Foundations of Physics, Vol. 33, No. 5, May 2003 (© 2003)

Higher Spin Fields and Non-Holonomic Constraints

E. C. G. Sudarshan[1]

Received December 5, 2002

I review theories and problems of inconsistencies in the description of higher spin wave equations.

1. INTRODUCTION

A relativistic particle has as dynamical variables the the energy-momentum, intrinsics spin, and the mass. Particles belong to irreducible representations of the Poincaré group (the inhomogeneous Lorentz group). A complete characterization and the behavior under Poincaré group is known for more than half a century. In the case of particles with non-zero mass one can also define a position variable.

Despite this we like to have a covariant description of these particles to incorporate other physical requirements. The interactions of a relativistic particle are best described in terms of a covariant amplitude. For example, the finite mass, spin-1/2 particles can be described by the two component Foldy–Wouthuysen–Tani representation, which gives a complete and consistent treatment of the coordinates and spin of a free particle.

But when interaction with an electromagnetic field are introduced, the beautiful results of the Dirac equation are not obtained by the non-covariant description: instead, one has to introduce a highly non-linear interaction. Furthermore, when we consider the quantum field theory of finite mass spin-1/2 particles, there are two physical requirements. The first one

[1] Center for Particle Physics, University of Texas, Austin, Texas; e-mail: sudarshan@physics.utexas.edu

0015-9018/03/0500-0707/0

concerns Kirchhoff's principle stating that emission and absorption have to be described by *same* interaction. This behavior is manifest as "crossing symmetry" of scattering amplitudes in quantum field theory. The second principle prescribes the connection between spin and statistics to be either Fermi–Einstein or Bose–Einstein statistics.

The above requirements are only met if one uses covariant wave equations for the description of particles. There is a double covariance: in addition to the wave functions of free particles as representations of the Poincaré group, the wave amplitudes are also described by means of covariant wave equations based upon space-time dependent finite dimensional (non-unitary) representations of the (homogeneous) Lorentz group. We, therefore, use co-variant wave equations; interactions including dual covariant amplitudes are described by local couplings.

In general, the above wave equations need more than one covariant amplitude. For example, the (linearized) wave equation for massive spinless particles involves a five-component field consisting of one scalar field and a covariant vector field. In order to reproduce the correct mass and spin, the wave equation must contain supplementary conditions which are not equations of "motion." The spin-0 Duffin–Kemmer field is described by the following set of variational equations:

$$i\partial^{\mu}\phi = m\phi_{\mu}, \qquad i\partial^{\mu}\phi_{\mu} = \phi. \tag{1}$$

Among these, only two are equations of motion:

$$i\,\frac{\partial}{\partial t}\,\phi = m\phi_0 + \nabla\cdot\boldsymbol{\phi}, \qquad i\,\frac{\partial}{\partial t}\,\phi_0 = -m\phi_0 + \nabla\cdot\boldsymbol{\phi}. \tag{2}$$

The other equations,

$$\boldsymbol{\phi} = \nabla\phi, \tag{3}$$

are not equations of motion; they do not contain time derivatives. Such constraints reduce the truly dynamical variables (and equations of motion) to be 2 $(2s+1)$. The constrains could be of two types. The holonomic constraints can be directly substituted into the true equation. But the non-holonomic constraints can not be incorporated easily. They tend to modify the true equations of motion and reduce the number of kinematic amplitudes. In quantum field theory this would mean a modification of the commutation (anti-commutation) relations and make kinematics depending on dynamics.

