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Hidden Variables or Positive Probabilities?

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Despite claims that Bell's inequalities are based on the Einstein locality condition, or equivalent, all derivations make an identical mathematical assumption that local hidden-variable theories produce a set of positive-definite probabilities for detecting a particle with a given spin orientation. The standard argument is that because quantum mechanics assumes that particles are emitted in a superposition of states the theory cannot produce such a set of probabilites. We examine a paper by Eberhard, and several similar papers, which claim to show that a generalized Bell inequality, the CHSH inequality, can be derived solely on the basis of the locality condition, without recourse to hidden variables. We point out that these authors nonetheless assumes a set of positive-definite probabilities, which supports the claim that hidden variables or "locality" is not at issue here, positive-definite probabilities are. We demonstrate that quantum mechanics does predict a set of probabilities that violate the CHSH inequality; however these probabilities are not positive-definite. Nevertheless, they are physically meaningful in that they give the usual quantum-mechanical predictions in physical situations. We discuss in what sense our results are related to the Wigner distribution.

1. INTRODUCTION

With the introduction of his celebrated inequalities in 1964, Bell (1964) provided the basis for an experimental test to distinguish quantum mechanics from local hidden-variable theories. Since that time the universal interpretation of the results has been that quantum mechanics violates Bell's inequalities because of its "nonlocal" character, whereas local hidden variable theories satisfy the inequalities because, as their name implies, they are "local."

The situation is actually not so transparent. Bohr taught us to be aware of ambiguous language. Although derivations of Bell's inequalities are evidently based on Einstein's "locality" condition, couched in various phrases such as "principle of separability" and so forth, mathematically all derivations make an identical assumption, specifically: *hidden-variable theories introduce a set of*

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a priori *positive-definite probabilities P that are not predicted by quantum mechanics*. In Bohm's classic version of the Einstein–Podolsky–Rosen experiment, e.g., a particle in a spin-singlet state decays into two daughter particles with zero total angular momentum (see, e.g., Sakurai's text (Sakurai, 1994) or Sudarshan and Rothman, 1993, henceforth SR). According to local hidden-variable theories there is an a priori positive-definite probability that the daughter particles will be detected with spins "up" along a chosen axis. Quantum mechanics, on the other hand, assumes that the daughter particles are in a superposition of states and so, by definition, there can be no a priori probability P such that their spins will be detected along a given direction.

Contrary to this view, in SR we pointed out that quantum mechanics *does* predict a set of a priori probabilities, in exactly the same way as do hidden-variable theories, but the quantum probabilities are not positive-definite. They are nevertheless meaningful in that when applied to physical situations they give the standard quantum-mechanical answers, in particular the usual violation of Bell's inequalities. Given the exact analogy in producing the two sets of probabilities the distinction between "local" hidden-variable theories and "nonlocal" quantum mechanics is dissolved. From this point of view one merely has two competing theories that give two different sets of probabilities; it is unsurprising that hidden-variables theories fail experimental tests of Bell's inequalities because they used the wrong set of probabilities for a quantum-mechanical problem.

The notion of "extended" probabilities dates back to Dirac and we have not been the only authors to suggest that they can resolve the EPR paradox (see Mückenheim, 1986; Mückenheim *et al*., 1986)[4] but, needless to say, the SR argument has not found widespread acceptance. Recently, several rather old papers, in particular one by Eberhard (1977) entitled "Bell's Theorem Without Hidden Variables," have come to our attention. Eberhard's paper is of interest because it claims to show that a more general version of Bell's inequalities, known as the CHSH inequality (after Clauser *et al*., 1969), is violated by quantum mechanics, and that the CHSH inequality can be demonstrated solely on the basis of the locality principle, *without the introduction of hidden variables*. (A slightly later paper by Peres, 1978, gives an almost identical argument; one by Stapp, 1985, is in some respects similar.) At first sight these proofs appear to assume little more than $2 < 2\sqrt{2}$. On closer inspection, however, we find that they "play into our hands," i.e., they may not make an explicit statement about hidden variables but they *do* assume a set of positive-definite probabilities. We now demonstrate this is so, reinforcing the contention in SR that, despite any words employed, the crucial

[4] Mückenheim (1986) suggests the need for negative probabilities to resolve the EPR paradox, but does not explicitly calculate the probability distribution. The study by Mückenheim *et al*. (1986) is a historical survey about the subject of negative probabilities.

mathematical assumption in derivations of Bell's inequalities is not locality but positive probability.

2. THE EBERHARD ARGUMENT

Eberhard (1977) considers two identical apparata, A and B, at two different locations. On apparatus A is a knob a that can be turned to two positions, 1 and 2. On apparatus B is a knob b that can also be turned to two positions, 1 and 2. With its knob at either position apparatus A can record a series of events. It is not important exactly what the events are, but we assume that for each event each apparatus can measure only one of the two possible outcomes, which for simplicity we take to be ± 1. When the knob a is in the position 1, we designate the outcome of the jth event as α_{1j}, with similar notation for position 2 and knob b. For each event we can thus in principle have $\alpha_{1j} = \pm 1$, $\alpha_{2j} = \pm 1$, $\beta_{1j} = \pm 1$, $\beta_{2j} = \pm 1$. However, for each measurement we will choose only one setting on each apparatus, so a given event will produce a pair of readings, such as $\alpha_1 = 1$, $\beta_2 = -1$. (Here and later we suppress the subscript j when it will not cause confusion.)

For a series of N measurements Eberhard next defines a quantity C, such that

$$C = \frac{1}{N} \sum_{j=1}^{N} \alpha_j \beta_j. \tag{2.1}$$

We see that $C = \langle \alpha_j \beta_j \rangle$, the statistical mean of the N products $\alpha_j \beta_j$. No restriction is placed on the fraction of the N measurements for which the α's and β's come out positive or negative, but note that each product $\alpha_j \beta_j = 1$ when α and β have the same sign and $\alpha_j \beta_j = -1$ when they have opposite signs. Thus C represents the fraction of events in which α and β have the same sign minus the fraction in which they have opposite sign.

Because each knob has two positions, there are four possible versions of C. That is, we can define

$$\begin{aligned} C_{11} &= \langle \alpha_1 \beta_1 \rangle \\ C_{12} &= \langle \alpha_1 \beta_2 \rangle \\ C_{21} &= \langle \alpha_2 \beta_1 \rangle \\ C_{22} &= \langle \alpha_2 \beta_2 \rangle \end{aligned} \tag{2.2}$$

(sum on j understood). Here, C_{11} is just the above statistical mean when both knobs a and b are in position 1, and so forth.

Now, for each event let

$$\gamma \equiv \alpha_1 \beta_1 + \alpha_1 \beta_2 + \alpha_2 \beta_1 - \alpha_2 \beta_2. \tag{2.3}$$

Then, the statistical mean of γ is just

$$\begin{aligned}\langle\gamma\rangle &= \frac{1}{N}\sum_{j=1}^{N}\gamma_j \\ &= \frac{1}{N}\sum_{j=1}^{N}(\alpha_1\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_1 - \alpha_2\beta_2) \\ &= C_{11} + C_{12} + C_{21} - C_{22}, \qquad (2.4)\end{aligned}$$

where in the second line we have again suppressed j.

