vectors $|\,j, m\rangle$ with spin projection $m = \pm 1/2$. In (1), the complex numbers

$$a = |a|\, e^{i\varphi_a}, \qquad b = |b|\, e^{i\varphi_b}$$

satisfy the normalization condition

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$$|a|^2 + |b|^2 = 1\,. \tag{2}$$

For the pure state $\mid \psi\rangle$, the density operator reads

$$\rho_\psi = \mid \psi\rangle\langle\psi \mid\,. \tag{3}$$

The positive densiy operator ρ_ψ of the pure spin-state satisfies the relations determining projector operator

$$\rho_\psi^\dagger = \rho_\psi\,, \qquad \rho_\psi^2 = \rho_\psi, \qquad \mathrm{Tr}\,\rho_\psi = 1\,. \tag{4}$$

The density matrix of pure state has rank equal unity and the unique nonzero eigenvalue equals to unity.

The density operator ρ of impure state is characterized by the impurity parameter

$$\mu_0 = \mathrm{Tr}\,\rho^2 < 1\,. \tag{5}$$

The example of density matrix of the pure spin state is the Hermitian matrix with two angle parameters

$$0 \leq \theta < \pi\,, \qquad 0 \leq \gamma < 2\,\pi\,,$$

which determine the point on the sphere S^2

$$\rho_\psi = \begin{pmatrix} \cos^2\theta & e^{i\gamma}\,\cos\theta\sin\theta \\ e^{-i\gamma}\,\cos\theta\sin\theta & \sin^2\theta \end{pmatrix}, \tag{6}$$

where

$$|a|^2 = \cos^2\theta\,, \qquad \gamma = \varphi_a - \varphi_b\,. \tag{7}$$

COMPOSITION LAW

In order to formulate a composition law of two density operators

$$\rho_1 = \mid \psi_1\rangle\langle\psi_1 \mid \tag{8}$$

and

$$\rho_2 = \mid \psi_2\rangle\langle\psi_2 \mid, \tag{9}$$

which corresponds to the superposition of two orthogonal state vectors $\mid \psi_1\rangle$ and $\mid \psi_2\rangle$ with complex coefficients c_1 and c_2 satisfying the condition

$$|c_1|^2 + |c_2|^2 = 1\,,$$

we use a normalized fiducial vector $\mid \psi_0\rangle$.

Two orthogonal pure spin-states have density matrices

$$\rho_{1,2} = \begin{pmatrix} \cos^2\theta_{1,2} & e^{i\gamma_{1,2}}\cos\theta_{1,2}\sin\theta_{1,2} \\ e^{-i\gamma_{1,2}}\cos\theta_{1,2}\sin\theta_{1,2} & \sin^2\theta_{1,2} \end{pmatrix}, \tag{10}$$

where

$$\theta_2 = \frac{\pi}{2} - \theta_1\,, \qquad \gamma_2 = \pi + \gamma_1\,. \tag{11}$$

Let us associate with the density operators ρ_1 and ρ_2 two normalized vectors

$$\mid \psi_1\rangle = \frac{\rho_1 \mid \psi_0\rangle}{\sqrt{\langle\psi_0 \mid \rho_1 \mid \psi_0\rangle}} \tag{12}$$

and

$$\mid \psi_2\rangle = \frac{\rho_2 \mid \psi_0\rangle}{\sqrt{\langle\psi_0 \mid \rho_2 \mid \psi_0\rangle}}\,. \tag{13}$$

If one considers a superposition of two arbitrary state vectors $\mid \psi_1\rangle$ and $\mid \psi_2\rangle$

$$\mid \psi\rangle = c_1 \mid \psi_1\rangle + c_2 \mid \psi_2\rangle\,, \tag{14}$$

one obtains a corresponding density operator which reads

$$\rho = |c_1|^2 \mid \psi_1\rangle\langle\psi_1 \mid + |c_2|^2 \mid \psi_2\rangle\langle\psi_2 \mid + c_1c_2^* \mid \psi_1\rangle\langle\psi_2 \mid + c_2c_1^* \mid \psi_2\rangle\langle\psi_1 \mid\,. \tag{15}$$