A simple example of this circumstance is obtained from the equation of motion of a particle in three dimensions coupled to a scalar source. The action density chosen is

$$\frac{m}{2}\,\dot{\mathbf{x}}^2+g(x)\,\mathbf{x}^2, \tag{4}$$

which can be written in the first order form

$$\mathbf{p}\cdot\dot{\mathbf{x}}-\frac{m}{2}\,\dot{\mathbf{x}}=g(x)\,\mathbf{x}^2. \tag{5}$$

The variational equations are

$$\dot{\mathbf{x}}=\mathbf{p}, \qquad \dot{\mathbf{p}}=\nabla(g(x)\,\mathbf{x}^2), \qquad \dot{g}(x)=0. \tag{6}$$

When $g(x)\equiv 0$ this is a free particle with a continuous spectrum; but otherwise it becomes a rotator in three dimensions with a characteristic discrete spectrum. The dynamics (interactions) however modified the kinematics (true degrees of freedom).

Theory of constrained dynamical systems for holonomic constraints was solved by Lagrange who introduced generalized coordinates and momenta equal in number to the degrees of freedom. But non-holonomic constraints had to wait until Dirac[(1)] introduced the modern theory of constraints. His work has been followed up by many people, who, partly working outside the Action Principle framework, tried to describe relativistic interaction and simultaneously circumvent the non-interaction theorem. Higher spin fields equations provide a natural evolution of non-holonomic constraints, but the reduction to the true equations of motion under usage of non-holonomic constraints amounts to unexpected difficulties.

2. RELATIVISTIC WAVE EQUATIONS

The light quantum was the first relativistic particle that we encountered more than a century ago. But it was first thought of in terms of the electromagnetic field. Introducing light quanta was required for the explanation of both the photoelectric and Compton effect. Yet, it was work by S. Bose on photon statistics that established the particle nature of a Bose gas. Subsequently, Heisenberg and Pauli quantized the Maxwell field to produce the field theory of electromagnetism. The Maxwell field and the associated field equations may be viewed as the first relativistic field.[(2)]

But the first explicit and new equation was the one discovered by Dirac. Dirac described a massive particle of spin-1/2 in terms of a four component first order equation. Soon it was realized that the second order relativistic wave equation first introduced by Schrödinger (and refereed to as the Klein–Gordon equation) was the appropriate relativistic equation for spinless massive particles). This second order equation can be linearized to get a five component first order wave equation for, again, spinless particles. This Duffin–Kemmer linearized form had a similar ten component analogues to describe massive spin-1 particles.[3]

Having obtained the above relativistic wave equations, there were extensive work by many authors, among them Belinfante, Bhabha, Corson, Harish-Chandra, Kemmer, Majorana, Fierz, and Pauli, toward finding generic wave equations. These equations do contain too many components and must involve supplementary constraints, since relativistic wave functions are finite dimensional representations of the Lorentz group and have many "spin" values for rotations in three space. As a consequence the wave equations for higher spins must have constraints eliminating the unwanted components.

The simplest non-trivial case is that for spin-3/2. The two Lorentz group representations of lowest dimensionalities which can be used for embedding spin-3/2, are of twelve, and eight components, respectively.[4] But to get an acceptable equation we need the 12-component representation linearly with a four component spinor. The easiest way to do this is to take the vector spinor ψ_μ with 16 components and write an equation in such a fashion that only eight components remain. The latter can be fixed by means of eight constraints, four of them holonomic, and the other four non-holonomic. There is a mass parameter with dynamic significance and a gauge parameter that can be chosen at will.

Then we have the possibility of having more than one spin or mass in the same wave equation. In the simplest cases, like, say, a spin-3/2 amplitude as a part of the 12 component representation of the Lorentz group, the spin-3/2 and spin-1/2 wave functions have opposite signs of the scalar products and hence of the corresponding probabilities. We can construct more sophisticated equations with a mass and spin spectrum. Simplest of these equations were found by Bhabha who observed that for spin-0, spin-1/2, and spin-1 wave equations,

$$\beta^\mu i\,\partial_\mu\psi - m\psi = 0, \tag{7}$$

the four-vector matrices β satisfy the relations

$$[\beta^\mu, \beta^\nu] = i \text{ const } S^{\mu\nu}, \tag{8}$$

where $S^{\mu\nu}$ are the six Lorentz group generators. Bhabha[2] proposed studying the class of higher spin equations which would satisfy this relativistic relation. He could find the generic solution since if we write