The locality condition enters the discussion when we attempt to put bounds on $\langle\gamma\rangle$. Recall that a knob will be set to either position 1 or 2 for each measurement. We assume that a measurement on A is independent of a measurement on B. The α's and β's are thus treated independently. This is the locality conditon.

At this point a digression is necessary. Eberhard states that only one setting of each knob (position 1 or 2) will be used for each measurement, and that thus only one α or β is recorded for each event. However, if this were indeed the case, then for each measurement only one term in γ would survive (one product $\alpha\beta$) and the upper bound on γ would be 1 (cf. Eqs. (2.3) and (2.7)). That the upper bound turns out to be 2 shows that *mathematically* all four possible terms $\alpha\beta$ are present in γ. Consequently, not only are the α's being taken to be independent of the β's but $\alpha_1(\beta_1)$ is being treated as independent of $\alpha_2(\beta_2)$. The rationale for including all α's and β's in γ simultaneously comes from a suggestion of Stapp (1971). Stapp and Eberhard (and Peres, 1978, in his nearly identical thought experiment), are actually considering all possible outcomes of the measurements in a hypothetical ensemble space. By doing so they intend to show that any conceivable outcome of the experiment is violated by quantum mechanics.

One can take several attitudes toward such a procedure. A first possible attitude is that it is illegitimate to speculate about the results of unperformed experiments. In other words, if one takes the quantity γ literally, the knobs must be set to two positions at once, a physical impossibility. A second view is that it is indeed legitimate to think about all possible outcomes of an experiment[5] and that if one does so, one is forced to conclude that quantum mechanics is nonlocal. In fact, there is a third possible viewpoint. As we will discuss later, the γ's are derivable from the "master probabilities" employed in a standard derivation of Bell's inequalities, quantities that are not directly measurable but nevertheless have physical consequences. Hence, both the Eberhard procedure and the standard derivation suffer from exactly the same ambiguities. For the moment it is not important which philosophy one adopts; we merely treat γ as a mathematical quantity, as Eberhard does. At the same time, however, we see that by treating

[5] This concept is often referred to as "counterfactual definiteness," after Stapp.

all the α's and β's as independent, mathematically the locality condition becomes indistinguishable from the general assumption of independent variables.

In any case, following Eberhard we assume 16 possible values for each γ. At this stage of the exposition, Eberhard goes through an elaborate argument to show that $\gamma \leq 2$ always. However, let us redistribute the terms in Eq. (2.3) and write

$$\gamma = \alpha_1(\beta_1 + \beta_2) + \alpha_2(\beta_1 - \beta_2). \tag{2.5}$$

Because β_1 and β_2 are equal or of opposite sign, if the first term is nonzero, the second term is zero and vice versa. Thus we can see trivially that $\gamma = \pm 2$ always and $|\gamma| = 2$.

But by the triangle inequality we know that

$$\left| \frac{1}{N} \sum_{j=1}^{N} (\alpha_1\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_1 - \alpha_2\beta_2) \right| \leq \frac{1}{N} \sum_{j=1}^{N} |(\alpha_1\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_1 - \alpha_2\beta_2)| \tag{2.6}$$

Yet from Eq. (2.4) and Eq. (2.3) this is by definition

$$|C_{11} + C_{12} + C_{21} - C_{22}| \leq \frac{1}{N} \sum_{j=1}^{N} |\gamma_j| = \frac{1}{N} \times N \times 2. \tag{2.7}$$

The CHSH inequality follows immediately:

$$|C_{11} + C_{12} + C_{21} - C_{22}| \leq 2, \tag{2.8}$$

or, in more compact notation,

$$|\mathcal{C}| \leq 2. \tag{2.9}$$

Eberhard next considers a quantum-mechanical experiment in which two photons are emitted in the directions of A and B by an atom between them. The photons are detected by polarizers; each $\alpha(\beta)$ is taken to be $+1$ when one polarization is detected and -1 when the other is detected. Unfortunately, at this point the paper becomes very unclear. Eberhard merely asserts without calculation that for each of the C's in Eq. (2.2), quantum mechanics predicts that "if the number of events N is large enough, then $C \cong \cos(2a - 2b)$," where $2a - 2b$ is twice the angle between the polarizers. Actually, no approximation is necessary. For spin-1/2 particles, the correct prediction is

$$\mathcal{C}_{\text{qm}} = 3\cos\theta - \cos 3\theta, \tag{2.10}$$

which we derive below, and in which θ is the angle between polarizers. (The result for photons will be the same if θ is taken to be twice the angle between polarizers.) Note that for $\theta = 45^\circ$ (2.10) gives $\mathcal{C}_{\text{qm}} = 2\sqrt{2} \geq 2$. Therefore, quantum mechanics violates the CHSH inequality, just as it does the Bell inequalities.

As mentioned earlier, the demonstration seems to assume almost nothing; no hidden variables, merely "locality," which implies that a certain mathematical quantity γ always equals ± 2. However, on closer inspection we find that more than an assumption of independent α's and β's is being made. In the first place, the value 2 on the right-hand side of Eq. (2.8) is entirely arbitrary and results merely from the choice of ± 1 as the "eigenvalues" for α and β. One could have equally well-chosen ± 1000. In that case, however, one would necessarily have to assume that the corresponding quantum experiment also had eigenvalues of ± 1000. This matter is not so serious, but it nevertheless illustrates that the CHSH inequality is not a purely mathematical assertion; a real measurement does lurk in the background.

The central issue lies elsewhere. Eberhard's version of CHSH inequality is a statement about the statistical mean of γ, and therefore it does deal with a probability distribution over the γ. Moreover, the frequency that a particular γ occurs is clearly taken to be positive. That probabilities should be positive-definite is usually regarded as self-evident, but because the assumption is the crux of the matter, we spend a moment examining it. (In the Appendix we detail where other authors have made the same assumption.)

As mentioned, there are 16 possible combinations of $\alpha_1\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_1 - \alpha_2\beta_2 (= \gamma)$, of which eight have the value $+2$ and eight have the value -2. In a sequence of N measurements, let us suppose that $+2$ occurs n_1 times and -2 occurs n_2 times, such that $n_1 + n_2 = N$. Then

$$\mathcal{C} = \frac{2}{N}[n_1 - n_2]. \tag{2.11}$$

If all frequencies are equal, i.e., $n_1 = n_2$, then $\mathcal{C} = 0$. If $n_2 = 0$, then $\mathcal{C} = 2$ and if $n_1 = 0$, then $\mathcal{C} = -2$. But here we have assumed that both n_1 and n_2 are positive-definite. If $n_2 < 0$, then $\mathcal{C} > 2$. In other words, the step leading to the last expression in Eq. (2.7) is valid only when $|n| = n$.