Each of the state vectors $\mid \psi_1\rangle$ and $\mid \psi_2\rangle$ can be multiplied by constant phase factors $e^{i\Theta_1}$ and $e^{i\Theta_2}$, respectively. The density operators

$$\mid \psi_1\rangle\langle\psi_1 \mid = \rho_1 \qquad \text{and} \qquad \mid \psi_2\rangle\langle\psi_2 \mid = \rho_2$$

do not depend on these phase factors. But the density operator of superposition (14) given by (15) depends on the relative phase

$$\varphi = \Theta_1 - \Theta_2\,.$$

To obtain the composition law of two density operators ρ_1 and ρ_2 corresponding to the superposition of state vectors (14), we define it using the superposition of vectors (12) and (13)

$$\rho = \Big(c_1 \mid \psi_1\rangle + c_2 \mid \psi_2\rangle\Big)\Big(c_1^*\langle\psi_1 \mid + c_2^*\langle\psi_2 \mid\Big)\,. \tag{16}$$

In terms of density operators of pure states ρ_1 and ρ_2, the density operator (16) reads

$$\rho = |c_1|^2\rho_1 + |c_2|^2\rho_2 + [c_1c_2^*\rho_1P_0\rho_2 + \text{h.c.}]\,\Big[\operatorname{Tr}(\rho_1P_0)\operatorname{Tr}(\rho_2P_0)\Big]^{-1/2}, \tag{17}$$

where

$$P_0 = | \psi_0\rangle\langle\psi_0 | \ .$$

The first two terms in (17) provide only impure density operator. The rest interference term provides a purification of the impure density operator. One can see that

$$\frac{\rho_1 P_0 \rho_2}{\left[\mathrm{Tr}\,(\rho_1 P_0)\,\mathrm{Tr}\,(\rho_2 P_0) \right]^{1/2}} = | \psi_1\rangle\langle\psi_2 | \ e^{i\varphi}, \tag{18}$$

where

$$e^{i\varphi} = \frac{\langle\psi_1 \mid \psi_0\rangle\langle\psi_0 \mid \psi_2\rangle}{|\langle\psi_1 \mid \psi_0\rangle\langle\psi_0 \mid \psi_2\rangle|}\,. \tag{19}$$

This means that the introduction of a fiducial vector gives us the possibility to describe the relative phase of the state vectors $| \psi_1\rangle$ and $| \psi_2\rangle$ in the composition law of the density operators. The role of the fiducial vector $| \psi_0\rangle$ is to select a vector each within the one-dimensional vector space determined, respectively, by the density operators ρ_1 and ρ_2.

To describe the phenomenon of a partial decoherence, one can generalize formula (17). The trace of the last interference term in (17) equals zero. By elaborating an additional parameter, visibility of the interference pattern in terms of the projectors can be associated with the extended composition rule

$$\rho = |c_1|^2\rho_1 + |c_2|^2\rho_2 + \gamma\,[c_1 c_2^*\rho_1 P_0 \rho_2 + \mathrm{h.c.}]\left[\mathrm{Tr}\,(\rho_1 P_0)\,\mathrm{Tr}\,(\rho_2 P_0) \right]^{-1/2}. \tag{20}$$

If the coefficient $\gamma = 1$, one has the pure state (17). If the coefficient $\gamma = 0$, one has the mixed state (20) without the interference pattern. The purity parameter of the mixed state (20) reads

$$\mu_0 = 1 - 2\left(1 - \gamma^2\right)|c_1 c_2|^2. \tag{21}$$

The parameter γ can be incorporated by using complex values of the angle in Eq. (19).

COMPOSITION LAW AND DEFORMED ASSOCIATIVE PRODUCT

Now we may incorporate the factor P_0 by using a K-deformed associative product on the space of operators by setting

$$A \cdot_K B := AKB\,; \qquad K = P_0\,,$$

as considered in [7]. In this more compact form, the extension to many projection operators can be easily written down and, for composition of n density operators, it reads [6]

$$\rho = \sum_{i,j=1}^{n} c_i c_j^* \frac{\rho_i \cdot_K \rho_j}{\left[\mathrm{Tr}\left(\rho_i \cdot_K \rho_j \cdot_K P_0\right)\right]^{1/2}} . \tag{22}$$

We used the relation:

$$\mathrm{Tr}\left(\rho_i P_0\right) \mathrm{Tr}\left(\rho_j P_0\right) = \mathrm{Tr}\left(\rho_i P_0 \rho_j P_0\right) . \tag{23}$$

The above formulas are written for generic spin j. The relative phases of the pure spin states can be included into relative phases of complex numbers c_i, c_j^*.