$$\beta^{\mu} = \text{const}\, S^{\mu 5}, \tag{9}$$

then β^{μ}, $S^{\mu\nu}$ together constitute the generators of the de Sitter group. Since the irreducible representations of the de Sitter group are known we can write down the mass spectrum of those wave functions which obey Bhabha equations. Bhabha also found however that the mass spectrum is inverted,

$$M(s) \sim (2s+1)^{-1}. \tag{10}$$

It has been found that the inverted spectrum of the Bhabha equations is a generic property of all wave equations (see the concluding section for the multi-mass Bhabha equation with an arbitrary mass ratio between spin-3/2 and spin-1/2 solutions).

3. THE GENERIC WAVE EQUATION

If Ψ is the relativistic wave function it obeys the wave equation[4]

$$(i\Gamma^{\mu}\partial_{\mu} - M)\,\Psi = 0, \tag{11}$$

where Γ^{μ} are vector matrices and M is a scalar matrix. The commutators of the Γ's with the Lorentz group generators are

$$[S^{\mu\nu}, \Gamma^{\lambda}] = i(\Gamma^{\mu} g^{\nu\lambda} - \Gamma^{\nu} g^{\mu\lambda}), \tag{12}$$

$$[S^{\mu\nu}, M] = 0. \tag{13}$$

Without any loss of generality, we could take Γ^{μ} and Ψ to be purely real. This is always possible by virtue of the Majorana matrices for the Dirac equation in which case one encounters all real Γ's defined according to

$$\Gamma^{\mu} = i\beta\gamma^{\mu}, \qquad M = m\gamma^{0}, \qquad (\gamma^{0})^{*} = -\gamma^{0}. \tag{14}$$

Since all representations of the Lorentz group can be built from the spinor $(1/2, 0) \oplus (0, 1/2)$, this remains true for *all representations* of the (extended) Lorentz group. In case we need to introduce an electric charge, we can double the components and use the antisymmetric matrix

$$Q = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{15}$$

For the Dirac equation the matrix M is antisymmetric and γ^μ are symmetric, ∂_μ is antisymmetric. The bilinear form

$$\bar{\Psi}\Gamma^0\Psi \tag{16}$$

is antisymmetric.

For a multispinor which transforms as the direct product of an even or odd number of spinors the symmetry property of Γ^0 will alternate—it will be anti-symmetric for integer spin (even rank multi-spinor) and symmetric for half-integer spin; similarly, M would be symmetric for integer spin and anti-symmetric for half-integer. For the wave equation in the form

$$\left\{\Gamma^0 \frac{\partial}{\partial t} + \mathbf{\Gamma}\cdot\nabla + M\right\}\Psi = 0, \tag{17}$$

the action principle[(4)] demands

$$[\Psi^T\Gamma^0\delta\Psi, \Psi] = \delta\Psi. \tag{18}$$

For Γ^0 non-singular we could derive the anti-commutation (commutation) rules

$$\{\Psi_j, \Psi_k\} = (\Gamma^0)^{-1}_{jk}. \tag{19}$$

4. CONSTRAINTS FOR HIGHER-SPIN EQUATIONS

Lorentz invariance of the wave equations demands[(4)] that

$$\Gamma^0 = (S^T)\,\Gamma^0 + 2S^T\Gamma^0 S + \Gamma^0 S^2, \tag{20}$$

where S is a generator of pure Lorentz ("boost") transformations. If λ is the lowest eigenvalue of Γ^0, this implies, in turn

$$2P(\lambda)\,S(1-P(\lambda))(\Gamma^0-\lambda)(1-P(\lambda))\,SP(\lambda) = \lambda P(\lambda)(1-2S^2)\,P(\lambda), \tag{21}$$

where $P(\lambda)$ projects to the lowest eigenvalue λ of Γ^0. Since the left hand side of this equation is non-negative for all fields (except pure Dirac fields), the matrix Γ^0 must be indefinite. With respect to commutation relations between the field components, the matrix Γ^0 is antisymmetric—and hence its indefiniteness is automatic. But anti-commutators are non-negative; hence the negative eigenvalue components of Ψ must be eliminated by supplementary conditions. So the correct equations for higher spin anti-commuting fields must have second class currents.

