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Entanglement in probability representation of quantum states and tomographic criterion of separability

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Abstract
Entangled and separable states of a bipartite (multipartite) system are studied in the tomographic representation of quantum states. Properties of tomograms (joint probability distributions) corresponding to entangled states are discussed. The connection with star-product quantization is presented. $U(N)$-tomography and spin tomography as well as the relation of the tomograms to positive and completely positive maps are considered. The tomographic criterion of separability (necessary and sufficient condition) is formulated in terms of the equality of the specific function depending on unitary group parameters and positive map semigroup parameters to unity. Generalized Werner states are used as an example.

Keywords: entanglement, separatibility, star-product, tomography

1. Introduction

The notion of entangled states (first discussed by Schrödinger [1]) has attracted much effort to find a criterion [2, 3] and quantitative characteristics of entanglement [4–12] (see also [13]). A criterion [2, 3] based on partial transpose transform of subsystem density matrix (complex conjugation of the subsystem density matrix or its time reverse) provides the necessary and sufficient condition of separability of the system of two qubits and the qubit–qutrit system [14]. The phase-space representation of the quantum states and time reverse transform (change of sign of the subsystem momenta) of the Wigner function in the case of a Gaussian state was applied to study the separability and entanglement of photon states in [11]. Recently, it was pointed out that the tomographic approach of reconstructing the Wigner function of the quantum state [15–17] can be developed to consider the positive probability distribution (tomogram) as an alternative to the density matrix (or wavefunction) because the complete set of tomograms contains the complete information on the quantum state [18]. This representation (called probability representation) was also constructed for spin states [19, 20] including a bipartite system of two spins [21]. Up to now the problem of entanglement has not been discussed in the tomographic representation. Some remarks on tomograms and entanglement of photon states in the process of Raman scattering were given in [22]. The tomographic approach has the advantage of dealing with positive probabilities and one deals with standard probability distributions which are positive and normalized.

We aim to reconstruct the properties of separable and entangled states of a multipartite system using tomographic probability distributions to describe their quantum states. The positive and completely positive maps of density matrices [23, 24] induce specific properties of the tomograms.

The connection of the separability with properties of the positive maps was studied intensively in [25]. We formulate the necessary conditions and a conjecture for sufficient conditions of separability and entanglement of multipartite systems in terms of properties of the quantum tomogram. Since the tomograms were shown [26] to be related to the star-product quantization procedure [27], we discuss entanglement and separability properties in terms of generic operator symbols: the tomographic symbols of generic spin operators were studied in [26]. Then we focus on properties of entanglement and separability of a bipartite system using spin tomograms ($SU(2)$-tomograms) and tomograms of the $U(N)$-group.

The idea of the approach is the following. The positive but not completely positive linear maps of a subsystem density matrix preserve the positivity of separable density matrix of the composite system. These maps also contain the maps which do not preserve the positivity of the initial density matrix of the entangled state of the composite system. This means that the set of all linear positive maps of the subsystem density matrix (this set is a semigroup) creates from the initial entangled positive density matrix of the composite system a set of Hermitian matrices including the matrices with negative eigenvalues. To detect the entanglement we use the tomographic symbols of the obtained Hermitian matrices. The tomographic symbols of state density matrices (state tomograms) are standard probabilities. In view of this the tomographic symbols of the obtained Hermitian matrices corresponding to an initial separable state preserve all the properties of the probability representation including positivity and normalization. But in the case of an entangled state the tomographic symbols of the obtained Hermitian matrices can take negative values. The different behaviour of tomograms of separable and entangled states of composite systems under action of the semigroup of positive maps of the subsystem density matrix provides the tomographic criterion of the separability.

The paper is organized as follows. In section 2 we review the generic properties of operator symbols. In section 3 we consider tomograms as image of density matrix. In section 4 we review the spin-tomography of spin states for one particle and two particles introducing also generalization of $SU(3)$-tomography of [28] to the case of $U(N)$-tomography in connection with properties of a spin density operator associated with adjoint representation of the $U(N)$-group. In section 5 we formulate the conditions which are necessary and sufficient for spin-state tomograms to correspond either to separable or entangled states. These considerations require use of the positive and noncompletely positive maps of [23, 29] which we review in this section. The example of the generalized Werner state [30] will be studied as an illustration of the approach in section 6.