The notion of "extended" (non–positive-definite) probabilities has been considered by a surprising number of prominent investigators, but the majority of physicists continue to regard them with distaste, if not revulsion. Nevertheless, the quantum violation of the bound on $\mathcal{C}$ is effectively due to the fact that quantum mechanics allows negative probabilities. In the next section, we examine this claim in greater detail.

3. QUANTUM MECHANICAL PROBABILITIES

Before deriving Eq. (2.10), it will be helpful to summarize the procedure for obtaining the standard Bell Inequalities in order to point out similarities to the CHSH–Eberhard experiment. The reader is referred to SR or Sakurai (1994)

Table I. Spin Combinations for Standard Bell Inequalities

Population	Particle 1	Particle 2
N_1	$(+++)$	$(---)$
N_2	$(++-)$	$(--+)$
N_3	$(+-+)$	$(-+-)$
N_4	$(-++)$	$(+--)$
N_5	$(+--)$	$(-++)$
N_6	$(-+-)$	$(+-+)$
N_7	$(--+)$	$(++-)$
N_8	$(---)$	$(+++)$

Note. Hidden-variable models assume that spin-1/2 particles can be emitted with $\pm$ spin along each of three axes, **a**, **b**, and **c**. The notation $(+++)$ etc., means spin up along all three axes. The eight possible spin combinations are shown. To ensure conservation of angular momentum, a particle of the type $(+++)$ must be paired with one of $(---)$ and so on.

for additional details; see also the Appendix. Like its successor, Bell's theorem is valid for local hidden-variable theories, which involve only classical probabilities. In a typical derivation such as Sakurai's one assumes that spin measurements may be made along any of three axes, **a**, **b**, and **c**. A system of decaying atoms emits N particles of which a certain fraction are taken to be, say, of the type $(\mathbf{a}+, \mathbf{b}+, \mathbf{c}+) \equiv (+++)$, which designates spin up along all three axes. To ensure zero total angular momentum, each emitted particle of type $(+++)$ must be paired with one of type $(---)$. There are eight such spin combinations in all, as listed in Table I.

The probability that $(+++)$ is emitted (and in the case of hidden variables, detected) is defined simply as $P(+++) = N(+++)/N$. One can immediately object that such a probability is unphysical because to determine it one requires three simultaneous spin measurements on a system of two particles, which is impossible. To eliminate this difficulty, one forms pairwise probabilities of the type $P(\mathbf{a}+, \mathbf{b}+) \equiv P(++)$, which represents the joint probability that the first particle will be found $+$ along **a** and the second particle $+$ along **b**. This is easily done. From the table, the total number of particles such that the first particle's spin is $+$ along **a** is $N(+-+) + N(+--)$, which must be paired with $N(-+-) + N(-++)$, the total number of particles for which the second particle's spin is $+$ along **b**. This combination is labeled $N_3 + N_5$. Next one forms traingle-type inequalities such as

$$N_3 + N_5 \leq (N_2 + N_5) + (N_3 + N_7), \tag{3.1}$$

which is obviously true, since we have just added positive numbers to $N_3 + N_5$.

Dividing by N gives by definition

$$P(\mathbf{a}+, \mathbf{b}+) \leq P(\mathbf{a}+, \mathbf{c}+) + P(\mathbf{c}+, \mathbf{b}+), \tag{3.2}$$

one of the Bell inequalities. Equation (3.2) involves only one measurement on each particle and so represents a physically realizable situation. Note that the "three-probabilities" $P(+++)$ were reduced to pairwise probabilities $P(++)$ by summing over the spins on the extraneous axis, in the above example **c**. We emphasize that, just as was the case for the CHSH inequality, the Bell inequality is valid only if the N's and hence the P's are taken to be positive-definite. In SR we demonstrated that one can form *quantum probabilities* $P(+++)$, analogous to the classical probabilities, then sum over the third argument exactly as above to get pairwise quantum probabilities $P(++)$ that violate (3.2) in the usual way.

By this point the reader will have noticed a similarity between the γ's in Eberhard's experiment and the three-probabilities here. Authors who derive the generalized Bell inequalities introduce γ as a measure of correlations between real and imagined experiments but, as mentioned, if one takes it literally, it amounts to having the apparatus knobs set on two positions simultaneously. This would seem to represent the same sort of physical impossibility as that of making three simultaneous spin measurements on two particles. Indeed, we will demonstrate in Section 4 that the two procedures are identical: Introducing an ensemble of hypothetical measurements is exactly equivalent to assuming a "master probability distribution" that requires more than two simultaneous spin measurements on two particles. Before doing so, however, we return to the Eberhard derivation.

Eberhard's experiment involves four axes, $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{b_1}$, $\mathbf{b_2}$, rather than three, but otherwise is almost identical to the standard derivation of Bell's inequalities and so it is not surprising that the above procedure can be followed to demonstrate a violation of the CHSH inequality. We first need to compute the quantum pairwise probabilities of the type just mentioned, $P(\mathbf{a}+, \mathbf{b}+)$. There are several ways to do this. Following SR, we write the quantum-mechanical projection operator for spin-1/2 particles as

$$\Pi(\mathbf{a}\pm) = \frac{1}{2}(\mathbf{1} \pm \boldsymbol{\sigma} \cdot \boldsymbol{a}). \tag{3.3}$$

In this equation, we are representing the Pauli spin matrices as a vector, $\boldsymbol{\sigma} = \hat{\mathbf{i}}\sigma_{\mathbf{x}} + \hat{\mathbf{j}}\sigma_{\mathbf{y}} + \hat{\mathbf{k}}\sigma_{\mathbf{z}}$. Thus $\boldsymbol{\sigma} \cdot \boldsymbol{a} = \sigma_x a_x + \sigma_y a_y + \sigma_z a_z$ represents a traceless 2×2 matrix, and $\mathbf{1}$ is the unit matrix. Now, the expectation value of any operator $\mathcal{O}$ can be written $\langle \mathcal{O} \rangle = \mathrm{Tr}(\rho \mathcal{O})$, where ρ is the density matrix $\equiv \mathrm{diag}(1/2, 1/2)$ for an initially unpolarized beam. The probability of finding the first particle in the $+$ state along $\mathbf{a}$ is thus $\mathrm{Tr}(\rho \Pi(\mathbf{a})) = 1/2$. Similarly, the joint probability $P(\mathbf{a}+, \mathbf{b}\pm)$ of finding the first particle in the $+$ state along $\mathbf{a}$ and the second particle in the $\pm$ state along **b** is

$$P(\mathbf{a}+,\ \mathbf{b}\pm) = \frac{1}{2}\mathrm{Tr}\Pi(\mathbf{a})\Pi(\mathbf{b}\pm)$$
$$= \frac{1}{8}\mathrm{Tr}\{(\mathbf{1} + \boldsymbol{\sigma}\cdot\boldsymbol{a})(\mathbf{1} \pm \boldsymbol{\sigma}\cdot\boldsymbol{b})\}$$
$$= \frac{1}{4}(1 \pm \mathbf{a}\cdot\mathbf{b}). \quad (3.4)$$

Here, the standard identity has been used (see Sakurai, 1994)

$$(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{1} + i\boldsymbol{\sigma}(\mathbf{a}\times\mathbf{b}). \quad (3.5)$$

Because the Pauli matrix is traceless, taking the trace of (3.5) yields $2\boldsymbol{a}\cdot\boldsymbol{b}$.

