



ELSEVIER

Available online at [www.sciencedirect.com](http://www.sciencedirect.com)

SCIENCE @ DIRECT®

Physics Letters A 317 (2003) 359–364

PHYSICS LETTERS A

[www.elsevier.com/locate/pla](http://www.elsevier.com/locate/pla)

# Time as a dynamical variable

G. Bhamathi, E.C.G. Sudarshan\*

*Center for Particle Theory, Department of Physics, University of Texas, Austin, TX 78712-1081, USA*

Received 16 June 2003; received in revised form 30 August 2003; accepted 7 September 2003

Communicated by P.R. Holland

## Abstract

Clocks are dynamical systems, notwithstanding our usual formalism in which time is treated as an external parameter. A dynamical variable which may be identified with time is explicitly constructed for a variety of simple dynamical systems. This variable is canonically conjugate to the Hamiltonian. The complications brought about by the semiboundedness of the Hamiltonian are taken into account and the spectral properties of the time operator are examined. For relativistic systems boosts affect the time variable in a familiar manner. It is pointed out that all physical clock times are cyclic, contrary to Newton's view of a 'uniformly flowing time'.

© 2003 Elsevier B.V. All rights reserved.

PACS: 03.65.Ta; 03.30.+p

Keywords: Time operator

## 1. Introduction

In the context of particle dynamics, the kinematics involves the specification of positions and momenta, but neither of time nor of the Hamiltonian. The Hamiltonian, which itself is a functional of the positions and momenta, enters at the level of dynamics, and time enters as a labeling parameter. While we have positions and momenta as conjugate pairs, there is no conjugate for the Hamiltonian, nor is there any for time, since time is only an external parameter. Despite this, there is a formal correspondence of the Hamiltonian  $H$  with the time, say, for example, in the Weiss–Schwinger

generalized variation of the action [1].

$$\delta A = \sum_n p_n \delta q_n - H \delta t. \quad (1)$$

Can this be made more than a formal correspondence?

In this context we must recall that all time-measuring instruments, clocks, are also physical systems and the pointer reading indicating time is a dynamical variable of *that* system. Almost all our clocks are *cyclic*; and the specification of time in the large involves a counting of the number of cycles plus the pointer reading. However, elaborate a clock is, it is always periodic, though there is no limit to the time interval for a cycle. Watches, clocks, calendars and 'perpetual calendars' are all cyclic. This contrasts with the Newtonian concept of time as being unbounded

\* Corresponding author.

E-mail address: [sudarshan@physics.utexas.edu](mailto:sudarshan@physics.utexas.edu)  
(E.C.G. Sudarshan).

in both directions and having the topology of an unbounded (straight) line.

In quantum mechanics the width of a metastable energy level is related inversely to its lifetime. In this case the lifetime is not related to the readings of an arbitrary clock but is related to the dynamical system itself. We have the relation

$$(\Delta E) \cdot (\Delta t) \sim \hbar \quad (2)$$

reminiscent of the Heisenberg uncertainty relation for canonically conjugate variables:

$$(\Delta p) \cdot (\Delta q) \geq \frac{\hbar}{2}. \quad (3)$$

It is therefore tempting to define a ‘dynamical time’ for any mechanical system which is measured intrinsically, and not by an external clock.

For this purpose we propose to investigate a number of simple dynamical systems. The discussion could be carried out for classical systems with Poisson brackets or quantum systems with quantum commutator brackets. To avoid needless repetition we will use classical notation, but we carry out the computations in a manner which is equally applicable to quantum mechanics. We shall also investigate the behavior of this ‘dynamical time’ under various transformations including relativistic time dilation.

Some concern is felt due to the fact that the Hamiltonian for physical systems is bounded from below:<sup>1</sup> how can there be a conjugate to the Hamiltonian? But we may recall that it is not necessary to have an unbounded Hamiltonian to define a time operator: only we need to remember that such a dynamical time cannot be integrated to generate a semigroup of energy translations.<sup>2</sup> The situation here is no different from defining a radial variable  $r \geq 0$  for which we have a conjugate radial momentum  $p_r$ , but it can only generate a semigroup and not a group of radial translations.

<sup>1</sup> W. Pauli is credited with the categorical assertion that we cannot define a time operator since the energies are bounded from below.