2. Operator algebra, associated functions, star-products

Let us discuss a map of operators acting in a Hilbert space of states onto functions. If we associate with every (bounded) operator $\hat{A}$ of an associative algebra of observables a function

$$f_A(x) = \mathrm{Tr}\,(\hat{A}\hat{U}(x)) \tag{1}$$

and we have the inverse relationship

$$\hat{A} = \int f_A(x)\hat{D}(x)\,\mathrm{d}x, \tag{2}$$

we call $f_A(x)$ a symbol for the operator. Here x stands for a collection of discrete or continuous variables which label the family of operators $\hat{U}(x)$ and the dual family $\hat{D}(x)$.

Operator multiplication and linear combinations can be expressed in terms of the functions $f_A(x)$. For scalar multiplication or linear combinations, the map induces the same operations on the functions. The operator multiplication is given by a star-product of the corresponding functions

$$f_{\hat{A}\hat{B}}(x) = \mathrm{Tr}\,(\hat{A}\hat{B}\hat{U}(x)) = f_A(x) \star f_B(x) \tag{3}$$

with

$$f_A(x) \star f_B(x) = \int f_A(x_1) f_B(x_2) K(x; x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{4}$$

where the kernel is given by the formula

$$K(x; x_1, x_2) = \mathrm{Tr}\,(\hat{U}(x)\hat{D}(x_1)\hat{D}(x_2)). \tag{5}$$

Thus the symbols $f_A(x)$ furnish a representation of the observable $\hat{A}$ and the star product is the image of the operator product of the observables.

In this paper we will be concerned with a particular class of symbols called the 'quantum tomograms' which give a family of nonnegative functions associated with canonical operators for the symplectic group and with spin systems. There exists an example of such a map for continuous photon quadratures. For one degree of freedom and a Weyl-ordered operator $\exp(\mathrm{i}\mu\hat{p} + \mathrm{i}\lambda\hat{q})$, we have the symbol (see also [31])

$$f(x, y) = \exp(\mathrm{i}\lambda x + \mathrm{i}\mu y). \tag{6}$$

The star-product of the symbols is given by the Moyal [32] exponential bracket

$$f(x, y) \star g(x, y) = \exp\left[\mathrm{i}\hbar\left(\frac{\partial}{\partial x_f}\frac{\partial}{\partial y_g} - \frac{\partial}{\partial y_f}\frac{\partial}{\partial x_g}\right)\right] \times f(x_f, y_f) g(x_g, y_g)|_{x_f = x_g = x, y_f = y_g = y}, \tag{7}$$

which gives an associative product. If we were to consider a state defined by a density distribution $\rho(x, y)$, the corresponding symbol is the Wigner–Moyal phase space distribution function. Moyal showed how to write the operator algebra (of bounded operators) in terms of these phase space functions and using the star-product given above.

The Wigner–Moyal phase space function is not pointwise positive, and hence it cannot be interpreted as a probability. However, it has the property that its line integral along any straight line $\mu x + \nu y - X = 0$ in the (xy)-plane is nonnegative.

These functions

$$\int\int f(x, y)\delta(\lambda x + \mu y - X)\,\mathrm{d}x\,\mathrm{d}y = w(\lambda, \mu, X) \tag{8}$$

are symplectic tomograms [17].

