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Generalized Euler angle parameterization for $U(N)$ with applications to $SU(N)$ coset volume measures

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Abstract

In a previous paper [J. Phys. A: Math. Gen. 35 (2002) 10467] an Euler angle parameterization for $SU(N)$ was given. Here we present a generalized Euler angle parameterization for $U(N)$. The formula for the calculation of the volume for $U(N)$, $\mathbb{C}P^N$ as well as other $SU(N)$ and $U(N)$ cosets, normalized to this parameterization, will also be given. In addition, the mixed and pure state product measures for N-dimensional density matrices under this parameterization will also be derived.

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1. Introduction

Having produced an Euler angle parameterization for $SU(N)$ we now turn our attention to explicitly writing down the Euler parameterization for the unitary group, $U(N)$ (which was hinted at in the $SU(N)$ work of [1,2]). Recall that $U(N)$ is a subgroup of $GL(N, \mathbb{C}) = GL_N(\mathbb{C})$, the group of all complex $N \times N$ matrices with non-vanishing determinant and requiring $2N^2$ parameters to represent. In this manner we can define $SU(N)$ to be a subgroup of $U(N)$, requiring $N^2 - 1$ parameters to represent, by adding the extra condition that any

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element of $SU(N)$ has unit determinant. We can therefore expect that not only will the Euler parameterization of $U(N)$ be easy to produce but also the group volume, once we exploit some simple group relationships between $SU(N)$ and $U(N)$.

The importance of such a parameterization and its corresponding volume equation, beyond that discussed in [3], is that it gives us the ability to calculate the measures and volumes for general N-dimensional pure and mixed-state density matrices, as well as the volumes of the manifolds of operations on pure and mixed states which produce entangled and separable states (which are directly related to the volume of separable and entangled states) without having to resort to extensive numerical computations as in [4,5].

2. Euler parameterization of $U(N)$

The idea behind the parameterization of $U(N)$ is straightforward. Referring to our previous work [1], for notations and details, as well as to Nakahara [6], Sattinger and Weaver [7], and others we know the following relationship holds between $SU(N)$, $U(N)$ and $\mathbb{C}P^N$:

$$\mathbb{C}P^N = \frac{SU(N+1)}{U(N)} = \frac{SU(N+1)}{SU(N)\times U(1)}. \tag{2.1}$$

The $U(1)$ in the denominator of the second equality is the $U(1)$ element from the $SU(N+1)$ group in the numerator, which we know from [1,2] to be:

$$U(1) \equiv U(1)_{SU(N+1)} = \mathrm{e}^{\mathrm{i}\lambda_{(N+1)^2-1}\beta}. \tag{2.2}$$

Using the $SU(N)$ parameterization work done previously [1], we can write down the Euler parameterization of $U(N)$ quite easily.

Recall from [1,2,7,8] we know we can write down a semi-direct sum for the Lie algebra for $SU(N)$ as

$$L(SU(N)) = L(K) \oplus L(P), \tag{2.3}$$

which yields a decomposition of the group,

$$V = K \cdot P, \tag{2.4}$$

where $V \in SU(N)$. From this work, we also know that $L(K)$ is comprised of the generators of the $SU(N-1)$ subalgebra of $SU(N)$, and therefore K will be the $U(N-1)$ subgroup obtained by exponentiating this subalgebra, $\{\lambda_1, \ldots, \lambda_{(N-1)^2-1}\}$, combined with λ_{N^2-1} and thus can be written as (see [1,2,9] for example):

$$K(N-1) = [SU(N-1)] \cdot \mathrm{e}^{\mathrm{i}\lambda_{N^2-1}\phi}, \tag{2.5}$$

where $[SU(N-1)]$ represents the $(N-1)^2-1$ term Euler angle parameterization of the $SU(N-1)$ subgroup.

We are now ready to look at the $U(N)$ group in general. For a $U \in U(N)$ we have from Eqs. (2.1) and (2.5) as well as from [1]:

$$U \equiv K(N) = [SU(N)] \cdot \mathrm{e}^{\mathrm{i}\lambda_{(N+1)^2-1}\beta}, \tag{2.6}$$

where

$$
\begin{aligned}
[SU(N)] &= \left(\prod_{2\le k\le N} A(k,j(N))\right)\cdot\left(\prod_{2\le k\le N-1} A(k,j(N-1))\right)\cdots(A(2,j(2)))\\
&\quad\times\ \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{N^2-(N-1)}}\ldots\mathrm{e}^{\mathrm{i}\lambda_{(N-1)^2-1}\alpha_{N^2-2}}\,\mathrm{e}^{\mathrm{i}\lambda_{N^2-1}\alpha_{N^2-1}}\\
&= \prod_{N\ge m\ge 2}\left(\prod_{2\le k\le m} A(k,j(m))\right)\\
&\quad\times\ \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{N^2-(N-1)}}\ldots\mathrm{e}^{\mathrm{i}\lambda_{(N-1)^2-1}\alpha_{N^2-2}}\,\mathrm{e}^{\mathrm{i}\lambda_{N^2-1}\alpha_{N^2-1}},\\
A(k,j(m)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{(2k-3)+j(m)}}\,\mathrm{e}^{\mathrm{i}\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}},\\
j(m) &= \begin{cases} 0, & m=N,\\ \sum_{0\le l\le N-m-1} 2(m+l), & m\ne N.\end{cases}
\end{aligned}
\tag{2.7}
$$

3. Volume of $U(N)$

From [1,10] the volume of $SU(N)$ is known to be

$$V_{SU(N)} = 2^{(N-1)/2}\pi^{(N-1)(N+2)/2}\sqrt{N}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right). \tag{3.1}$$

If we use Eq. (2.2) then from [1] we can define the following volume for $U(1)_{SU(N+1)}$:

$$
\begin{aligned}
V_{U(1)_{SU(N+1)}} \equiv V_{U(1)_{\lambda_{(N+1)^2-1}}} &= (N+1)*\int_0^{\pi\sqrt{2/(N+1)((N+1)-1)}} \mathrm{d}\alpha_{(N+1)^2+1}\\
&= \pi\sqrt{\frac{2(N+1)}{N}}.
\end{aligned}
\tag{3.2}
$$

From Eq. (2.6) we can write

$$V_{U(N)} = V_{SU(N)}\times V_{U(1)_{\lambda_{(N+1)^2-1}}} \tag{3.3}$$

and thus using Eqs. (3.1) and (3.2) we have

$$
\begin{aligned}
V_{U(N)} &= 2^{(N-1)/2}\pi^{(N-1)(N+2)/2}\sqrt{N}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right)*\pi\sqrt{\frac{2(N+1)}{N}}\\
&= 2^{N/2}\pi^{N(N+1)/2}\sqrt{N+1}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right)
\end{aligned}
\tag{3.4}
$$

for $N\ge 2$. Note that when $N=1$ we generate the volume for the $U(1)_{SU(2)}$ group element. Since there can be many different $U(1)$'s with different volumes, the fact that $U(N)$, when

$N = 1$ gives the $SU(2)$ group element volume demands that we limit the use of Eq. (3.4) to $N \geq 2$.