<sup>2</sup> The radial momentum operator is not self-adjoint and has deficiency indices (1, 0) so that it is not even essentially self-adjoint.

## 2. Non-relativistic free particles

Consider a free particle of mass  $m$  with coordinates  $\mathbf{q}$ , momenta  $\mathbf{p}$ , and non-relativistic Hamiltonian

$$H = \frac{p^2}{2m}. \quad (4)$$

The equations of motion

$$\dot{\mathbf{q}} = \frac{1}{m}\mathbf{p}, \quad \dot{\mathbf{p}} = 0 \quad (5)$$

can be integrated to obtain

$$\mathbf{q}(t) = \mathbf{q}(0) + \frac{1}{m}\mathbf{p}t, \quad \mathbf{p}(t) = \mathbf{p}(0). \quad (6)$$

Hence

$$m(\mathbf{q}(t) - \mathbf{q}(0)) = t\mathbf{p}(0). \quad (7)$$

Define  $\tau$  by

$$m\mathbf{q} = \tau\mathbf{p} \quad (8)$$

then, by virtue of the equations of motion

$$\tau \equiv \tau(\mathbf{q}, \mathbf{p}) = t. \quad (9)$$

More specifically

$$\begin{aligned} \tau &= \frac{m}{4} \left[ \frac{1}{p^2}(\mathbf{q} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{q}) + (\mathbf{q} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{q}) \frac{1}{p^2} \right] \\ &= \frac{1}{8} \left[ \frac{1}{E}(\mathbf{q} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{q}) + (\mathbf{q} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{q}) \frac{1}{E} \right] \end{aligned} \quad (10)$$

may be taken as the definition of a dynamical time. The expression for dynamical time must be explicitly symmetrised in its non-commuting factors to get a hermitian symmetric operator. This expression has been studied before by Aharonov and Bohm [2]. More references are given for various proposals in [3]. Under rotations  $\tau$  is unchanged but under translations  $\mathbf{q} \rightarrow \mathbf{q} + \mathbf{a}$

$$\tau \rightarrow \tau + \left[ \frac{1}{p^2}(\mathbf{a} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{a}) + (\mathbf{a} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{a}) \frac{1}{p^2} \right]. \quad (11)$$

This unsatisfactory situation can be remedied by taking *two* free particles masses  $m_1, m_2$  with canonical coordinates and momenta  $\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2$  and defining

$$\mathbf{q} = (\mathbf{q}_1 - \mathbf{q}_2), \quad \mathbf{p} = \mu \left( \frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2} \right),$$

$$\tau = \frac{\mu}{4} \left\{ \frac{1}{\left(\frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2}\right)^2} \times \left[ \mathbf{q} \cdot \left(\frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2}\right) + \left(\frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2}\right) \cdot \mathbf{q} \right] + \left[ \mathbf{q} \cdot \left(\frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2}\right) + \left(\frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2}\right) \cdot \mathbf{q} \right] \times \frac{1}{\left(\frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2}\right)^2} \right\}, \quad (12)$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

This definition is also rotationally invariant. Under a Galilean boost:

$$\mathbf{q} \rightarrow \mathbf{q}, \quad \mathbf{p}_1 \rightarrow \mathbf{p}_1 + m_1 \mathbf{v}, \quad \mathbf{p}_2 \rightarrow \mathbf{p}_2 + m_2 \mathbf{v} \quad (13)$$

we find that the dynamical time transforms according to

$$\tau \rightarrow \tau.$$

In all these cases we see that while the Hamiltonian changes,

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \rightarrow \frac{(\mathbf{p}_1 + m_1 \mathbf{v})^2}{2m_1} + \frac{(\mathbf{p}_2 + m_2 \mathbf{v})^2}{2m_2} \quad (14)$$

$\tau(\mathbf{q}, \mathbf{p})$  remains unchanged.