We are primarily concerned with spin tomograms (symbols of spin density matrices) associated with finite-dimensional state densities. For an $n \times n$ matrix algebra, we define the unitary spin tomogram

$$W(\{m\}, g) = \langle\{m\}|\mathcal{D}(g)\rho\mathcal{D}^{\dagger}(g)|\{m\}\rangle, \tag{9}$$

where g is the unitary group element which can be parametrized by trigonometric functions, ρ is the density matrix, $\{m\}$ is a complete set of labels to describe the unitary group $U(N)$ representations and $\mathcal{D}(g)$ is the unitary matrix representation of the $U(N)$-group. The tomogram has the following simple property:

$$\sum_{\{m\}} W(\{m\}, g) = 1 \tag{10}$$

for all g and for all normalized density matrices. The tomogram is the standard probability distribution. In fact the tomogram coincides with the diagonal matrix element of the unitary transformed density matrix. But the diagonal matrix elements of the density matrix in arbitrary basis are probabilities. Thus the above equation is the normalization condition for the probability distribution.

Since

$$\sum_{\{m\}} |\{m\}\rangle\langle\{m\}| = 1,$$

this is the trace

$$\sum_{\{m\}} \mathrm{Tr}(|\{m\}\rangle\langle\{m\}|\mathcal{D}(g)\rho\mathcal{D}^{\dagger}(g)) = \mathrm{Tr}\,(\mathcal{D}(g)\rho\mathcal{D}^{\dagger}(g))$$
$$= \mathrm{Tr}\ \rho = 1. \tag{11}$$

Moreover, the unitary spin tomograms are positive for each $\{m\}$. Since $U(N)$ acts transitively on ρ, any negative value of $W(\{m\}, g)$ would point to the ρ not being positive definite.

3. Quantum tomogram as image of density matrix

We are interested in the tomogram of bipartite (and multipartite) systems. We would like to recognize simply separable and separable states from their tomograms; clearly a separable bipartite density matrix could have a separable spin tomograms. This ability to detect separability or lack of positivity of a (pseudo) density matrix in terms of simple trigonometric functions of one or more angles is a new approach exploiting tomography.

For a bipartite system with dimensions $n_1 = 2j_1 + 1$ and $n_2 = 2j_2 + 1$, one has $n_1n_2 \times n_1n_2$ matrices. The $U(n_1n_2)$ fundamental realization remains irreducible when it is decomposed with respect to $U(n_1) \otimes U(n_2)$ except for an overall phase. For $n_1 = n_2 = 2$ corresponding to a bipartite system of two qubits, we have the fundamental four-dimensional representation of $U(4)$. A generic representation of $U(4)$ needs two labels of each component, one of which is associated to $U(1)$. So we have two alternative labellings: $|\frac{1}{2}, m_1\rangle \times |\frac{1}{2}, m_2\rangle$ or $|4; \{m\}\rangle$ (m_1 and m_2 are spin projections). Using the former for the generic case of $(2j_1 + 1) \times (2j_2 + 1)$ bipartite system, we have

$$W(\{m\}, g) = \langle\{m\}|\mathcal{D}(g)\rho\mathcal{D}^{\dagger}(g)|\{m\}\rangle, \tag{12}$$

where g is in $U(4)$. The $W(\{m\}, g)$ may be thought of as probabilities since they are nonnegative and sum to unity for all g.

We have the relation

$$\sum_{\{m\}} |W(\{m\}, g)| = 1. \tag{13}$$

This relation is necessary and sufficient condition for positivity of Hermitian operator with unit trace.

For the bipartite system, the set $\{m\}$ is (m_1, m_2).

For $SU(2)$ representation, expressed in terms of Pauli matrices,

$$g \to \mathcal{D}(g) = \exp\left(\frac{\mathrm{i}}{2}\,\psi\sigma_3\right)\exp\left(\frac{\mathrm{i}}{2}\,\theta\sigma_2\right)\exp\left(\frac{\mathrm{i}}{2}\,\varphi\sigma_3\right), \tag{14}$$

but since in (12) we are taking the diagonal matrix elements, it follows that ψ does not enter the evaluation of the tomogram. The other two angles θ and ϕ define a direction (vector $\vec{v}$) on the unit sphere. If we call these vectors $\vec{v}_1$ and $\vec{v}_2$ for the two $U(2)$ groups, we can rewrite this unitary spin tomogram in the form of spin tomograms [21]

$$W(m_1, m_2, g) \to W(m_1, m_2, \vec{v}_1, \vec{v}_2). \tag{15}$$

The simple separability of the density matrix will make a symbol of the density matrix to be seen as a product. The simple separability of the density matrix $\rho = \rho_1 \otimes \rho_2$ will make the tomogram also be seen as a product. This is the simply separable case and can be recognized by inspection. For generic separable cases of density matrices expressed in terms of a convex set of simple separable states, the decomposition is not trivial.