With this information in hand, we can now look at the differential volume elements, and corresponding volumes of the full range of $SU(N)$ and $U(N)$ cosets that are of interest in physics, beginning with the fundamental manifolds which define pure and mixed states.

4. Differential volume elements for pure and mixed states

Now that we have an Euler angle parameterization for both $SU(N)$ and $U(N)$, for $N \geq 2$, we are now in a position to look at the group representations of pure and mixed states in terms of our parameterizations.

In general, the manifold of pure states is given by the sequence of maps:

$$U(N-1) \mapsto SU(N) \mapsto \mathbb{C}P^{N-1}. \tag{4.1}$$

These are related to the "Grassmannian" manifolds, which are defined as

$$\mathbb{C}P^{N-1} \equiv G(N,1) = \frac{U(N)}{U(1) \times U(N-1)} = \frac{SU(N)}{U(N-1)}. \tag{4.2}$$

On the other hand, the manifold for mixed states (here for rank N density matrices with non-degenerate and non-singular eigenvalues) is given by [11,12]:

$$\mathcal{M}_{\mathrm{ms}} = \Omega_{N-1} \times \frac{SU(N)}{(U(1))^{N-1}}, \tag{4.3}$$

where Ω_{N-1} can be seen as the $(N-1)$-dimensional solid angle (with appropriate ranges) derived from the eigenvalues of a suitably parameterized $(N-1)$-dimensional sphere (see [3,12]), and the factor $(U(1))^{N-1}$ is the maximal torus spanned by the exponentiation of the Cartan subalgebra of the group:

$$\begin{aligned}(U(1))^{N-1} &= U(1)_{SU(2)} \times U(1)_{SU(3)} \times \cdots \times U(1)_{SU(N)} \\ &= U(1)_{\lambda_3} \times U(1)_{\lambda_8} \times \cdots \times U(1)_{\lambda_{N^2-1}}.\end{aligned} \tag{4.4}$$

One may also notice that $\mathcal{M}_{\mathrm{ms}}$ is stratified by noting that

$$\frac{SU(N)}{(U(1))^{N-1}} \cong \mathbb{C}P^{N-1} \ltimes \mathbb{C}P^{N-2} \ltimes \cdots \ltimes \mathbb{C}P^{1}, \tag{4.5}$$

where $\ltimes$ denotes the (possibly) non-trivial topological product of the spaces. These cosets are called flag manifolds and the given topological product follows from the fact that the $SU(N)$ groups are products of odd-dimensional spheres (see [3] and references within).

Now, in order to do any "physically" meaningful calculation on either manifold we require their measures; measures that can be derived by using the Euler angle parameterizations of $SU(N)$ and $U(N)$. It is to this question that we now turn our attention to.

4.1. Pure state measure

We know that pure states are in $\mathbb{C}P^N$ and from the previous sections that

$$\mathbb{C}P^N = \frac{SU(N+1)}{U(N)} = \frac{SU(N+1)}{SU(N) \times U(1)_{SU(N+1)}}. \tag{4.6}$$

Using the differential volume element for $SU(N)$ from [1] we can immediately write down the pure state measure as

$$\begin{aligned} \mathrm{d}V_{\mathrm{ps}} &= \frac{\mathrm{d}V_{SU(N+1)}}{\mathrm{d}V_{SU(N)} \times \mathrm{d}V_{U(1)_{SU(N+1)}}} = \frac{K_{SU(N+1)}\,\mathrm{d}\alpha_{(N+1)^2-1}\cdots\mathrm{d}\alpha_1}{K_{SU(N)}\,\mathrm{d}\alpha_{N^2-1}\cdots\mathrm{d}\alpha_1 \times \mathrm{d}\alpha_{(N+1)^2-1}} \\ &= \left(\prod_{2\le k\le N+1} Ker(k, j(N+1))\right)\mathrm{d}\alpha_{2N}\cdots\mathrm{d}\alpha_1 \end{aligned} \tag{4.7}$$

where from [1]:

$$Ker(k, j(N+1)) = \begin{cases} \sin(2\alpha_2), & k = 2, \\ \cos(\alpha_{2(k-1)})^{2k-3}\sin(\alpha_{2(k-1)}), & 2 < k < N+1, \\ \cos(\alpha_{2N})\sin(\alpha_{2N})^{2N-1}, & k = N+1, \end{cases} \tag{4.8}$$

with the following ranges:

$$\begin{aligned} &0 \le \alpha_1 \le \pi \quad \text{and} \quad 0 \le \alpha_2 \le \frac{\pi}{2}, \qquad 0 \le \alpha_{2j} \le \frac{\pi}{2}, \\ &0 \le \alpha_{2j-1} \le 2\pi \quad \text{for } 2 \le j \le N. \end{aligned} \tag{4.9}$$

Note that these ranges are from the *covering* ranges for $SU(N+1)$ and not from $SU(N+1)/Z_{N+1}$ which are used to calculate the invariant volume for $SU(N+1)$ (see the appendices in [1] for more details). On the other hand, one *could* use the $SU(N+1)/Z_{N+1}$ ranges:

$$0 \le \alpha_{2j} \le \frac{\pi}{2}, \qquad 0 \le \alpha_{2j-1} \le \pi \quad \text{for } 1 \le j \le N \tag{4.10}$$

but then one would need to add a normalization factor of 2^{N-1} in front of the product in Eq. (4.7) in order to generate the correct volume for $\mathbb{C}P^N$.

4.1.1. Example calculation: two qubit pure state measure

t is interesting to note that Eq. (4.7) for $N = 3$ is *equivalent* to the "natural" measure (referred to in [2]) derived from the Hurwitz parameterization (see [13] and references within). To begin, we define a general vector of a random four-dimensional unitary matrix $U(4)$ as

$$|\Psi(\boldsymbol{\eta})\rangle = \begin{pmatrix} \cos(\theta_3) \\ \sin(\theta_3)\cos(\theta_2)\,\mathrm{e}^{\mathrm{i}\phi_3} \\ \sin(\theta_3)\sin(\theta_2)\cos(\theta_1)\,\mathrm{e}^{\mathrm{i}\phi_2} \\ \sin(\theta_3)\sin(\theta_2)\sin(\theta_1)\,\mathrm{e}^{\mathrm{i}\phi_1} \end{pmatrix}, \tag{4.11}$$

where $0 \leq \theta_i \leq \pi/2$ and $0 \leq \phi_i \leq 2\pi$ $(i = 1, 2, 3)$, and $\boldsymbol{\eta} = \{\theta_i, \phi_i\}$. From this vector one can calculate the corresponding Fubini-Study metric (here given as in [14]):

$$g_{\mu\nu} = \tfrac{1}{2}(\mathcal{F}_{\mu\nu} + \mathcal{F}_{\nu\mu}), \tag{4.12}$$

where in this case

$$\mathcal{F}_{\mu\nu}(\boldsymbol{\eta}) = \left\langle \frac{\partial}{\partial\eta_\mu}\Psi(\boldsymbol{\eta})|(\mathbb{1}_4 - |\Psi(\boldsymbol{\eta})\rangle\langle\Psi(\boldsymbol{\eta})|)|\frac{\partial}{\partial\eta_\nu}\Psi(\boldsymbol{\eta})\right\rangle, \tag{4.13}$$

the square root of the determinant of which yields the invariant measure for $\mathbb{C}P^3$ *under this representation*:

$$\begin{aligned} \mathrm{d}V_{\mathrm{ps}} &= \mathrm{Det}[\sqrt{g}] \\ &= \cos(\theta_1)\sin(\theta_1)\cos(\theta_2)\sin(\theta_2)^3\cos(\theta_3)\sin(\theta_3)^5\,\mathrm{d}\theta_3\,\mathrm{d}\phi_3\cdots\mathrm{d}\theta_1\,\mathrm{d}\phi_1. \end{aligned} \tag{4.14}$$