In all these cases we deduce:

$$[\tau(\mathbf{q}, \mathbf{p}), H(\mathbf{q}, \mathbf{p})]_{\text{P.B.}} = 1$$

or

$$\tau(\mathbf{q}, \mathbf{p})H(\mathbf{q}, \mathbf{p}) - H(\mathbf{q}, \mathbf{p})\tau(\mathbf{q}, \mathbf{p}) = i\hbar. \quad (15)$$

It follows that the Heisenberg and Robertson–Schrödinger [4] inequalities also hold:

$$\Delta\tau \cdot \Delta H \geq \frac{\hbar}{2}, \quad (16)$$

$$(\Delta\tau)^2(\Delta H)^2 - \Delta^2(\tau H) \geq \frac{\hbar^2}{4}. \quad (17)$$

### 3. Relativistic free particles

For relativistic free particles the dynamical variables obey different mutual relations. (We choose  $c = 1$  for rotational simplification.) Then

$$\omega_1 = \sqrt{p_1^2 + m_1^2}, \quad \omega_2 = \sqrt{p_2^2 + m_2^2}, \quad (18)$$

$$\mathbf{v}_1 = \frac{\mathbf{p}_1}{\omega_1}, \quad \mathbf{v}_2 = \frac{\mathbf{p}_2}{\omega_2}. \quad (19)$$

For one free particle

$$\tau = \frac{1}{2} \left\{ \mathbf{q}, \frac{\mathbf{v}}{v^2} \right\} = \frac{1}{2} \left\{ \mathbf{q}, \frac{\omega \mathbf{p}}{p^2} \right\} \quad (20)$$

which reduces to

$$\tau \rightarrow \frac{1}{2} \left\{ \mathbf{q}, \frac{m \mathbf{p}}{p^2} \right\} \quad (21)$$

in the non-relativistic limit. The curly brackets  $\{, \}$  denote anticommutators which take care of the necessary symmetrisation with respect to the non-commuting operators. For getting a translation and rotation invariant time variable, define

$$\tau = \frac{1}{2} \left\{ \mathbf{q}, \frac{\mathbf{v}_1 - \mathbf{v}_2}{(\mathbf{v}_1 - \mathbf{v}_2)^2} \right\} = \frac{1}{2} \left\{ \mathbf{q}, \frac{\mathbf{p}_1/\omega_1 - \mathbf{p}_2/\omega_2}{(\mathbf{p}_1/\omega_1 - \mathbf{p}_2/\omega_2)^2} \right\}. \quad (22)$$

It is easily verified that  $\tau$  so defined is conjugate to the Hamiltonian

$$H = \omega_1 + \omega_2. \quad (23)$$

The time variable so defined is not invariant under transformation to a moving frame. This cannot be remedied since the Hamiltonian is the Poisson bracket of the boost  $\mathbf{K}$  and the translator  $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$  yields the Hamiltonian [5]

$$[K_j, P_k]_{\text{P.B.}} = \delta_{jk} H. \quad (24)$$

In fact, this, in turn, leads to the relativistic time dilation.

### 4. Relativistic time dilation

The velocity for free relativistic particles is given by

$$\mathbf{v} = \omega^{-1} \mathbf{p}, \quad \omega = \sqrt{p^2 + m^2}. \quad (25)$$

The relativistic velocity addition theorem for the component along the direction of a boost is:

$$\mathbf{v}' = \frac{\mathbf{u} + \mathbf{v}}{1 + \mathbf{u}\mathbf{v}}. \quad (26)$$

Since the transverse momenta are not changed while the energy  $\omega$  increases, the transverse velocity components decrease as  $\omega^{-1}$ . For sufficiently large boosts,

$$\frac{\mathbf{v}}{\sqrt{1 - u^2}} \ll 1 \quad (27)$$

the longitudinal component (along the direction of the boost) of the velocity goes as

$$\mathbf{v} \rightarrow (1 - u^2)^{(1/2)} \mathbf{v}. \quad (28)$$

So the time operator

$$\begin{aligned} \tau &= \frac{1}{2} \left\{ \mathbf{q}, \frac{\mathbf{v}_1 - \mathbf{v}_2}{(\mathbf{v}_1 - \mathbf{v}_2)^2} \right\} \\ &= \frac{1}{2} \left\{ \mathbf{q}, \frac{\mathbf{p}_1/\omega_1 - \mathbf{p}_2/\omega_2}{(\mathbf{p}_1/\omega_1 - \mathbf{p}_2/\omega_2)^2} \right\} \end{aligned} \quad (29)$$

goes as  $(1 - u^2)^{-\frac{1}{2}} \equiv \gamma(u)$ .