In the case of a bipartite system, we consider generic dynamic mappings, which probably associate with every density matrix for any of the subsystems another density matrix (for each map):

$$\rho^{(1)} \to L_{\mathrm{s}}^{(1)}\rho^{(1)}, \qquad \rho^{(2)} \to L_{\mathrm{s}}^{(2)}\rho^{(2)}. \tag{16}$$

There are many ways of displaying $L_{\mathrm{s}}^{(1)}$ and $L_{\mathrm{s}}^{(2)}$: the familiar ways are in terms of supermatrix with four indices each [23]; or more simply by an ordinary matrix in n_1^2 or n_2^2 dimension by writing the L as a matrix of these 'vectors' [29, 33]. The dynamic map induces the corresponding map of symbols of the density matrices including the map of tomographic symbols.

4. Tomograms for maps

Let us denote the generic map on the bipartite density matrix by

$$\rho \to L\rho, \qquad \rho^{(1)} \otimes \rho^{(2)} \to L^{(1)}\rho^{(1)} \otimes L^{(2)}\rho^{(2)}. \tag{17}$$

Such a map is simply separable and maps a simply separable state into a simply separable state, and generic separable states into generic separable states. The generic separable state is a convex set of the simple separable states. A map is said to be positive if

$$\rho > 0 \to L\rho > 0. \tag{18}$$

The tensor product of two positive maps $L^{(1)}$ and $L^{(2)}$ is not necessarily positive: i.e., if

$$L^{(1)}\rho^{(1)} \geqslant 0, \qquad L^{(2)}\rho^{(2)} \geqslant 0, \qquad \rho \geqslant 0,$$

the Hermitian matrix $(L^{(1)} \otimes L^{(2)})\rho$ is not always positive. In the case of a tensor product of completely positive maps one has the positive map of the composite system density matrix.

A convex set of such tensor products also provides a positive map of the composite system density matrix. But if one of the maps (or both) in the tensor product is not completely positive the result can yield the nonpositive density matrix of the composite system. The transpose transform is a positive but not completely positive map. Another noncompletely positive map for N-dimensional matrices is given by the relation

$$\rho \to -x\rho + \frac{1+x}{N} I. \tag{19}$$

Here x is a parameter. In fact, if $L^{(1)} \otimes L^{(2)}$ on the composite density distribution is not positive so that the unitary spin tomograms $\langle m_1, m_2|U(g)L^{(1)} \times L^{(2)}\rho U^\dagger(g)|m_1, m_2\rangle$ are not positive for some g, we can conclude that ρ was *not* separable. Of course, most $L^{(1)} \times L^{(2)}\rho$ may yield positive spin tomograms; we need to specify a sequence of specific maps $L^{(1)}$, $L^{(2)}$ which will show definitely when ρ is entangled and when it is not.

We have already seen that the spin tomograms exhibit all the features of a standard probability distribution with nonnegative probability adding to unity. So for every g, we have a joint probability distribution of two random spin projections

$$p(m_1, m_2) = \langle m_1, m_2|\mathcal{D}(g)\rho\mathcal{D}^\dagger(g)|m_1, m_2\rangle \geqslant 0, \qquad \sum_{m_1,m_2} p(m_1, m_2) = 1. \tag{20}$$