In passing we note that since $\Psi^T \Gamma^0 \Psi$ is an invariant bilinear in Ψ, it must be symmetric for half-integer spin fields and anti-symmetric for integer spin-fields. This already follows from rotational invariance (which may or may not be Lorentz invariant!). So the action principle demands that integer spin field must obey commutation relations (Fermi's fields). This basic result that integer spin-fields must be Bose fields and half integer spin fields must be Fermi fields was published more than twenty five years ago. It is the fundamental theorem on the connection between spin and statistics. Here it is derived from *rotational invariance only, unlike* the celebrated Pauli spin-statistic theorem. One of the corollaries is that *electrons* in condensed matter must be *fermions*, and *phonons* in lattice must be *bosons*.

5. THE SPIN-3/2 EQUATION: RARITA–SCHWINGER FIELDS

For the particular case of spin-3/2 there are two types of fields which may be considered, namely $D(1, 1/2) \oplus D(1/2, 1)$ with 12 components, and $D(3/2, 0) \oplus D(0, 3/2)$ with 9 components. Consistency demands that these by themselves are not sufficient. We have the following conditions:

1. The 12 component $D(1, 1/2) \oplus D(1/2, 1)$ gives an indefinite Γ^0.
2. Coupling the 8 component $D(3/2, 0) \oplus D(0, 3/2)$ has also Γ^0 indefinite.
3. Coupling the 4 component $D(1/2, 0) \oplus D(0, 1/2)$ to the 12 component $D(1, 1/2) \oplus D(1/2, 1)$ can make the system consistent by generating two necessary (second class) constraints.
4. The 16-component amplitude may be written following Rarita and Schwinger in terms of a vector spinor, ψ_μ, which transforms as $D(1/2, 1/2) \otimes D(1/2, 0) \oplus D(0, 1/2)$, and a carefully crafted mass matrix M.
5. There is a one parameter transformation of the form

$$\psi^\lambda \to \psi^\lambda - c\gamma^\lambda(\gamma_\nu \psi^\nu), \tag{22}$$

all of which give equivalent Rarita–Schwinger equations.[5]

Action density in terms of ψ_λ has the form

$$\frac{i}{4}(\psi^\lambda)^T \left\{ \beta\gamma^\mu g_{\lambda\sigma} + W(\delta^\mu_\lambda \gamma_\sigma + \delta^\mu_\sigma \gamma_\lambda) - \frac{1}{2}(3W^2 + 2W + 1)\,\gamma_\lambda \gamma^\mu \gamma_\sigma \right\} \psi^\sigma. \tag{23}$$

6. KINEMATICS DEPENDS ON THE DYNAMICS

To discuss minimal electromagnetic interaction by the replacement $i\partial_\mu \rightarrow (i\partial_\mu - eQA_\mu) = \pi_\mu$, where A_μ is the vector potential of an external field and Q is the antisymmetric imaginary matrix of electric charge. We have fixed the arbitrary parameters. The equations of motion now produce a second class constraint

$$(-\tfrac{2}{3}\boldsymbol{\gamma}\cdot\boldsymbol{\Pi} + m)\,\boldsymbol{\gamma}\cdot\boldsymbol{\Pi} = \boldsymbol{\Pi}\cdot\boldsymbol{\psi} + \tfrac{1}{3}\boldsymbol{\gamma}\cdot\boldsymbol{\Pi}\boldsymbol{\gamma}\cdot\boldsymbol{\psi}. \tag{24}$$

This leads to the modified anti-commutator relations

$$\{\psi_k^{3/2}(x), \psi_l^{3/2}(y)\} = (\delta_{kn} + \tfrac{1}{3}\gamma_k\gamma_n)(\delta_{nk} + \tfrac{2}{3}\pi_n \Delta \pi_l)\,\delta(x-y), \tag{25}$$

with

$$\Delta = (m^2 - \tfrac{2}{3}eQ\boldsymbol{\sigma}\cdot\mathbf{B}), \qquad \mathbf{B} = \nabla\times\mathbf{A}. \tag{26}$$

This anti-commutator is *local*. But here the problem is when

$$|eB| \gg \tfrac{3}{2}M^2, \tag{27}$$

the anti-commutator is still local but indefinite.