The *equivalent* aspect of our statement comes in when one explicitly evaluates Eq. (4.7) for $N = 3$:

$$\begin{aligned} \mathrm{d}V_{\mathrm{ps}} &= \frac{\mathrm{d}V_{SU(4)}}{\mathrm{d}V_{SU(3)} \times \mathrm{d}V_{U(1)_{SU(4)}}} = \left(\prod_{2\leq k\leq 4} Ker(k, j(4))\right)\mathrm{d}\alpha_6\cdots\mathrm{d}\alpha_1 \\ &= \sin(2\alpha_2)\cos(\alpha_4)^3\sin(\alpha_4)\cos(\alpha_6)\sin(\alpha_6)^5\,\mathrm{d}\alpha_6\cdots\mathrm{d}\alpha_1 \\ &= 2\sin(\alpha_2)\cos(\alpha_2)\cos(\alpha_4)^3\sin(\alpha_4)\cos(\alpha_6)\sin(\alpha_6)^5\,\mathrm{d}\alpha_6\cdots\mathrm{d}\alpha_1, \end{aligned} \tag{4.15}$$

where the ranges on the α_i's are from Eq. (4.9). Obviously there's some contradiction between this measure and the one given in Eq. (4.14) but any concern it may raise should be eliminated in the following work.

To begin we note that Eq. (4.15) can also be derived in the following manner that follows the arguments found in [15]. First we define a pure state as

$$\rho'_{\mathrm{d}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{4}(\mathbb{1}_4 - \sqrt{6}\lambda_{15}) \tag{4.16}$$

and then apply a $U \in SU(4)$ to yield

$$\rho = U\rho'_{\mathrm{d}}U^\dagger = \tfrac{1}{4}(\mathbb{1}_4 - \sqrt{6}U\lambda_{15}U^\dagger). \tag{4.17}$$

Recalling that a general two qubit density matrix has the form:

$$\rho = |\Phi(\boldsymbol{\alpha})\rangle\langle\Phi(\boldsymbol{\alpha})| = \tfrac{1}{4}(\mathbb{1}_4 + \sqrt{6}\mathbf{n}\cdot\boldsymbol{\lambda}) \tag{4.18}$$

we can therefore solve for the components of $\mathbf{n}$ and in turn $\Phi(\boldsymbol{\alpha})$, via evaluating $n_j = \Phi^\dagger\lambda_j\Phi$ for $j = 1, \ldots, 15$. Doing these calculations yields (dropping an overall multiplicative phase term dependent on the λ_{15} element of the Cartan subalgebra found in $U(3)$):

$$|\Phi(\boldsymbol{\alpha})\rangle = \begin{pmatrix} \sin(\alpha_6)\cos(\alpha_4)\cos(\alpha_2)\,\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_3+\alpha_5)} \\ -\sin(\alpha_6)\cos(\alpha_4)\sin(\alpha_2)\,\mathrm{e}^{\mathrm{i}(\alpha_1-\alpha_3-\alpha_5)} \\ -\sin(\alpha_6)\sin(\alpha_4)\,\mathrm{e}^{-\mathrm{i}\alpha_5} \\ \cos(\alpha_6) \end{pmatrix}. \tag{4.19}$$

Calculating and taking the determinant of the Fubini-Study metric as before but now *under this representation* yields the following invariant measure for $\mathbb{C}P^3$:

$$\mathrm{d}V_{\mathrm{ps}} = \sin(2\alpha_2)\sin(\alpha_4)\cos(\alpha_4)^3\sin(\alpha_6)^5\cos(\alpha_6). \tag{4.20}$$

One can see that $|\Phi(\boldsymbol{\alpha})\rangle$ is similar to $|\Psi(\boldsymbol{\eta})\rangle$ but not equal. Therefore in comparing the two measures we can only note the following:

(1) The factor of 2 in Eq. (4.15) equates to having the range of α_1 run from 0 to 2π rather than its original range set $0 \le \alpha_1 \le \pi$ given in Eq. (4.9) thus allowing one to *conceptually* equate θ_i with α_{2i} and ϕ_i with a functional form of the α_{2i-1}'s.

(2) Eq. (4.14) can be generalized to $\mathbb{C}P^N$ (see [13]):

$$\mathrm{d}V_{\mathrm{ps}} = \prod_{k=1}^{N-1}\cos(\theta_k)\sin(\theta_k)^{2k-1}\,\mathrm{d}\theta_k\,\mathrm{d}\phi_k \tag{4.21}$$

which obviously *does not* have the same form as Eq. (4.7), but due to the invariance of the integral:

$$\int_0^{\pi/2}\sin(\xi)^m\cos(\xi)\,\mathrm{d}\xi = \int_0^{\pi/2}\sin(\xi)\cos(\xi)^m\,\mathrm{d}\xi \tag{4.22}$$

does yield the same invariant volume (see the next section and [13]).

Thus the difference between the two pure state measures is just in the way one initially chooses the distribution of the angles $\boldsymbol{\eta}$ and $\boldsymbol{\alpha}$ in the space $\mathbb{C}P^3$ (and in $\mathbb{C}P^N$ in general). Since we are most concerned with unitary operators in $SU(N)$ acting upon pure state density matrices and not within the more general $U(N)$ group, we feel that our representation of the pure state measure is more useful with regards to the overall Euler parameterization of $SU(N)$ and $U(N)$ than the one given in Eq. (4.21).

4.2. Mixed-state product measure

From Eq. (4.3) we can see that, in general, one can write down the mixed-state product measure for $\rho = U\rho_{\mathrm{d}}U^\dagger$ as

$$\mathrm{d}V_{\mathrm{ms}} = \mathrm{d}\mu \times \mathrm{d}\left(\frac{G}{H}\right), \tag{4.23}$$

where $\mathrm{d}\mu$ defines a measure in the $(N-1)$-dimensional symplex of eigenvalues of ρ_{d} and $\mathrm{d}(G/H)$, where $G = SU(N)$ and $H = U(1)_{SU(2)} \times U(1)_{SU(3)} \times \cdots \times U(1)_{SU(N)}$, defines a "truncated" Haar measure which is responsible for the choice of eigenvectors of ρ that ensures $\mathrm{d}\mu$ is "rotational invariant."