An inversion analytic formula for spin-S particle may be adopted from [28]:

$$\rho = \sum_{L=0}^{2S} \sum_{M=-L}^{L} \hat{T}_{L,M}^{(S)} \frac{2L+1}{\sqrt{4\pi(2S+1)}} \times \int d\Omega\, Y_{LM}^*(\theta, \phi) \sum_{k=-S}^{S} w(k, \theta, \phi)\langle Sk; L0|Sk\rangle, \tag{21}$$

where $\hat{T}_{L,M}^{(S)}$ is the irreducible tensor operator for the $SU(2)$ group

$$\hat{T}_{L,M}^{(S)} = \sqrt{\frac{2L+1}{2S+1}} \sum_{m,m'=-S}^{S} \langle Sm; LM|Sm'\rangle |S, m'\rangle\langle S, m|. \tag{22}$$

This formula connects spin tomogram $w(k, \theta, \varphi)$ with the density matrix.

An analogous formula can be written for two spins.

The spin tomograms furnish an overcomplete set of quantities determining the density matrix completely. Specific analytic inverse formulae have been derived but it suffices that we recognize if $W(\{m\}, \vec{v}_1, \vec{v}_2)$ are known at more than n^4 directions $\vec{v}_1, \vec{v}_2$ to determine the density matrix completely. It follows that there is an infinite number of sum rules involving the tomograms.

5. Overcomplete states in quantum theory and quantum entanglement

In the related case of an overcomplete set of coherent states in quantum optics, the distribution function $\Phi(z)$ is related to the density matrix by a Fourier integral [31, 34]

$$\Phi(z) = \int d^2\zeta\, \langle\zeta|\rho|\zeta\rangle \exp(\zeta^*\zeta) \exp(z\zeta^* - z^*\zeta). \tag{23}$$

This distribution $\Phi(z)$ is not necessarily a generalized function, much less a simple function, nor is it pointwise nonnegative. It is, however, precisely when $\Phi(z)$ becomes negative that typically quantum effects like sub-Poissonian photo-count distributions are. In a related context, we found that the unitary spin tomogram

$$W(\{m\}, L\rho; g) < 0 \tag{24}$$

indicates quantum entanglement. We include a dependence on the density matrix into the argument of the tomogram.

We conclude this section by noting that for the bipartite spin system a computationally simple test of separability is given by the identity

$$\sum_{\{m\}} W(\{m\}, L\rho; g) = 1 \tag{25}$$

modified to read

$$F_\rho(L, g) = \sum_{\{m\}} |W(\{m\}, L\rho; g)| = 1, \tag{26}$$

where $L\rho$ is given by equation (17), $g \in U((2j_1+1)(2j_2+1))$.

Whenever this arithmetical sum is not unity, we know that ρ is entangled. In this case, the maximum value of the sum can be considered as a measure of the state entanglement.

6. Two illustrative examples

For a two-qubit system, this is easily illustrated.

Take the density matrix

$$\rho = \begin{pmatrix} R_{11} & 0 & 0 & R_{12} \\ 0 & \rho_{11} & \rho_{12} & 0 \\ 0 & \rho_{21} & \rho_{22} & 0 \\ R_{21} & 0 & 0 & R_{22} \end{pmatrix}, \qquad \mathrm{Tr}\,\rho = 1, \tag{27}$$

$$\rho_{12} = \rho_{21}^*, \quad R_{12} = R_{12}^*, \quad \rho_{11}, \rho_{22}, R_{11}, R_{22} \geqslant 0, \qquad \rho_{11}\rho_{22} \geqslant \rho_{12}\rho_{21}; \quad R_{11}R_{22} \geqslant R_{12}R_{21}. \tag{28}$$

Construct the spin tomograms evaluated for the states

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle, \qquad |\tfrac{1}{2}, -\tfrac{1}{2}\rangle, \qquad |-\tfrac{1}{2}, \tfrac{1}{2}\rangle, \qquad |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle.$$

Construct the two sums only, (25) and (26), respectively.