This makes the theory *inconsistent*.[3] Further, since **B** can be made as large as we please by going to sizable Lorentz frame, the theory also violates relativistic invariance. Similar but more complicated constraints arise for theories of spin-5/2, etc.

Let us recapitulate what we have discovered. When we take a spin-3/2 wave equation constructed to be consistent for the free field and introduce in it *minimal* electromagnetic interaction with an *external* field, we find that the theory is no longer consistent.

Hagen[6] has studied the coupling of a *scalar* external field and showed that it develops inconsistency too. One might inquire if we deal not with one field but many fields strung together as a super-field could the electromagnetic interactions be consistent? The answer is *not* known.

It must be emphasized that the framework was of a spin-3/2 field by itself and in minimal electromagnetic interaction. Several modern theories treat collections of particles with different masses and spins; and, in particular, super-fields. It is concisely that within some such context, higher spin fields in interaction can be consistently treated.

The demonstrated inconsistency is not obtained if the higher spin particles are bound states of more elementary units; for example, we have many complex nuclei with higher spin which interact with electromagnetic field. In these cases the structure of the action is different but the one we have studied.

7. PHENOMENOLOGY WITH THE RARITA–SCHWINGER FIELD

This raises the question of possible use of higher spin fields to incorporate hadronic resonances. The Poincaré transformations may be defined by a Rarita–Schwinger field or other representations of the Lorentz group. It would be worthwhile searching for such systematizations.

Classification schemes according multiplets of various groups have been widely employed in nuclear as well as in particle physics, the most familiar of them being Wigner super-multiplets of isospin–spin $SU(2) \times SU(2)$ scheme in nuclear physics.[7] The extended supersymmetry based upon a graded Lie algebra was used by Iachello and Arima to make super-supermultiplets of nuclei.[8]

Even more striking is the use of the spin–isospin symmetry $SU(6)$ to classify the ground states of hadrons.[9] It puts pseudo-scalar and vector mesons in the 35-dimensional adjoint representation while the nucleon and the spin-$3/2$ ground state baryons are part of the 56-dimensional representation. Although this scheme led to some useful sum-rules, after all, pseudoscalar mesons had to be coupled to baryons (whether of same spin or different) only through derivative (not Yukawa) couplings in order to respect chiral symmetry. Hence, strictly speaking, even at the kinematic level of Yukawa coupling, the particle classification needed to include the orbital angular momentum. This could be done, for example, in assigning particles to $SU(6) \times SO(3)$ group multiplets,[10] an option we made use of here.

8. CONCLUDING REMARKS

In summary, we recognize that there are second class constraints, where the choice of canonical variables depends on the dynamics. In the spin-$3/2$ Rarita–Schwinger field we find that the true spin-$3/2$ dynamical fields obeyed anti-commutation rules which depended exclusively on the external field to which it is coupled, which is part of the dynamics. We have outlined a number of attempts to introduce higher spin fields with multiple masses into a quantum field theory. The Bhabha equations which formed a de Sitter ($SO(4, 1)$), multiplet have an inverted mass spectrum. The Rarita–Schwinger spin-$3/2$ field equations are consistent for free fields, but develop inconsistencies when an electromagnetic or scalar field is coupled to it. Bhabha had developed an extended Rarita–Schwinger field[11] together with an additional Dirac field which described a spin-3/2 particle and a spin-1/2 particle with an arbitrary mass ratio. Unfortunately, this system also develops inconsistencies when external electromagnetic interactions are included. Although in the $SU(6) \times O(3)$ case there was the inconsistency in writing down Yukawa couplings, the scheme was nonetheless useful in obtaining mass and coupling sum rules. Back to the entering question of Sec. 6, the inconsistencies of constraint Rarita–Schwinger fields do not exclude the usefulness of unconstrained Rarita–Schwinger, or more generally, higher-spin fields in classifying hadron spectra, as done in Ref. 12. This would widen the spirit of the non-relativistic $SU(4)$ nuclear-, or the $SU(6)$ hadron super-multiplet classification toward relativistic classification schemes.