We note that this relativistic time dilation obtains for the transverse components also

$$\tau \rightarrow \gamma \tau. \quad (30)$$

## 5. Dynamical time for interacting systems

We revert to non-relativistic particles, but now introduce an interaction between them. The simplest case is that of a linear coupling corresponding to a constant force. Write

$$\begin{aligned} H &= \frac{\mathbf{p}_1^2 + \mathbf{p}_2^2}{2m} + \lambda(\mathbf{q}_1 - \mathbf{q}_2) \cdot \mathbf{n} \\ &= \frac{P^2}{2M} + \frac{p^2}{2\mu} + \lambda \mathbf{q} \cdot \mathbf{n}, \end{aligned} \quad (31)$$

where  $\mathbf{n}$  is an arbitrary direction. The system is no longer rotationally invariant. The equations of motion are

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{p}, & \dot{\mathbf{Q}} &= \mathbf{P}, & \dot{\mathbf{P}} &= 0, \\ \dot{\mathbf{p}} \cdot \mathbf{n} &= \lambda \mu, & \dot{\mathbf{p}} \times \mathbf{n} &= 0. \end{aligned} \quad (32)$$

Hence

$$\tau = \lambda^{-1} \mathbf{p} \cdot \mathbf{n} \quad (33)$$

is the appropriate time variable.

If the interaction is bilinear in the relative momentum and relative coordinate

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} + \frac{\lambda}{2}(\mathbf{q} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{q}). \quad (34)$$

This is rotationally invariant. A dynamical time is given by the relation

$$\tau = \frac{\mu}{4} \left[ \frac{1}{p^2}(\mathbf{q} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{q}) + (\mathbf{q} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{q}) \frac{1}{p^2} \right] \quad (35)$$

which is translation and rotation invariant; and it is also Galilean invariant.

## 6. Cyclic time

For a Galilean system with a harmonic interaction in the relative variables

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} + \frac{1}{2} \lambda q^2 \quad (36)$$

we have harmonic time dependence of both  $\mathbf{q}$  and  $\mathbf{p}$  with  $\mathbf{P}$  as a constant of motion. Then

$$\begin{aligned} \tau &= \sqrt{\frac{\mu}{\lambda}} \tan^{-1} \left\{ \frac{\sqrt{\lambda\mu}}{4} \left[ \frac{1}{p^2}(\mathbf{q} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{q}) \right. \right. \\ &\quad \left. \left. + (\mathbf{q} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{q}) \frac{1}{p^2} \right] \right\} \end{aligned} \quad (37)$$

is conjugate to the Hamiltonian. This is a multiple valued *cyclic* variable. This is quite appropriate to represent faithfully most clocks that we have. (Even when there is a counter attached to the clock, since the counter register has only a finite number of digits, the time is still cyclic!) An alternate choice is

$$\tau' = \frac{M}{4} \left[ \frac{1}{P^2}(\mathbf{Q} \cdot \mathbf{P} + \mathbf{P} \cdot \mathbf{Q}) + (\mathbf{Q} \cdot \mathbf{P} + \mathbf{P} \cdot \mathbf{Q}) \frac{1}{P^2} \right]. \quad (38)$$

## 7. The semigroup of energy translations

Since  $\tau$  satisfies the commutative relation

$$\tau H - H \tau = i, \quad (39)$$

$\tau$  acts as a translation generator with respect to the Hamiltonian. If it were possible to integrate it to a

finite transformation, the energy would be shifted. So a state vector of the form

$$|\psi\rangle = \int_0^{\infty} f(E)|E\rangle dE \quad (40)$$

will get transformed into

$$|\psi\rangle = e^{-i\epsilon\tau}|\psi\rangle = \int_0^{\infty} f(E)|E - \epsilon\rangle dE. \quad (41)$$

But for  $\epsilon \geq 0$  the state  $|\psi\rangle$  cannot exist. So it must not be possible to integrate the infinitesimal transformation. This is because  $\tau$  is not self-adjoint. But the family of operators  $e^{-i\epsilon\tau}$ ,  $\epsilon \geq 0$  constitute a semigroup of operators. For  $|\mathbf{p}\rangle \rightarrow 0$  the time operator is singular which brings about only the semigroup being realized.<sup>3</sup> The lack of self-adjointness is demonstrated by the existence of a square integrable solution for  $\tau|\Phi\rangle = +i|\Phi\rangle$  but not for  $\tau|\Phi\rangle = -i|\Phi\rangle$ .