Equation (3.4) is simply a sophisticated way of writing Malus' law. The first factor of 1/2 in (3.4) gives the probability of detecting a particle in the + state along the **a** axis. The remaining factor $1/2(1+\boldsymbol{a}\cdot\boldsymbol{b}) = 1/2(1+\cos\theta)$, where θ is the angle between polarizers. For photons(where θ is taken to be the double angle) this then represents the usual decrease in intensity with $\cos^2\theta$. For a Bohm-type experiment, which assumes an (antisymmetric) spin-singlet state, one should choose the $-$ on the right of (3.4) when computing $P(\mathbf{a}+, \mathbf{b}+)$ to conserve angular momentum. With either sign, by inserting (3.4) into (3.2), it is straightforward to show that quantum mechanics violates Bell's inequalities.

For the Eberhard experiment we take the knob setting a_1, a_2, b_1, b_2 to represent the position of the polarizers on the measuring devices. Recall that his quantities $C = \langle\alpha\beta\rangle$ represented the fraction of events in which α and β had the same sign minus the fraction in which they had opposite signs, irrespective of whether an individual spin is + or $-$. Evidently the equivalent quantum expression is $1/2(1+\mathbf{a}\cdot\mathbf{b}) - 1/2(1-\mathbf{a}\cdot\mathbf{b})$. Then

$$\mathcal{C}_{\mathrm{qm}} = \mathbf{a_1}\cdot\mathbf{b_1} + \mathbf{a_1}\cdot\mathbf{b_2} + \mathbf{a_2}\cdot\mathbf{b_1} - \mathbf{a_2}\cdot\mathbf{b_2}. \quad (3.6)$$

If the axes are chosen to be coplanar such that $\mathbf{a_1}\cdot\mathbf{b_1} = \mathbf{a_1}\cdot\mathbf{b_2} = \mathbf{a_2}\cdot\mathbf{b_1} = \cos\theta$ and $\mathbf{a_2}\cdot\mathbf{b_2} = \cos 3\theta$, then (3.6) gives exactly (2.10), which violates the CHSH inequality for $\theta = 45^\circ$.

The derivation of (2.10) just given involved only pairwise probabilities and did not go beyond standard quantum mechanics. With the projection-operator formalism, however, it is not difficult to write down the joint probability for four "simultaneous" spin measurements among four axes. An example would be $P(++++)$, in analogy to the classical three-probability mentioned earlier that appears in the derivation of Bell's inequality. Extending (3.4) to four arguments we take

$$P(\lambda\mathbf{a_1}, \mu\mathbf{a_2}, \nu\mathbf{b_1}, \tau\mathbf{b_2}) = \frac{1}{2}\mathrm{Tr}\{\Pi(\lambda\mathbf{a_1})\Pi(\mu\mathbf{a_2})\Pi(\nu\mathbf{b_1})\Pi(\tau\mathbf{b_2})\}, \quad (3.7)$$

where λ, μ, ν, τ are chosen as ± 1 to represent up or down. For the symmetric case this is

$$P(\lambda \mathbf{a_1}, \mu \mathbf{a_2}, \nu \mathbf{b_1}, \tau \mathbf{b_2}) = \frac{1}{32}\mathrm{Tr}\{(\mathbf{1} + \lambda \boldsymbol{\sigma} \cdot \boldsymbol{a}_1)(\mathbf{1} + \mu \boldsymbol{\sigma} \cdot \boldsymbol{a}_2)(\mathbf{1} + \nu \boldsymbol{\sigma} \cdot \boldsymbol{b}_1) \times (\mathbf{1} + \tau \boldsymbol{\sigma} \cdot \boldsymbol{b}_2)\}. \tag{3.8}$$

We will need the antisymmetric expression later to make the subtraction just done above. Assuming that a measurement of $+$ on knob a requires $-$ on knob b, the antisymmetric case will have the same expression as (3.8) with the signs on the b's reversed. We calculate only the symmetric case and state the results for the antisymmetric case as needed.

Working out (3.8) and making frequent use of identity (3.5) yields

$$\begin{aligned} P(\lambda \mathbf{a_1}, \mu \mathbf{a_2}, \nu \mathbf{b_1}, \tau \mathbf{b_2}) = \frac{1}{16}\{&1 + \lambda\mu \mathbf{a_1} \cdot \mathbf{a_2} + \lambda\nu \mathbf{a_1} \cdot \mathbf{b_1} + \lambda\tau \mathbf{a_1} \cdot \mathbf{b_2} \\ &+ \mu\nu \mathbf{a_2} \cdot \mathbf{b_1} + \mu\tau \mathbf{a_2} \cdot \mathbf{b_2} + \nu\tau \mathbf{b_1} \cdot \mathbf{b_2} \\ &+ i\lambda\mu\nu(\mathbf{a_1} \times \mathbf{a_2}) \cdot \mathbf{b_1} + i\lambda\mu\tau(\mathbf{a_1} \times \mathbf{a_2}) \cdot \mathbf{b_2} \\ &+ i\lambda\nu\tau(\mathbf{b_1} \times \mathbf{b_2}) \cdot \mathbf{a_1} + i\mu\nu\tau(\mathbf{b_1} \times \mathbf{b_2}) \cdot \mathbf{a_2} \\ &+ \lambda\mu\nu\tau[(\mathbf{a_1} \cdot \mathbf{a_2})(\mathbf{b_1} \cdot \mathbf{b_2}) + \mathbf{2}(\mathbf{a_1} \times \mathbf{a_2}) \cdot (\mathbf{b_1} \times \mathbf{b_2})]\}. \end{aligned} \tag{3.9}$$

Notice that this expression is complex because of the imaginary elements of σ_y. If we desire a real result to eventually make contact with the usual quantum predictions, we can easily eliminate the imaginary terms. Note that $\Pi(\lambda \mathbf{a_1})\Pi(\mu \mathbf{a_2}) \times \Pi(\nu \mathbf{b_1})\Pi(\tau \mathbf{b_2})$ has been written in an arbitrary order; it is not symmetric in the arguments. There are 4! permutations of the arguments in this expression, 12 even and 12 odd. In (3.9), each imaginary term is a triple scalar product, which is invariant under even permutations and changes sign under odd permutations. Thus these terms vanish under symmetrization, as does the double cross product in the last line. The symmetrized version of (3.9) is

$$\begin{aligned} P(\lambda \mathbf{a_1}, \mu \mathbf{a_2}, \nu \mathbf{b_1}, \tau \mathbf{b_2}) = \frac{1}{16}\{&1 + \lambda\mu \mathbf{a_1} \cdot \mathbf{a_2} + \lambda\nu \mathbf{a_1} \cdot \mathbf{b_1} + \lambda\tau \mathbf{a_1} \cdot \mathbf{b_2} \\ &+ \mu\nu \mathbf{a_2} \cdot \mathbf{b_1} + \mu\tau \mathbf{a_2} \cdot \mathbf{b_2} + \nu\tau \mathbf{b_1} \cdot \mathbf{b_2} \\ &+ \frac{1}{3}\lambda\mu\nu\tau[(\mathbf{a_1} \cdot \mathbf{a_2})(\mathbf{b_1} \cdot \mathbf{b_2}) + (\mathbf{a_1} \cdot \mathbf{b_1})(\mathbf{a_2} \cdot \mathbf{b_2}) \\ &+ (\mathbf{a_1} \cdot \mathbf{b_2})(\mathbf{b_1} \cdot \mathbf{a_2})]\}, \end{aligned} \tag{3.10}$$

which is entirely real.[6]