Now as [4,5,13,16–19] and others have noted, $\mathrm{d}\mu$ is defined via the probability distribution induced on the $(N-1)$-dimensional symplex but there can be more than one possible $\mathrm{d}\mu$ that is applicable for a given system since there can be more than one usable probability distribution. As Hall noted:

> ... [I]f [mixed states] described by density operators are allowed, the requirement of unitary invariance (thus there is no preferred measurement basis for extracting information) only implies that the probability measure over the set of possible states is a function of the density operator eigenvalue spectrum alone. Hence a unique probability measure can be specified only via some further principle or restriction, to be motivated on physical or conceptual grounds [20].

Therefore

> An ensemble of general states of a quantum system is in general described by a probability measure over the density operators of the system. given that probability measures transform in the same way as volume elements under coordinate transformations, and that volume elements are in general properties of metric spaces, this suggests that the distribution of density operators corresponding to a "minimal knowledge" (i.e. most random ensemble of possible states) ensemble may be obtained from the normalized volume element induced by some natural metric on the set of density matrices [20].

Thus, the volume measure is defined by the choice of metric, and since the metric is invariant under unitary transformations, defining $\mathrm{d}\mu$ comes down to determining which metric is the most appropriate in defining a statistical distance between two density matrices; especially when one adds the additional requirement that the metric satisfy certain criteria for entanglement measures (see for example [21–25] and references within).

Since there are multiple choices for distance measures between two density matrices, and therefore $\mathrm{d}\mu$, and since we want to keep our discussion as general as possible, we shall defer further discussion on $\mathrm{d}\mu$ to other papers (for example those previously cited) and just use the most general form of $\mathrm{d}\mu$ in the spirit equation (4.3) and given in [5,17,18] and references within; the Dirichlet distribution:

$$\mathrm{d}\mu = \frac{\Gamma(s_1+\cdots+s_N)}{\Gamma(s_1)\cdots\Gamma(s_N)}\Lambda_1^{s_1-1}\cdots\Lambda_{N-1}^{s_{N-1}-1}\left(1-\sum_{j=1}^{N-1}\Lambda_j\right)^{s_N-1}\mathrm{d}\Lambda_1\cdots\mathrm{d}\Lambda_{N-1}, \tag{4.24}$$

where $\sum\Lambda_j = 1$ and $1>\Lambda_j>0$ are just the eigenvalues of ρ_{d}. The Dirichlet distribution provides a means of expressing quantities that vary randomly, independent of each other, yet obeying the condition that there sum remains fixed. In our case, $s_j \equiv s > 0$ thus

$$\mathrm{d}\mu = \frac{\Gamma(Ns)}{N\Gamma(s)}\Lambda_1^{s-1}\cdots\Lambda_N^{s-1}\,\mathrm{d}\Lambda_1\cdots\mathrm{d}\Lambda_N = \alpha_s\Lambda_1^{s-1}\cdots\Lambda_N^{s-1}\,\mathrm{d}\Lambda_1\cdots\mathrm{d}\Lambda_N, \tag{4.25}$$

where the ranges for the Λ_j, we conjecture, are equal to

$$1\geq\Lambda_N\geq\frac{1}{N}\quad\text{and}\quad 0\leq\Lambda_2,\ldots,\Lambda_{N-1}\leq\frac{1}{N}. \tag{4.26}$$

These ranges disagree with those given by Slater in [26] for $N = 4$ but under integration, the difference between those in [26] and ours is a multiplicative factor of N upon the kernel. The benefit of the ranges given here is that they are easily generalized, while those in [26] are not.

With $\mathrm{d}\mu$ so defined, we are now free to look at the flag manifold G/H. By using Eqs. (2.2) and (2.7) we can see that the coset G/H can be expressed as

$$\begin{aligned}\frac{G}{H} &= \frac{SU(N)}{U(1)_{SU(2)} \times U(1)_{SU(3)} \times \cdots \times U(1)_{SU(N)}} \\ &= \frac{\prod_{N \geq m \geq 2} \left(\prod_{2 \leq k \leq m} A(k, j(m))\right) \mathrm{e}^{\mathrm{i}\lambda_3 \alpha_{N^2-(N-1)}} \cdots \mathrm{e}^{\mathrm{i}\lambda_{(N-1)^2-1} \alpha_{N^2-2}} \, \mathrm{e}^{\mathrm{i}\lambda_{N^2-1} \alpha_{N^2-1}}}{\mathrm{e}^{\mathrm{i}\lambda_3 \alpha_{N^2-(N-1)}} \cdots \mathrm{e}^{\mathrm{i}\lambda_{(N-1)^2-1} \alpha_{N^2-2}} \, \mathrm{e}^{\mathrm{i}\lambda_{N^2-1} \alpha_{N^2-1}}} \\ &= \prod_{N \geq m \geq 2} \left(\prod_{2 \leq k \leq m} A(k, j(m))\right), \end{aligned} \tag{4.27}$$

where $A(k, j(m))$ is defined in Eq. (2.7). This coset representation comes from the following observation; it allows us to write down the "truncated" Haar measure $\mathrm{d}(G/H)$ as

$$\begin{aligned}\mathrm{d}\left(\frac{G}{H}\right) &= \mathrm{d}\left(\frac{SU(N)}{U(1)_{SU(2)} \times U(1)_{SU(3)} \times \cdots \times U(1)_{SU(N)}}\right) \\ &= \frac{\mathrm{d}V_{SU(N)}}{\mathrm{d}V_{U(1)_{SU(2)}} \times \mathrm{d}V_{U(1)_{SU(3)}} \times \cdots \times \mathrm{d}V_{U(1)_{SU(N)}}} \\ &= \frac{K_{SU(N)} \, \mathrm{d}\alpha_{N^2-1} \cdots \mathrm{d}\alpha_1}{\mathrm{d}\alpha_{N^2-(N-1)} \cdots \mathrm{d}\alpha_{N^2-1}} = K_{SU(N)} \mathrm{d}\alpha_{N(N-1)} \cdots \mathrm{d}\alpha_1, \end{aligned} \tag{4.28}$$

where from [1]:

$$\begin{aligned} K_{SU(N)} &= \prod_{N \geq m \geq 2} \left(\prod_{2 \leq k \leq m} Ker(k, j(m))\right), \\ Ker(k, j(m)) &= \begin{cases} \sin(2\alpha_{2+j(m)}), & k = 2, \\ \cos(\alpha_{2(k-1)+j(m)})^{2k-3} \sin(\alpha_{2(k-1)+j(m)}), & 2 < k < m, \\ \cos(\alpha_{2(m-1)+j(m)}) \sin(\alpha_{2(m-1)+j(m)})^{2m-3}, & k = m \end{cases} \end{aligned} \tag{4.29}$$

and $j(m)$ is from Eq. (2.7).