For two j-spin states, the composition law of nonorthogonal density operators holds by the replacement of the operator P_0 with the projector

$$P_j = \mid j,j\rangle\,\langle j,j \mid \tag{24}$$

and using angle φ given by Eq. (19). Thus, for the j-spin case, one has [4]

$$\begin{aligned}
\rho_\varphi = & \Big\{ |c_1|^2 \rho_1 + |c_2|^2 \rho_2 + \Big(c_1 c_2^* e^{i\varphi} \rho_1 \mid j,j\rangle\,\langle j,j \mid \rho_2 + \mathrm{h.c.}\Big) \\
& \times \Big[\mathrm{Tr}\left(\rho_1 P_j\right)\ \mathrm{Tr}\left(\rho_2 P_j\right) \Big]^{-1/2} \Big\} \\
& \times \Big\{ 1 + \Big[c_1 c_2^* e^{i\varphi}\, \mathrm{Tr}\, \rho_1 \mid j,j\rangle\,\langle j,j \mid \rho_2 + c_1 c_2^* e^{-i\varphi}\, \mathrm{Tr}\, \rho_2 \mid j,j\rangle\,\langle j,j \mid \rho_1 \Big] \\
& \times \left[\mathrm{Tr}\left(\rho_1 P_j\right)\ \mathrm{Tr}\left(\rho_2 P_j\right)\right]^{-1/2} \Big\}^{-1} .
\end{aligned} \tag{25}$$

One can check that, for arbitrary projectors ρ_1 and ρ_2, the Hermitian operator ρ_φ given by (25) satisfies the relations

$$\rho_\varphi^2 = \rho_\varphi ; \qquad \mathrm{Tr}\, \rho_\varphi^2 = 1 .$$

Thus, for the spin superposition state in the case

$$|c_1|^2 + |c_2|^2 = 1 ,$$

we obtain the density matrix with parameters θ and γ given by the relations

$$\cos^2\theta = \frac{|c_1|^2 \cos^2\theta_1 + |c_2|^2 \cos^2\theta_2 + \left(c_1 c_2^* e^{i\varphi} + c_1^* c_2 e^{-i\varphi}\right) \cos\theta_1 \cos\theta_2}{1 + \left(c_1 c_2^* e^{i\varphi} + c_1^* c_2 e^{-i\varphi}\right) \cos\left(\theta_1 - \theta_2\right)} \tag{26}$$

and

$$\begin{aligned}
\cos\theta \sin\theta\, e^{i\gamma} = & \frac{|c_1|^2 \cos\theta_1 \sin\theta_1\, e^{i\gamma_1} + |c_2|^2 \cos\theta_2 \sin\theta_2\, e^{i\gamma_2}}{1 + \left(c_1 c_2^* e^{i\varphi} + c_1^* c_2 e^{-i\varphi}\right) \cos\left(\theta_1 - \theta_2\right)} \\
& + \frac{c_1 c_2^* e^{i\varphi} \cos\theta_1 \sin\theta_2\, e^{i\gamma_2} + c_1^* c_2 e^{-i\varphi} \sin\theta_1 \cos\theta_2\, e^{i\gamma_1}}{1 + \left(c_1 c_2^* e^{i\varphi} + c_1^* c_2 e^{-i\varphi}\right) \cos\left(\theta_1 - \theta_2\right)} .
\end{aligned} \tag{27}$$

The main result of our paper is the formulation of the composition law for density operators of pure spin states, which yields the rule of purification of the impure mixture of quantum states. This composition law connects the Schrödinger and von Neumann descriptions of pure spin states and provides a rule of the pure-state superposition in terms of density operators.

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