We see that the sums are equal to unity and the positivity condition

$$W(\{m\}, L^{(1)} \otimes L^{(2)}\rho, g) \geqslant 0 \tag{29}$$

can be checked. One can use either partial transpose transform for the second spin subsystem and unity transform for the first spin subsystem or apply to the second spin subsystem the map (19). The induced violation of positivity of the density matrix for two qubits is necessary and sufficient condition for entanglement. The computations are elementary and give the first sum rule (25) but generic density matrix does not always satisfy the other sum rule (26) or the positivity condition. When they are satisfied we get an unentangled state as may be shown by redefining the basis for the two qubits. In fact one can take as a probe the unitary group element g which is close to the unitary transform diagonalizing the Hermitian matrix obtained after action of the map (e.g., the partial transpose transform) on the initial positive density matrix. The function (26) calculated for these elements g is larger than unity for both discussed maps.

A simpler and more transparent case is the generalized Werner model with density matrix

$$\rho = \tfrac{1}{4}(1 + \mu_1\sigma_1 \otimes \tau_1 + \mu_2\sigma_2 \otimes \tau_2 + \mu_3\sigma_3 \otimes \tau_3). \quad (30)$$

Here the density matrix is expressed in terms of tensor products of two sets of Pauli matrices σ_k and τ_k $(k = 1, 2, 3)$, which are chosen in standard form.

Its eigenvalues are

$$1 - \mu_1 - \mu_2 - \mu_3, \qquad 1 + \mu_1 + \mu_2 - \mu_3,$$
$$1 + \mu_1 - \mu_2 + \mu_3, \qquad 1 - \mu_1 + \mu_2 + \mu_3.$$

These points form the vertices of a regular tetrahedron. The partially time-reversed density matrix is

$$\tilde{\rho} = \tfrac{1}{4}(1 - \mu_1\sigma_1 \otimes \tau_1 - \mu_2\sigma_2 \otimes \tau_2 - \mu_3\sigma_3 \otimes \tau_3), \quad (31)$$

which may be viewed as

$$L^{(1)} \otimes L^{(2)}\rho \qquad \text{with } L^{(1)}\rho^{(1)} = \rho^{(1)},$$
$$L^{(2)}\rho^{(2)} = 1 - \rho^{(2)}. \quad (32)$$

The eigenvalues of this are

$$1 + \mu_1 + \mu_2 + \mu_3, \qquad 1 + \mu_1 - \mu_2 - \mu_3,$$
$$1 - \mu_1 + \mu_2 - \mu_3, \qquad 1 - \mu_1 - \mu_2 + \mu_3.$$

These form an inverted tetrahedron and they have the common domain which is a regular octahedron. The spin tomograms can be written down by inspection and we may verify that all the relations are satisfied by any point inside the octahedron for ρ and for $L^{(1)} \otimes L^{(2)}\rho$ but the second and third fail when it lies outside.

To conclude, we summarize the main result of the work.

We have found the criterion of separability given by equation (26). The criterion can be extended to the multiparticle spin system. The criterion can be called the 'tomographic criterion' of separability. The tomographic criterion can be considered also for symplectic tomograms of multimode photon states. We conjecture that the formulated tomographic test is a sufficient condition of the state separability. The condition of separability is sufficient because there always exists a unitary group element by means of which any Hermitian matrix can be diagonalized. This means that the tomographic symbol of the nonpositive Hermitian matrix has nonpositive values for some unitary group parameters. The suggested criterion is connected with properties of the constructed function (26) which for a given density matrix depends on the unitary group parameters g and the parameters of positive map semigroup L. For the separable density matrix the dependence on unitary group parameters and the semigroup parameters disappears and the function becomes constant equal to unity. For entangled states the function differs from unity and depends on both group and semigroup parameters. The suggested criterion can be considered as some complementary test of separability together with other criteria available in the literature (see, for example, [13, 14]). We point out that the suggested criterion differs from the usually available ones in the nature of the necessary numerical calculations. To apply this criterion one needs to calculate the sum of moduli of diagonal matrix elements of the product of three matrices. One of the matrices is Hermitian and the other two are unitary ones. This procedure does not require calculation of the eigenvalues of a matrix. The structure of positive (including not completely positive) map semigroup with elements L needs extra investigation.

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