There is an entirely different kind of difficulty with higher spin fields which is manifest already at the wave equation level, even before quantization. It applies equally to integer spin and half integer spin wave equations. The constraints in the higher spin field equations has the dependent fields with external sources coupled to the field. Solving for this introduces nonlocal functions of the sources. This may be interpreted as amplitudes that propagate outside the light cone. Such a behavior of the solutions can be interpreted as "non-causal" (faster than light) propagating modes. This result was first obtained by Joseph Weinberg(13) in his doctoral thesis of the University of California, Berkley under the supervision of Robert Oppenheimer. This was not published. It was rediscovered by Velo and Zwanziger(14) who one credited with showing that higher spin fields are inconsistent. This conclusion holds equally for integer and half-integer higher spin fields. This problem has nothing to do with kinematics depending on dynamics. Yet, the "axiomatic brotherhood" would like to propagate the view that the Velo–Zwanziger result superseded the field inconsistency we have demonstrated and of course they have forgotten J. Weinberg's work *several decades ago*.

REFERENCES

1. P. A. M. Dirac, "Generalized Hamiltonian dynamics," *Can. J. Math.* **2**, 129–148 (1950).
2. For an introduction, see: I. M. Duck and E. C. G. Sudarshan, *One Hundred Years of Planck's Quantum* (World Scientific, Singapore, 2000).
3. For background on relativistic wave equations, see: E. M. Corson, *Tensors, Spinors, and Relativistic Wave Equations* (Blackville, London, 1953).
4. K. A. Johnson and E. C. G. Sudarshan, "Inconsistency of the local field theory of charged spin-3/2 particles," *Ann. Phys.* **13**, 126–145 (1961).
5. W. Rarita and J. Schwinger, "On a theory of particles with half-integer spin," *Phys. Rev.* **60**, 61 (1941).

6. C. R. Hagen, "New inconsistencies in the quantization of spin-3/3 fields," *Phys. Rev. D* **4**, 2204–2206 (1971).
7. E. P. Wigner, "On the consequences of the symmetry of the nuclear Hamiltonian on the spectroscopy of nuclei," *Phys. Rev.* **51**, 106–119 (1937).
8. A. Arima and F. Iachello, "Extension of the integrating boson model to odd-A nuclei," *Phys. Rev. C* **14**, 761–763 (1976).
9. F. Gursey and L. A. Radicati, "Spin and unitary spin independence of strong interaction," *Phys. Rev. Lett.* **13**, 173–175 (1964).
10. E. C. G. Sudarshan and K. T. Mahanttappa, "$SO(4)\times O(3)$ structure of strong interacting particles," *Phys. Rev. Lett.* **14**, 163–166 (1965).
11. H. J. Bhabha, "Equations with two distinct masses," *Phil. Mag.* **43**, 33 (1952).
12. M. Kirchbach, M. Moshinsky, and Yu. F. Smirnov, "Baryons in O(4) and vibron model," *Phys. Rev. D* **64**, 114005 (2001).
13. J. Weinberg, Ph.D. thesis, University of California Berkley, 1943. J. Weinberg and S. Kusaka (unpublished).
14. G. Velo and D. Zwanziger, "Propagation and quantization of Rarita–Schwinger waves in an external electromagnetic potential," *Phys. Rev.* **186**, 1337–1341 (1969).