## 8. Spectral properties of the time operator

We have already seen that the dynamical variable  $\tau$  may be made Hermitian by suitable symmetrization:

$$\begin{aligned} \tau &= \frac{1}{4} \left\{ \frac{1}{p^2} (\mathbf{p} \cdot \mathbf{q} + \mathbf{q} \cdot \mathbf{p}) + (\mathbf{p} \cdot \mathbf{q} + \mathbf{q} \cdot \mathbf{p}) \frac{1}{p^2} \right\} \\ &\equiv \frac{1}{p^2} \left( \mathbf{p} \cdot \mathbf{q} - \frac{i}{2} \right) \end{aligned} \quad (42)$$

but even this operator is not self-adjoint. Every complex value  $z$  is an eigenvalue with a normalizable eigenvector

$$\psi_z(p) = \sqrt{p} e^{-iz'p^2/2}, \quad z' = z - \frac{i}{2} \quad (43)$$

as long as  $z'$  is in the upper-half plane. The non-self-adjoint property may also be verified by noting that  $z = i$  gives a normalizable solution but  $z = -i$  does not. So its deficiency indices are  $(1, 0)$  and *cannot be extended* to a self-adjoint operator. But the states  $\{\psi_z(p)\}$  may be used to expand any function  $F(p^2)$  which vanishes for  $p^2 < 0$  the expansion is *non-unique* since the function  $\psi_z(p)$  are *overcomplete*. If

the weight function  $\omega(z)$  is analytic in the half plane  $\text{Re } z > -\frac{1}{2}$ , the function  $F(p^2)$  so computed vanishes for  $p^2 < 0$ . This meets the requirement that the energy is bounded from below for free particles. A clear discussion of the problems of non-self adjoint operator of radial momentum is given in [3].

## 9. Summary

We have defined a dynamical variable conjugate to the Hamiltonian for several typical two-particle systems. This dynamical variable may be considered as the dynamical time. For Galilean systems this time is Galilean invariant, whether the system is interacting or not. For relativistic two-particle system there is a typical time dilation, by virtue of the non-commutativity of boosts with the Hamiltonian. When a bounded dynamical system is used as the clock, the obtained time variable is cyclic. We have also shown that the semibounded Hamiltonian leads to a non-self-adjoint time variable. The spectral properties of the non-self-adjoint time are also studied.

## Acknowledgements

We are grateful to the referee for his comments and especially for pointing out Ref. [3].

## Appendix A. Relativistic kinematics

The relativistic transformations of the momenta and coordinates for a boost along the third axes are given by

$$\begin{aligned} p_1 &\rightarrow p_1, & p_2 &\rightarrow p_2, & p_3 &\rightarrow \gamma(p_3 + \beta\omega), \\ q_1 &\rightarrow q_1 - \beta\left(\frac{p_1 q_3}{\omega}\right), & q_2 &\rightarrow q_2 - \beta\left(\frac{p_2 q_3}{\omega}\right), \\ q_3 &\rightarrow \frac{q_3}{(\omega + \beta p_3)}, \\ \omega &= \sqrt{p^2 + m^2}. \end{aligned}$$

These are verified to be finite and canonical.

<sup>3</sup> See footnote 2.

**References**

- [1] (a) P. Weiss, Proc. R. Soc. A 169 (1938) 102;  
(b) J. Schwinger, Phys. Rev. 82 (1951) 914;  
(c) E.C.G. Sudarshan, N. Mukunda, Classical Dynamics. A Modern Perspective, Wiley, New York, 1974.
- [2] Y. Aharonov, D. Bohm, Phys. Rev. 122 (1961) 1640.
- [3] J.G. Muga, R. Sala-Mayato, I.L. Egusquiza (Eds.), Time in Quantum Mechanics, Springer, Berlin, 2002.
- [4] (a) W. Heisenberg, Z. Phys. 43 (1927) 122;  
(b) J.P. Robertson, Phys. Rev. A 35 (1930) 667;  
(c) E. Schrödinger, Ber. Kgl. Akad. Wiss. (1930) 296.
- [5] M. Reed, B. Simon, Functional Analysis I, Academic Press, San Diego, 1980.