Now the ranges for the α's can again be either the *covering* ranges defined for the first $N(N-1)$ α's of the Euler parameterization of $SU(N)$ (see [1]) *or* the $SU(N)/Z_N$ ranges:

$$0 \leq \alpha_{2j} \leq \frac{\pi}{2}, \qquad 0 \leq \alpha_{2j-1} \leq \pi \quad \text{for } 1 \leq j \leq \frac{N(N-1)}{2}, \tag{4.30}$$

which would necessitate adding a normalization factor of $2^{(N-1)(N-2)/2}$ to $K_{SU(N)}$ in Eq. (4.28).

Depending on which set of ranges are used, a general mixed-state product measure can thus be written as

$$\begin{aligned} \mathrm{d}V_{\mathrm{ms}} &= \alpha_s \Lambda_1^{s_1-1} \cdots \Lambda_{N-1}^{s_{N-1}-1} \left(1 - \sum_{j=1}^{N-1} \Lambda_j\right)^{s_N-1} \mathrm{d}\Lambda_1 \cdots \mathrm{d}\Lambda_{N-1} \\ &\quad \times \xi \cdot K_{SU(N)} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{N(N-1)}, \end{aligned} \tag{4.31}$$

where the Λ_i are the *non-zero* eigenvalues of the corresponding N-dimensional diagonal density matrix ρ_{d} (see [1,4,5,11,13,26,27] for more details) and ξ is the necessary normalization constant (equal to 1 if one uses the covering ranges for $SU(N)$ and equal to $2^{(N-1)(N-2)/2}$ if one uses the generic $SU(N)/Z_N$ coset ranges used in calculating the group volume [1,2]).

4.2.1. *Example calculation: two qubit mixed-state product measure*

For two qubits, Eqs. (4.23), (4.25) and (4.28) yield

$$\begin{aligned} \mathrm{d}V_{\mathrm{ms}} &= \mathrm{d}\mu \times \mathrm{d}\left(\frac{SU(4)}{U(1)_{SU(2)} \times U(1)_{SU(3)} \times U(1)_{SU(4)}}\right) \\ &= \alpha_s \Lambda_1^{s-1} \Lambda_2^{s-1} \Lambda_3^{s-1} \left(1 - \sum_{i=1}^{3} \Lambda_i\right)^{s-1} \mathrm{d}\Lambda_1 \cdots \mathrm{d}\Lambda_3 \times \xi \cdot K_{SU(4)}\, \mathrm{d}\alpha_{12} \cdots \mathrm{d}\alpha_1 \\ &= \alpha_s \Lambda_1^{s-1} \Lambda_2^{s-1} \Lambda_3^{s-1} \Lambda_4^{s-1}\, \mathrm{d}\Lambda_1 \cdots \mathrm{d}\Lambda_4 \\ &\quad \times \xi \cdot \sin(2\alpha_2) \sin(\alpha_4) \cos(\alpha_4)^3 \sin(\alpha_6)^5 \cos(\alpha_6) \\ &\quad \times \sin(2\alpha_8) \sin(\alpha_{10})^3 \cos(\alpha_{10}) \sin(2\alpha_{12})\, \mathrm{d}\alpha_{12} \cdots \mathrm{d}\alpha_1, \end{aligned} \tag{4.32}$$

where we have used the $SU(4)$ differential volume element from [2] in the last step. The ranges of integration for the α_i parameters has already been discussed; ideally they should be the *covering* ranges for $SU(4)$ from [2] so that $\xi = 1$. As for the ranges on $\mathrm{d}\mu$, recall that for two qubits, ρ_{d} is given by [1,2]:

$$\rho_{\mathrm{d}} = \begin{pmatrix} \sin^2(\theta_1)\sin^2(\theta_2)\sin^2(\theta_3) & 0 & 0 & 0 \\ 0 & \cos^2(\theta_1)\sin^2(\theta_2)\sin^2(\theta_3) & 0 & 0 \\ 0 & 0 & \cos^2(\theta_2)\sin^2(\theta_3) & 0 \\ 0 & 0 & 0 & \cos^2(\theta_3) \end{pmatrix}, \tag{4.33}$$

where

$$\frac{\pi}{4} \le \theta_1 \le \frac{\pi}{2}, \qquad \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \le \theta_2 \le \frac{\pi}{2}, \qquad \frac{\pi}{3} \le \theta_3 \le \frac{\pi}{2}, \tag{4.34}$$

thus we have

$$1 \ge \Lambda_4 \ge \tfrac{1}{4}, \qquad 0 \le \Lambda_1 \le \tfrac{1}{4}, \qquad 0 \le \Lambda_2 \le \tfrac{1}{4}, \qquad 0 \le \Lambda_3 \le \tfrac{1}{4} \tag{4.35}$$

for the ranges of integration on $\mathrm{d}\mu$. Notice that one could have also used Eq. (4.26) with $N = 4$ to achieve these ranges, but it is instructive to see their explicit derivation.

5. Volume of $\mathbb{C}P^N$ and $SU(N)/(U(1))^{N-1}$

Now we are in a position to give two different methods for calculating the volume for the pure and mixed-state manifolds of general N-dimensional quantum systems. The pure state manifold volume is quite simple and already well known; it is the volume of $\mathbb{C}P^N$, while the mixed-state manifold volume, as we have seen in the previous section, is the product of two different measures—one of which is dependent on the initial distribution of states on the $(N-1)$-dimensional symplex. Therefore in the mixed-state case we shall only worry about calculating the volume contribution from the second measure; the "truncated" Haar measure, since the volume from the $N-1$ symplex measure can be equated to a general multiplicative constant determined by the initial distribution of states on the $N-1$ symplex (see for example [26]):

$$V_{\text{mixed states}} = V_{\text{symplex}} \times V_{SU(N)/(U(1))^{N-1}} = \omega V_{SU(N)/(U(1))^{N-1}}. \tag{5.1}$$

For example, for the two qubit case described previously, a naive calculation of ω can be seen to be equal to:

$$\begin{aligned}\omega &= \alpha_s \int_{1/4}^{1}\int_0^{1/4}\int_0^{1/4}\int_0^{1/4} \Lambda_1^{s-1}\Lambda_2^{s-1}\Lambda_3^{s-1}\Lambda_4^{s-1}\,\mathrm{d}\Lambda_1\,\mathrm{d}\Lambda_2\,\mathrm{d}\Lambda_3\,\mathrm{d}\Lambda_4\\ &= \frac{\Gamma(4s)}{4\Gamma(2)}\left(\frac{4^{-4s}(-1+4^s)}{s^4}\right)\end{aligned} \tag{5.2}$$

for when $s > 0$ (note that the integration ranges on Λ_4 were reversed in order to keep ω positive).