[6] It is not actually necessary to symmetrize (3.9). One can leave it as a complex expression, but when the sum over the extraneous arguments is performed as in (3.11), the imaginary terms cancel and the result will be entirely real, as before. However, the complex four-probability is not symmetric in the arguments.

Table II. Four Probabilities

$P(++++) = P(----) =$	$\frac{1}{16}\{1 + \mathbf{a_1}\cdot\mathbf{a_2} + \mathbf{a_1}\cdot\mathbf{b_1} + \mathbf{a_1}\cdot\mathbf{b_2} + \mathbf{a_2}\cdot\mathbf{b_1} + \mathbf{a_2}\cdot\mathbf{b_2} + \mathbf{b_1}\cdot\mathbf{b_2} + \Delta\}$
$P(-+++) = P(+---) =$	$\frac{1}{16}\{1 - \mathbf{a_1}\cdot\mathbf{a_2} - \mathbf{a_1}\cdot\mathbf{b_1} - \mathbf{a_1}\cdot\mathbf{b_2} + \mathbf{a_2}\cdot\mathbf{b_1} + \mathbf{a_2}\cdot\mathbf{b_2} + \mathbf{b_1}\cdot\mathbf{b_2} - \Delta\}$
$P(+-++) = P(-+--) =$	$\frac{1}{16}\{1 - \mathbf{a_1}\cdot\mathbf{a_2} + \mathbf{a_1}\cdot\mathbf{b_1} + \mathbf{a_1}\cdot\mathbf{b_2} - \mathbf{a_2}\cdot\mathbf{b_1} - \mathbf{a_2}\cdot\mathbf{b_2} + \mathbf{b_1}\cdot\mathbf{b_2} - \Delta\}$
$P(++-+) = P(--+-) =$	$\frac{1}{16}\{1 + \mathbf{a_1}\cdot\mathbf{a_2} - \mathbf{a_1}\cdot\mathbf{b_1} + \mathbf{a_1}\cdot\mathbf{b_2} - \mathbf{a_2}\cdot\mathbf{b_1} + \mathbf{a_2}\cdot\mathbf{b_2} - \mathbf{b_1}\cdot\mathbf{b_2} - \Delta\}$
$P(+++-) = P(---+) =$	$\frac{1}{16}\{1 + \mathbf{a_1}\cdot\mathbf{a_2} + \mathbf{a_1}\cdot\mathbf{b_1} - \mathbf{a_1}\cdot\mathbf{b_2} + \mathbf{a_2}\cdot\mathbf{b_1} - \mathbf{a_2}\cdot\mathbf{b_2} - \mathbf{b_1}\cdot\mathbf{b_2} - \Delta\}$
$P(++--) = P(--++) =$	$\frac{1}{16}\{1 + \mathbf{a_1}\cdot\mathbf{a_2} - \mathbf{a_1}\cdot\mathbf{b_1} - \mathbf{a_1}\cdot\mathbf{b_2} - \mathbf{a_2}\cdot\mathbf{b_1} - \mathbf{a_2}\cdot\mathbf{b_2} + \mathbf{b_1}\cdot\mathbf{b_2} + \Delta\}$
$P(+-+-) = P(-+-+) =$	$\frac{1}{16}\{1 - \mathbf{a_1}\cdot\mathbf{a_2} + \mathbf{a_1}\cdot\mathbf{b_1} - \mathbf{a_1}\cdot\mathbf{b_2} - \mathbf{a_2}\cdot\mathbf{b_1} + \mathbf{a_2}\cdot\mathbf{b_2} - \mathbf{b_1}\cdot\mathbf{b_2} + \Delta\}$
$P(+--+) = P(-++-) =$	$\frac{1}{16}\{1 - \mathbf{a_1}\cdot\mathbf{a_2} - \mathbf{a_1}\cdot\mathbf{b_1} + \mathbf{a_1}\cdot\mathbf{b_2} + \mathbf{a_2}\cdot\mathbf{b_1} - \mathbf{a_2}\cdot\mathbf{b_2} - \mathbf{b_1}\cdot\mathbf{b_2} + \Delta\}$

Note. Shown are the four-probabilities from symmetric wavefunction as computed from Eq. (3.10). The quantity

$$\Delta \equiv \frac{1}{3}[(\mathbf{a_1}\cdot\mathbf{a_2})(\mathbf{b_1}\cdot\mathbf{b_2}) + (\mathbf{a_1}\cdot\mathbf{b_1})(\mathbf{a_2}\cdot\mathbf{b_2}) + (\mathbf{a_1}\cdot\mathbf{b_2})(\mathbf{b_1}\cdot\mathbf{a_2})].$$

Note that these probabilities sum to one. The four-probabilities for the antisymmetric wave function can be obtained by flipping last two sings, i.e., $P(++++)_{\mathrm{AS}} = P(++--)_{\mathrm{S}}$, $P(-+++)_{\mathrm{AS}} = P(-+--)_{\mathrm{S}}$, etc.

It is now easy to read off the various four-probabilities, $P(++++)$, $P(----)$, etc., for each case merely by choosing the required signs of λ, μ, ν, τ. The 16 possibilities are listed for convenience in Table II. Note that these four-probabilities do sum to one and therefore in that respect behave as ordinary probabilities. However, although it is perhaps not evident from inspection, several of these probabilities can become negative. We plot $P(+++-)$ and $P(+-+-)$ in Fig. 1. The antisymmetric P's can be obtained from the symmetric ones merely by flipping the signs on the two b's.