5.1. Volume of $\mathbb{C}P^N$

Using the results for $U(N)$ we can immediately write down the general volume for $\mathbb{C}P^N$. Using Eqs. (2.1), (3.1) and (3.4) we have

$$V_{\mathbb{C}P^N} = \frac{V_{SU(N+1)}}{V_{U(N)}} = \frac{2^{N/2}\pi^{N(N+3)/2}\sqrt{N+1}\prod_{k=1}^{N}(1/k!)}{2^{N/2}\pi^{N(N+1)/2}\sqrt{N+1}\prod_{k=1}^{N-1}(1/k!)} = \frac{\pi^N}{N!}. \tag{5.3}$$

We see that this result also comes from the integration of Eq. (4.7) over the ranges given in Eq. (4.9) or Eq. (4.10):

$$\begin{aligned}&\int\cdots\int_{\alpha\text{ ranges}}\left(\prod_{2\le k\le N+1} Ker(k, j(N+1))\right)\mathrm{d}\alpha_{2N}\cdots\mathrm{d}\alpha_1\\ &\quad= 2^{N-1}\pi^N\prod_{2\le k\le N+1}\mathbb{V}(k, N+1),\end{aligned} \tag{5.4}$$

where $\mathbb{V}(k, N+1)$ is, from [1]:

$$\mathbb{V}(k, N+1) = \begin{cases} 1, & k=2,\\ \dfrac{1}{2(k-1)}, & 2<k\le N+1.\end{cases} \tag{5.5}$$

Expansion of this product yields

$$\prod_{2\le k\le N+1} \mathbb{V}(k, N+1) = 1 \times \frac{1}{4} \times \frac{1}{6} \times \cdots \times \frac{1}{2N} = \frac{1}{2^{N-1}N!} \tag{5.6}$$

which when multiplied by the $2^{N-1}\pi^N$ factor gives the volume for $\mathbb{C}P^N$ as previously calculated. One can also see that integration of Eq. (4.21) using the ranges given for $\boldsymbol{\eta}$ in Eq. (4.11) will also generate Eq. (5.3) (see [13] for example).

We should also note the remarkable results:

$$\sum_{n=0}^{\infty} \mathrm{Vol}(\mathbb{C}P^n) = \sum_{n=0}^{\infty} \frac{\pi^n}{n!} = \mathrm{e}^{\pi} \approx 23.147, \qquad \lim_{k\to\infty} \prod_{n=0}^{k} \mathrm{Vol}(\mathbb{C}P^n) \to 0. \tag{5.7}$$

Thus, in terms of our pure state manifold discussion, we can conclude that as one increases the dimensionality of the system, there will always be a non-zero pure state volume. In the spirit of this result we should also note an interesting introduction to the importance of the pure state manifold $\mathbb{C}P^N$ for large N, especially with regard to quantum entanglement, can be found in [28].

5.2. Volume of $SU(N)/(U(1))^{N-1}$

Recall that the action of a group in the adjoint representation produces interesting orbits; the manifolds of which are called *generalized flag manifolds*, and appear very often in geometric quantization, density matrices, entangled states, etc. These manifolds can be represented by the coset $SU(N)/U(1)^{N-1}$; obviously then the volume of such manifolds are quite important to our work. By using Eqs. (3.2) and (4.4) we can write down the general volume for such a coset, $SU(N)/U(1)_{SU(2)} \times U(1)_{SU(3)} \times \cdots \times U(1)_{SU(N)}$, as

$$\begin{aligned} V&\left(\frac{SU(N)}{U(1)_{\lambda_3} \times U(1)_{\lambda_8} \times \ldots \times U(1)_{\lambda_{N^2-1}}}\right) \\ &= \frac{V_{SU(N)}}{V_{U(1)_{\lambda_3}} \times V_{U(1)_{\lambda_8}} \times \cdots \times V_{U(1)_{\lambda_{N^2-1}}}} \\ &= \frac{2^{(N-1)/2}\pi^{(N-1)(N+2)/2}\sqrt{N}\prod_{k=1}^{N-1}(1/k!)}{2\pi * \sqrt{3}\pi * \cdots * \pi\sqrt{2N/(N-1)}} \\ &= \frac{2^{(N-1)/2}\pi^{(N-1)(N+2)/2}\sqrt{N}\prod_{k=1}^{N-1}(1/k!)}{\prod_{l=1}^{N-1}\pi\sqrt{2(l+1)/l}} = \frac{\pi^{N(N-1)/2}\sqrt{N}\prod_{k=1}^{N-1}(1/k!)}{\prod_{l=1}^{N-1}\sqrt{(l+1)/l}}. \end{aligned} \tag{5.8}$$

But we know

$$\prod_{l=1}^{N-1}\sqrt{\frac{l+1}{l}} = \sqrt{\frac{2}{1}} * \sqrt{\frac{3}{2}} * \cdots * \sqrt{\frac{N}{N-1}} = \sqrt{N}. \tag{5.9}$$

Thus we can see that

$$V\left(\frac{SU(N)}{U(1)_{\lambda_3}\times U(1)_{\lambda_8}\times\cdots\times U(1)_{\lambda_{N^2-1}}}\right)=\pi^{N(N-1)/2}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right), \tag{5.10}$$

which is in agreement with [3]. This volume can also be generated via integrating the "truncated" Haar measure, using the appropriate ranges and normalization conditions, given in Eq. (4.28) (done in detail in [1]).

For completeness, and with Eq. (4.5) in mind, we should note the following:

$$\pi^{N(N-1)/2}\equiv\prod_{k=1}^{N-1}\pi^k. \tag{5.11}$$

Thus Eq. (5.10) has the following, equivalent representation:

$$V\left(\frac{SU(N)}{U(1)_{\lambda_3}\times U(1)_{\lambda_8}\times\cdots\times U(1)_{\lambda_{N^2-1}}}\right)=\pi^{N(N-1)/2}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right)=\prod_{k=1}^{N-1}\frac{\pi^k}{k!}=\prod_{k=1}^{N-1}V_{\mathbb{C}P^k}. \tag{5.12}$$

So the volume of our flag manifold is nothing more than the product of the volumes of the complex projective space $\mathbb{C}P^k$ where $k\le N$ [3]. Notice also that as N increases the volume of the flag manifold approaches, but *never equals*, zero (see Eq. (5.7)). It is an asymptotic limit which converges to zero from the left on $\mathbb{R}^1$. Thus, since one *usually* chooses a non-zero probability distribution on the $N-1$ symplex defining $\mathrm{d}\mu$ (see Eq. (4.25)), we can conclude that, as in the pure state case, the mixed-state volume measure will never equal zero *unless* V_{symplex} *does*!

6. Other $SU(N)$ and $U(N)$ coset volumes

Beyond the full pure and mixed-state manifolds there are numerous other sub-manifolds that are of interest in physics; the volumes of which have already been calculated (see, for example [3,10,19,29–31] and references within). These sub-manifolds and their volumes give us both a way to confirm our methodology, as well as offering a systematic, rather than numeric, way of computing such quantities. From this work, we will then be able to calculate the manifolds that contain the set of entangled and mixed states (either pure or mixed) for specific quantum systems [32]. It should be understood though that the following volume calculations are specific to the $SU(N)$ and $U(N)$ Euler angle parameterization that we have developed and its corresponding normalizations via the Cartan subalgebra being used. The general question of volume normalization of a manifold, especially when one begins to talk about coset manifolds with specific elements of the Cartan subalgebra being removed will be the subject of a future paper.

6.1. Volume of $SU(N)/SU(P) \times SU(Q)$

To begin, we would like to be able to write down the general volume of the $SU(N)/SU(P) \times SU(Q)$ coset where $N+1 \geq P+Q$ and $P, Q \neq 1$. To do this we can use Eq. (3.1) to generate

$$\frac{V_{SU(N)}}{V_{SU(P)} \times V_{SU(Q)}} = \frac{2^{(N-1)/2}\pi^{(N-1)(N+2)/2}\sqrt{N}\prod_{k=1}^{N-1}(1/k!)}{2^{(P-1)/2}\pi^{(P-1)(P+2)/2}\sqrt{P}\prod_{k=1}^{P-1}(1/k!)} \times 2^{(Q-1)/2}\pi^{(Q-1)(Q+2)/2}\sqrt{Q}\prod_{k=1}^{Q-1}(1/k!)$$
$$= 2^{((N+1)-(P+Q))/2}\pi^{(N(N+1)-P(P+1)-Q(Q+1)+2)/2} \times \sqrt{\frac{N}{PQ}}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right)\prod_{k=1}^{P-1}k!\prod_{k=1}^{Q-1}k!. \tag{6.1}$$

When $N+1 = P+Q$ we have

$$\frac{V_{SU(N)}}{V_{SU(P)} \times V_{SU(Q)}} = 2^{((N+1)-(N+1))/2}\pi^{((P+Q-1)(P+Q)-P(P+1)-Q(Q+1)+2)/2} \times \sqrt{\frac{P+Q-1}{PQ}}\prod_{k=1}^{P+Q-2}\left(\frac{1}{k!}\right)\prod_{k=1}^{P-1}k!\prod_{k=1}^{Q-1}k! = \pi^{(P-1)(Q-1)} \times \sqrt{\frac{P+Q-1}{PQ}}\prod_{k=1}^{P+Q-2}\left(\frac{1}{k!}\right)\prod_{k=1}^{P-1}k!\prod_{k=1}^{Q-1}k!. \tag{6.2}$$

6.1.1. Example calculation: volume of $SU(4)/SU(2) \times SU(2)$

Defining $N = 4$, and $P = Q = 2$, we get from Eq. (6.1) the volume of the coset $SU(4)/SU(2) \times SU(2)$:

$$\frac{V_{SU(4)}}{V_{SU(2)} \times V_{SU(2)}} = 2^{(5-4)/2}\pi^{(4(5)-2(3)-2(3)+2)/2}\sqrt{\frac{4}{(2)(2)}}\prod_{k=1}^{4-1}\left(\frac{1}{k!}\right)\prod_{k=1}^{2-1}k!\prod_{k=1}^{2-1}k!$$
$$= \sqrt{2} * \pi^5 * \frac{1}{3!} * 1! * 1! = \frac{\pi^5}{6\sqrt{2}}. \tag{6.3}$$

Which is equivalent to the volume of $SU(4)$, $\sqrt{2}\pi^9/3$, divided by the square of the volume of $SU(2)$, $2\pi^2$, as expected. It should also be noted that this is the volume of the manifold that is comprised of all *non-local* transformations which can be implemented on a two qubit system.

6.2. Volume of $SU(N)/U(P) \times U(1)$

Beyond the general volume of $\mathbb{C}P^N$, general flag manifold, and the previous $SU(N)$ coset, we would like to be able to write down the general volume of the $SU(N)/U(P) \times U(1)$ coset where $N-1 \geq P+1$ and $P \neq 1$. To do this we can use Eqs. (3.1) and (3.4) as follows:

$$\frac{V_{SU(N)}}{V_{U(P)} \times V_{U(1)}} = \frac{2^{(N-1)/2}\pi^{(N-1)(N+2)/2}\sqrt{N}\prod_{k=1}^{N-1}(1/k!)}{2^{P/2}\pi^{P(P+1)/2}\sqrt{P+1}\prod_{k=1}^{P-1}(1/k!) \times V_{U(1)}}. \tag{6.4}$$

The problem we now face is how to define $U(1)$. If we use Eq. (3.2), here now defined for $SU(N)$, we would generate

$$\begin{aligned}\frac{V_{SU(N)}}{V_{U(P)} \times V_{U(1)_{SU(N)}}} &= \frac{2^{(N-1)/2}\pi^{(N-1)(N+2)/2}\sqrt{N}\prod_{k=1}^{N-1}(1/k!)}{2^{P/2}\pi^{P(P+1)/2}\sqrt{P+1}\prod_{k=1}^{P-1}(1/k!) * \pi\sqrt{2N/(N-1)}} \\ &= \frac{2^{(N-1)/2}\pi^{(N-1)(N+2)/2}\sqrt{N-1}\prod_{k=1}^{N-1}(1/k!)}{2^{(P+1)/2}\pi^{(P^2+P+2)/2}\sqrt{P+1}\prod_{k=1}^{P-1}(1/k!)} \\ &= 2^{((N-1)-(P+1))/2}\pi^{((N^2+N-2)-(P^2+P+2))/2} \\ &\quad \times \sqrt{\frac{N-1}{P+1}}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right)\prod_{k=1}^{P-1}k!.\end{aligned} \tag{6.5}$$

If we demand that $N-1 = P+1$, we can simplify the product terms:

$$\begin{aligned}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right)\prod_{k=1}^{P-1}k! &= \prod_{k=1}^{P+1}\left(\frac{1}{k!}\right)\prod_{k=1}^{P-1}k! \\ &= \frac{1! * 2! * \cdots * (P-2)! * (P-1)!}{1! * 2! * \cdots * (P-2)! * (P-1)! * P * (P+1)!} \\ &= \frac{1}{P!(P+1)!}\end{aligned} \tag{6.6}$$

as well as the powers and other factors. Therefore, for this case we have

$$\begin{aligned}\frac{V_{SU(N)}}{V_{U(P)} \times V_{U(1)_{SU(N)}}} &= \frac{2^{((N-1)-(P+1))/2}\pi^{((N^2+N-2)-(P^2+P+2))/2}}{P!(P+1)!}\sqrt{\frac{N-1}{P+1}} \\ &= \frac{\pi^{2N-3}}{(N-2)!(N-1)!} = \frac{\pi^{2P+1}}{P!(P+1)!}.\end{aligned} \tag{6.7}$$

Depending on which parameter is used.