From these four-probabilities one can form the quantity $\mathcal{C}_{\mathrm{qm}}$ in Eq. (3.6) in exact analogy to the procedure used for deriving the Bell inequalities. To compute $P(\mathbf{a_1}+, \mathbf{b_1}+)$, e.g., we only care that the first particle will be found $+$ along a_1 and the second particle will be found $+$ along $\mathbf{b_1}$. As before, we count all such possibilities by summing over the two extraneous arguments, $\mathbf{a_2}$ and $\mathbf{b_2}$. Thus, for the symmetric wavefunction,

$$P(\mathbf{a_1}+, \mathbf{b_1}+) = P(+\_\_+\_\_) = P(++++) + P(+++-) + P(+-+-) + P(+-++). \tag{3.11}$$

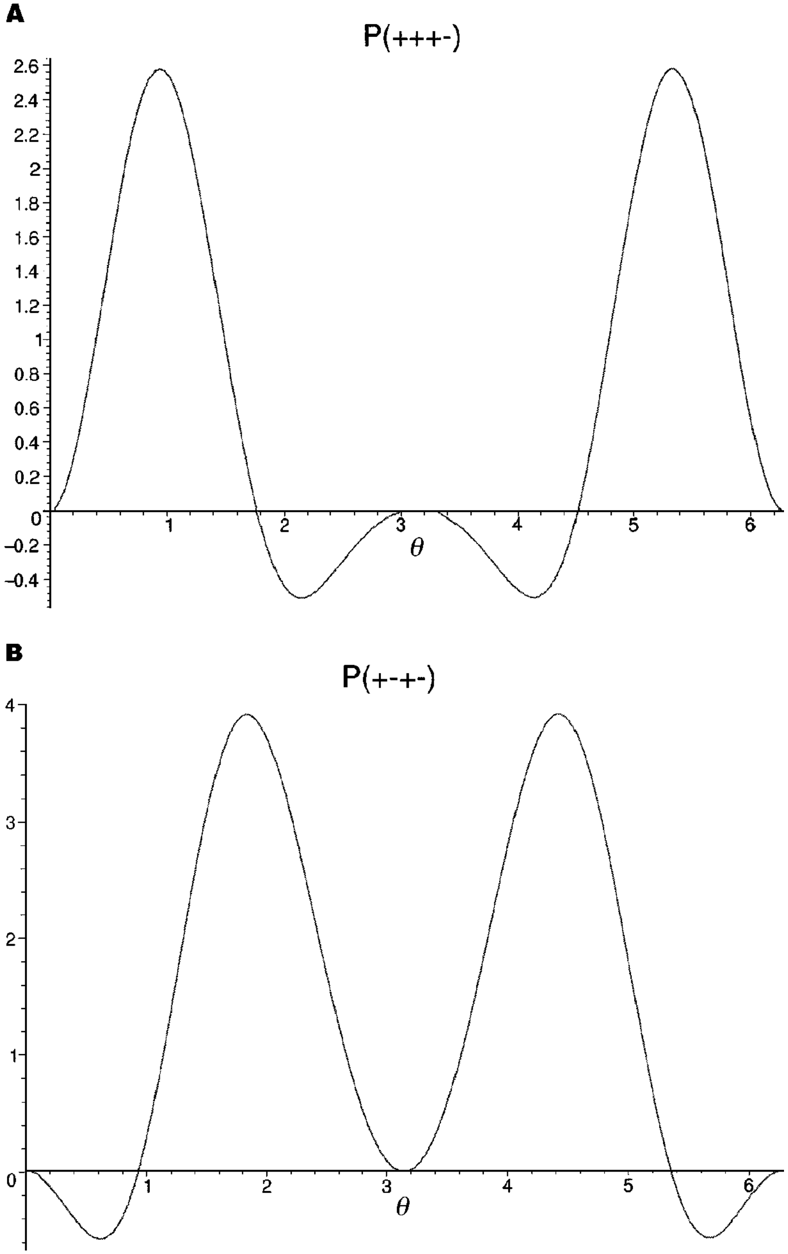

Fig. 1. Four-probabilities from Table III. (**a**) Plot of $16P(+++-)$. (**b**) Plot of $16P(+-+-)$. Note that these quantities become negative.

Table III. Four Probabilities as Functions of Polarizer Angles

$$
\begin{aligned}
P(++++) = P(----) &= \frac{1}{16}\{1 + 3\cos\theta + 2\cos 2\theta + \cos 3\theta + \Delta\} = \frac{1}{16}\{4C^3 + 4C^2 - 1 + \Delta\}\\
P(-+++) = P(+---) &= \frac{1}{16}\{1 - \cos\theta + \cos 3\theta - \Delta\} = \frac{1}{16}\{4C^3 - 4C + 1 - \Delta\}\\
P(+-++) = P(-+--) &= \frac{1}{16}\{1 + \cos\theta - \cos 3\theta - \Delta\} = \frac{1}{16}\{-4C^3 + 4C + 1 - \Delta\}\\
P(++-+) = P(--+-) &= \frac{1}{16}\{1 - \cos\theta + \cos 3\theta - \Delta\} = \frac{1}{16}\{4C^3 - 4C + 1 - \Delta\}\\
P(+++-) = P(---+) &= \frac{1}{16}\{1 + \cos\theta - \cos 3\theta - \Delta\} = \frac{1}{16}\{-4C^3 + 4C + 1 - \Delta\}\\
P(++--) = P(--++) &= \frac{1}{16}\{1 + 2\cos 2\theta - 3\cos\theta - \cos 3\theta + \Delta\}\\
&= \frac{1}{16}\{-4C^3 + C^2 - 1 + \Delta\}\\
P(+-+-) = P(-+-+) &= \frac{1}{16}\{1 - \cos\theta - 2\cos 2\theta + \cos 3\theta + \Delta\}\\
&= \frac{1}{16}\{4C^3 - 4C^2 - 4C + 3 + \Delta\}\\
P(+--+) = P(-++-) &= \frac{1}{16}\{1 + \cos\theta - 2\cos 2\theta - \cos 3\theta + \Delta\}\\
&= \frac{1}{16}\{-4C^3 - 4C^2 + 4C + 3 + \Delta\}
\end{aligned}
$$

Note. Shown are the same four-probabilities as in Table II for the configuration $\mathbf{a_1} \cdot \mathbf{b_1} = \mathbf{a_1} \cdot \mathbf{b_2} = \mathbf{b_1} \cdot \mathbf{a_2} = \cos\theta$ and $\mathbf{a_2} \cdot \mathbf{b_2} = \cos 3\theta$. Now $\Delta = 1/3(\cos^2\theta + \cos^2 2\theta + \cos\theta\cos 3\theta)$. With the identities $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ all the probabilities can be written in terms of one parameter, $\cos\theta \equiv C$. This form makes it more plausible that some of the P can become negative.

Reading off these P's from Table II and performing the sum yields

$$\frac{1}{4}(1 + \mathbf{a_1} \cdot \mathbf{b_1}), \tag{3.12}$$

which is exactly Eq. (3.4). For the antisymmetric wave function one obtains $1/4(1 - \mathbf{a_1} \cdot \mathbf{b_1})$. Similar expressions are obtained for the other three pairwise probabilities. Clearly, subtracting the antisymmetric expressions from the symmetric ones and adding the four terms leads back to Eq. (3.6) for $\mathcal{C}_{\mathrm{qm}}$. This procedure must work because the four-probabilities are symmetric in all the arguments; summing over any of them produces an equal number of terms of opposite sign, which cancel out, leaving the usual quantum pairwise probabilities.

4. DISCUSSION AND CONCLUSIONS

We have shown that, like the Bell inequalities, the CHSH inequality assumes positive-definite probabilities and that quantum mechanics breaks both inequalities effectively because it introduces negative weights to the measurements. These negative four-probabilities enter the derivation in exactly the same way as the classical three-probabilities entered the derivation of the Bell's inequalities. If they are unphysical, it is not necessarily because they are negative, but because it is impossible to make four simultaneous spin measurements on two particles. By the same token, it is impossible to make three simultaneous spin measurements on two particles. In any case, neither the classical three-probabilities found in Bell's theorem, nor the four-probabilities that figure here are actually measured. Both merely serve as "master distributions" from which to derive the usual pairwise probabilities, classical and quantum, which are both positive-definite. To reiterate our earlier remarks, from this point of view it is not surprising that the Bell and CHSH inequalities are violated by experimental tests; they merely used the wrong set of probabilities for a quantum-mechanical problem.

Although one might choose to reject negative probabilities as unphysical, one should not reject the notion of master probability distributions in favor of correlations between real and imaginary experiments because the two procedures are identical! Recall again that Eberhard's quantity C_{11} was $C_{11} = \frac{1}{N}\sum_{j=1}^{N}\alpha_{1j}\beta_{1j}$, which represented the fraction of events $\alpha_1\beta_1$ that had the same sign minus the fraction that had opposite sign. Thus by definition we can write

$$C_{11} = P(\mathbf{a_1}+, \mathbf{b_1}+) + P(\mathbf{a_1}-, \mathbf{b_1}-) - [P(\mathbf{a_1}+, \mathbf{b_1}-) + P(\mathbf{a_1}+, \mathbf{b_1}-)]. \quad (4.1)$$

Now, in exact analogy with the procedure of Section 3 we imagine that these pairwise probabilities can be derived from a master distribution involving all four axes $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{b_2}$, $\mathbf{b_3}$. In that case, as in Eq. (3.11), $P(++) = P(\mathbf{a_1}+, \mathbf{b_1}+) = P(++++) + P(+++-) + P(+-+-) + P(++-+)$, with analogous expressions for $P(--)$, $P(+-)$, and $P(-+)$. There are thus 16 terms that contribute to C_{11}, similarly for C_{12}, C_{21}, and C_{22}. Writing out all 64 terms yields for $\mathcal{C} = \langle\gamma\rangle$:

$$\begin{aligned}\mathcal{C} = 2\{&P(++++) + P(----) + P(+++-) + P(---+)\\ &+ P(+-++) + P(-+--) + P(+--+) + P(-++-)\\ &- P(++-+) - P(--+-) - P(-+++) - P(+---)\\ &- P(++--) - P(--++) - P(+-+-) - P(-+-+)\}. \end{aligned} \quad (4.2)$$

These P's are general and may be taken to be either classical or quantum. Notice half enter with positive sign and half with negative. If all the probabilities are equal, then $\mathcal{C} = 0$. If those that enter with negative sign are zero, then $\mathcal{C} = 2$, and if those that enter with positive sign are zero, then $\mathcal{C} = -2$. All this is in complete agreement with the analysis of Section 2. Clearly, if the P's are positive-definite then $\mathcal{C} \leq 2$, but if the probabilities are allowed to become negative then this is violated. If the P's are assumed to be quantum, they take on the values given by Table II. In this case, inserting those values into (4.2) gives exactly (3.6), as before.

This demonstration shows clearly that the γ's can be derived from a master probability distribution that involves simultaneous spin measurements along four axes. The *only* difference between the classical and quantum cases is that in the former we assume the probabilities are positive-definite. The master distributions themselves cannot be regarded as any more or any less meaningful than the space of hypothetical measurements, because the procedures are exactly equivalent. Indeed, we see that there is no difference between the Eberhard procedure and the usual derivation of Bell's inequalities.

There remains the problem of interpretation. Most people insist that probability be defined in terms of relative frequency of events, in which case it must be positive-definite. In quantum mechanics, however, although one can define the expectation value in terms of the square of the wave amplitude, which corresponds to a relative-frequency interpretation, an alternate procedure is available. The expectation value may also be taken as a function of the dynamical variables under consideration, e.g., position and momentum. Classically, one might consider a Maxwellian distribution of particles in phase space; integrating over position or momentum would give the marginal probability distribution for the conjugate variable. But in quantum mechanics, the uncertainty principle precludes precise simultaneous knowledge of noncommuting variables. If one attempts to associate a function with a distribution over noncommuting variables, such that an integration over one of them gives the correct marginal distribution for the other, then one finds that the distribution function must in places become negative. This is the well-known Wigner distribution (Wigner, 1932).

In the case of spin, the different components of angular momentum do not commute; hence no ordinary (positive-definite) probability distribution can be defined over the various components simultaneously. Any distribution will share with the Wigner distribution the property that it will become negative in some region of "phase space." For example, in the spin-1/2 systems we have been considering, the probability of finding S_z in the $+$ state, and S_x in the $+$ state is given by taking the trace of the product of the projection operators, as we have done earlier. Now, given a state with $S_x = +$, the probability is 1/2 for finding $S_z = +$, and 1/2 for $S_z = -$. Suppose, however, that many measurements show $S_z = +$, always, but that $S_x = +$ appears with probability λ and $S_x = -$ appears with probability $1 - \lambda (0 \leq \lambda \leq 1)$. The probability for finding $S_z = -$ must then be $(1/2)\lambda + (1/2)(1 - \lambda) = 1/2$. On the one hand, the probability of $S_z = -$ must equal zero. On the other hand, no mixture of $S_x = +$ and $S_x = -$ can give a zero probability for $S_z = -$.

This is quite a general property of noncommuting variables and has little to do with quantum mechanics. In such situations the best that one can ask for is that the probability distribution give the correct marginal distribution for one of the variables, in our case one component of angular momentum. This is what has been found in the present paper. The probability distribution for simultaneous measurements along three or more axes are not positive-definite, but the marginal distributions that give correlations between two spin components are, and are in accord with the standard predictions of quantum mechanics.

The main point of this paper has been that assumptions beyond locality do enter into derivations of Bell's inequalities. It is worth mentioning yet another tacit assumption: that space is flat. The notion of parallel and antiparallel spins is only well defined for flat space where the measurement axes (the "z" axes) can be taken to be fixed everywhere relative to one another. In curved space there is no universal definition of parallel and one can only compare spins in distant locations by parallel transport of the measurement axes (von Boreszkowski and Mensky, 2000). In the case of nonnegligible gravitational fields, then, the "nonlocal" EPR correlation between two particles, to the extent that they can be said to exist at all, must be the result of parallel transport, a local phenomenon.

Returning to probabilities, we find ourselves in a strange situation. If one insists that probabilities remain positive-definite, we are forced to use vague and imprecise concepts, such as "local" or "nonlocal" to describe the outcome of the EPR experiment. On the other hand, we are able to formulate the precise mathematical conditions necessary for the violation of the Bell and CHSH inequalities, although at the cost of introducing negative probabilities. Most investigators would say that a unified, physical interpretation of negative probabilities is, in fact, exactly what is currently lacking. To be sure, Feynman conceded (see Mückenheim *et al.*, 1986, and Feynman 1948; also Sudarshan, 1963; Mehta and Sudarshan, 1965) that all the results of quantum mechanics can be analyzed in terms of negative probabilities but he remained skeptical about the utility of such an approach and that a useful meaning could be attached to it. Nevertheless, many of the interpretational problems associated with negative probabilities stem from an insistence on viewing them within the framework of relative frequencies. This is clearly "no go." We have shown that a more natural framework for their interpretation arises when one considers the expectation value as a measure of probability over noncommuting variables. One can even go further than we have and consider complex probability measures (Srinivasan and Sudarshan, 1994, 1996), which also involve expectation values. Under such circumstances it is well to bear in mind that imaginary numbers are more similar to rotations than to real numbers. One should also bear in mind the very word "imaginary," an obsolete relic of their original status.

Note added: Since this paper was initially posted, José Cereceda has come to essentially the same conclusions (see quant-ph/0010091).

APPENDIX

Many researchers appear unwilling to accept that *any* assumptions beyond locality are employed in the derivations of Bell's inequalities. We now list a few of the proofs we have found and point out explicitly where the assumption of positive probabilities enters.

Bell, 1964. In Bell's original proof, he defines two quantities $A(\vec{a}, \lambda) = \pm 1$, $B(\vec{b}, \lambda) = \pm 1$. He defines a normalized probability distribution $\rho(\lambda)$, such that $\int d\lambda\, \rho(\lambda) = 1$. The expectation value of the spin components $\vec{\sigma}_1 \cdot \vec{a}$ and $\vec{\sigma}_2 \cdot \vec{b}$ is

$$P(\vec{a}, \vec{b}) = \int d\lambda\, \rho(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda), \tag{A.1}$$

which he shows can be written (his Eq. (14)) as

$$P(\vec{a}, \vec{b}) = -\int d\lambda\, \rho(\lambda) A(\vec{a}, \lambda) A(\vec{b}, \lambda). \tag{A.2}$$

When another vector $\vec{c}$ is involved, one has

$$P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c}) = -\int d\lambda\, \rho(\lambda)[A(\vec{a}, \lambda)A(\vec{b}, \lambda) - A(\vec{a}, \lambda)A(\vec{c}, \lambda)]. \tag{A.3}$$

Bearing in mind that $A(\vec{b}, \lambda) = 1/A(\vec{b}, \lambda)$ one can rewrite this as

$$P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c}) = \int d\lambda\, \rho(\lambda) A(\vec{a}, \lambda) A(\vec{b}, \lambda)[A(\vec{b}, \lambda)A(\vec{c}, \lambda) - 1]. \tag{A.4}$$

Bell then asserts

$$|P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c})| \leq \int d\lambda\, \rho(\lambda)[A(\vec{b}, \lambda)A(\vec{c}, \lambda) - 1], \tag{A.5}$$

where, of course, $|A(\vec{a}, \lambda)A(\vec{b}, \lambda)| = 1$. However, stricly speaking the triangle inequality gives

$$|P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c})| \leq \int d\lambda\, |\rho(\lambda)|[A(\vec{b}, \lambda)A(\vec{c}, \lambda) - 1], \tag{A.6}$$

which is equal to (A.5) only when $|\rho| = \rho$, i.e., when $\rho \geq 0$.

Clauser et al., 1969. The CHSH paper makes the same assumption at the identical point in their derivation, in their first (unnumbered) equation.

Peres, 1978. Peres' derivation is almost identical to Eberhard's and makes the same assumption of positive weights in the same step, i.e. between Steps 1 and 2 of Eq. (2.7) of this paper.

Stapp, 1971. Stapp's proof is very similar to Bell's. He arrives at an expression (below his Eq. (8))

$$\sqrt{2} \leq \frac{1}{N} \sum_j |n''_{2j} n'_{2j} - 1|, \tag{A.7}$$

where $n''_{2j} = \pm 1$ and $n'_{2j} = \pm 1$. He then shows this leads to the contradiction $\sqrt{2} \leq 1$. However, if the n's are ± 1, then the summand can only have values 0, 2. If N_1 and N_2 are the frequencies with which these two values occur, and $N_1 + N_2 = N$, then the right hand side can be written as

$$\frac{1}{N}[N_1 \times 0 + N_2 \times 2] = \frac{2N_2}{N} = \frac{2(N - N_1)}{N} = 2\left(1 - \frac{N_1}{N}\right). \tag{A.8}$$

As in the Eberhard argument, a contradiction can always be avoided by taking N_1 negative.

Stapp, 1985. Stapp establishes a contradiction by demonstrating (his Eq. 8) that

$$\frac{1}{n}\sum_{i=1}^{n}[\sqrt{2}\, r_{Ai}(\hat{\lambda}_a) + r_{Bi}(\hat{\lambda}_a) + r_{Bi}(\hat{\lambda}_b)]^2 > (\sqrt{2} - 2)^2, \tag{A.9}$$

where $r_{Ai}(\hat{\lambda}_a) = \pm 1$, $r_{Bi}(\hat{\lambda}_a) = \pm 1$, and $r_{Bi}(\hat{\lambda}_b) = \pm 1$. However, since the r's are ± 1, the summand can have only one of three values: $(\sqrt{2})^2$, $(2 + \sqrt{2})^2$, and $(2 - \sqrt{2})^2$. Then the above expression can be written as

$$\frac{1}{n}[n_1(\sqrt{2})^2 + n_2(\sqrt{2} + 2)^2 + n_3(2 - \sqrt{2})^2], \tag{A.10}$$

where n_1, n_2, n_3 are the frequencies with which the three terms occur and $n_1 + n_2 + n_3 = n$. Squaring out and combining terms yields

$$\frac{2(n_1 + n_2 + n_3)}{n} + \frac{2n_2(2 + \sqrt{2})}{n} + \frac{2n_3(2 - \sqrt{2})}{n}. \tag{A.11}$$

Assuming n and n_3 positive, this expression can become negative if

$$n_2 < \frac{-(n + n_3(2 - \sqrt{2}))}{2 + \sqrt{2}}, \tag{A.12}$$

in other words, if n_2 is sufficiently negative.

Bell, 1971. A proof that has been cited as qualitatively different than the others is Bell's 1971 proof. This proof is basically the same as the CHSH proof. In Bell's version the probability density is also explicitly taken to be positive-definite. The only difference is that now $|A(\vec{a}, \lambda)| \leq 1$ and $|B(\vec{b}, \lambda)| \leq 1$. (In our notation this corresponds to $|\alpha_i| \leq 1$ and $|\beta_i| \leq 1$.) This change merely strengthens the upper bound on the classical correlations. That is, in our Eq. (2.5), whereas previously $|\gamma| = 2$, now $|\gamma| \leq 2$. The rest of the derivation is consequently unaffected and the CHSH inequality continues to hold. Furthermore, our demonstration of the equivalence of the Eberhard procedure with the "master probability distribution" procedure is also unaffected, since Eq. (4.2) made no assumption about the values of the P.

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