Now, if in using Eq. (3.2), we now define $U(1)$ for $SU(M)$, $M < N$, we would generate

$$\begin{aligned}\frac{V_{SU(N)}}{V_{U(P)} \times V_{U(1)_{SU(M)}}} &= \frac{2^{(N-1)/2}\pi^{(N-1)(N+2)/2}\sqrt{N}\prod_{k=1}^{N-1}(1/k!)}{2^{P/2}\pi^{P(P+1)/2}\sqrt{P+1}\prod_{k=1}^{P-1}(1/k!) * \pi\sqrt{2M/(M-1)}} \\ &= 2^{((N-1)-(P+1))/2}\pi^{((N^2+N-2)-(P^2+P+2))/2} \\ &\quad \times \sqrt{\frac{N(M-1)}{M(P+1)}}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right)\prod_{k=1}^{P-1}k!\end{aligned} \tag{6.8}$$

and if we demand $N-1=P+1$, we can simplify, yielding

$$\begin{aligned} &= \frac{2^{((N-1)-(P+1))/2}\pi^{((N^2+N-2)-(P^2+P+2))/2}}{P!(P+1)!}\sqrt{\frac{N(M-1)}{M(P+1)}} \\ &= \frac{\pi^{2N-3}}{(N-2)!(N-1)!}\sqrt{\frac{N(M-1)}{M(N-1)}}, \end{aligned} \tag{6.9}$$

which reduces to Eq. (6.7) when $M=N$. Therefore, depending on which $U(1)$ we use, we will generate a different volume; the ratio between any two being equal to

$$\frac{V_{SU(N)}/V_{U(P)}\times V_{U(1)_{SU(X)}}}{V_{SU(N)}/V_{U(P)}\times V_{U(1)_{SU(Y)}}} = \sqrt{\frac{Y(X-1)}{X(Y-1)}}. \tag{6.10}$$

6.2.1. Example calculation: volumes of $SU(4)/U(2)\times U(1)_{SU(i)}$ for $i=2,3,4$

Defining $N=4$ and $P=2$ (thus satisfying $N-1=P+1$) we get from Eq. (6.9) the volume of the coset $SU(4)/U(2)\times U(1)_{SU(i)}$ when $i=2$:

$$\frac{V_{SU(4)}}{V_{U(2)}\times V_{U(1)_{SU(2)}}} = \frac{\pi^{2*4-3}}{(4-2)!(4-1)!}\sqrt{\frac{4(2-1)}{2(4-1)}} = \frac{\pi^5}{12}\sqrt{\frac{2}{3}} = \frac{\pi^5}{6\sqrt{6}}, \tag{6.11}$$

when $i=3$

$$\frac{V_{SU(4)}}{V_{U(2)}\times V_{U(1)_{SU(3)}}} = \frac{\pi^{2*4-3}}{(4-2)!(4-1)!}\sqrt{\frac{4(3-1)}{3(4-1)}} = \frac{\pi^5}{12}\sqrt{\frac{8}{9}} = \frac{\pi^5}{9\sqrt{2}} \tag{6.12}$$

and when $i=4$ the volume of the coset $SU(4)/U(2)\times U(1)_{SU(4)}$, using Eq. (6.7) now, is

$$\frac{V_{SU(4)}}{V_{U(2)}\times V_{U(1)_{SU(4)}}} = \frac{\pi^{2*4-3}}{(4-2)!(4-1)!} = \frac{\pi^5}{12}. \tag{6.13}$$

6.3. Volume of $SU(N)/U(P)\times U(Q)$

Now we would like to be able to write down the general volume of the $SU(N)/U(P)\times U(Q)$ coset for $N-1\geq P+Q$ and $P, Q\neq 1$. To do this we can use Eqs. (3.1) and (3.4) as follows:

$$\frac{V_{SU(N)}}{V_{U(P)}\times V_{U(Q)}} = \frac{2^{(N-1)/2}\pi^{(N-1)(N+2)/2}\sqrt{N}\prod_{k=1}^{N-1}(1/k!)}{2^{P/2}\pi^{P(P+1)/2}\sqrt{P+1}\prod_{k=1}^{P-1}(1/k!) \times 2^{Q/2}\pi^{Q(Q+1)/2}\sqrt{Q+1}\prod_{k=1}^{Q-1}(1/k!)}. \tag{6.14}$$

Simplification yields

$$\frac{V_{SU(N)}}{V_{U(P)}\times V_{U(Q)}} = \frac{2^{(N-1)/2}\pi^{(N-1)(N+2)/2}\sqrt{N}\prod_{k=1}^{N-1}(1/k!)}{2^{(P+Q)/2}\pi^{(P(P+1)+Q(Q+1))/2}\sqrt{3}3} \times (P+1)(Q+1)\prod_{k=1}^{P-1}(1/k!)\prod_{k=1}^{Q-1}(1/k!)$$
$$= 2^{((N-1)-(P+Q))/2}\pi^{((N-1)(N+2)-P(P+1)-Q(Q+1))/2} \times \sqrt{\frac{N}{(P+1)(Q+1)}} \times \prod_{k=1}^{N-1}\left(\frac{1}{k!}\right)\prod_{k=1}^{P-1}k!\prod_{k=1}^{Q-1}k!. \qquad (6.15)$$

For the special case when $N-1 = P+Q$ we can go further and eliminate the N dependence in the above volume, thus yielding (in one possible representation):

$$\frac{V_{SU(N)}}{V_{U(P)}\times V_{U(Q)}} = \pi^{P+Q+PQ}\sqrt{\frac{P+Q+1}{(P+1)(Q+1)}}\prod_{k=1}^{P+Q}\left(\frac{1}{k!}\right)\prod_{k=1}^{P-1}k!\prod_{k=1}^{Q-1}k!$$
$$= \pi^{P+Q+PQ}\sqrt{\frac{P+Q+1}{(P+1)(Q+1)}}\prod_{k=P}^{P+Q}\left(\frac{1}{k!}\right)\prod_{k=1}^{Q-1}k!. \qquad (6.16)$$

6.3.1. Example calculation: volume of $SU(9)/U(4)\times U(4)$

Defining $N = 9$, and $P = Q = 4$, thus satisfying $N-1 = P+Q$, we get from Eq. (6.16) the volume of the coset $SU(9)/U(4)\times U(4)$ to be equal to

$$\frac{V_{SU(9)}}{V_{U(4)}\times V_{U(4)}} = \pi^{(4+4+4*4)}\sqrt{\frac{4+4+1}{(4+1)(4+1)}}\prod_{k=4}^{4+4}\left(\frac{1}{k!}\right)\prod_{k=1}^{4-1}k$$
$$= \frac{\pi^{24}}{58\,525\,286\,400\,000}, \qquad (6.17)$$

which is what one would get if they used Eqs. (3.1) and (3.4) separately.

6.4. Volume of $SU(N)/\prod_{i=1}^{x}U(P_i)\times\prod_{j=1}^{y}U(1)_{SU(Z_j)}$

We are now ready to write down the volume for the most general of cosets that we are interested in, $SU(N)/\prod_{i=1}^{x}U(P_i)\times\prod_{j=1}^{y}U(1)_{SU(Z_j)}$, where

$$\sum_{i=1}^{x}P_i + \sum_{j=1}^{y}1 = \sum_{i=1}^{x}P_i + y \leq N-1, \qquad P_i \neq 1 \qquad (6.18)$$

and

$$U(1)_{SU(Z_j)} \in \{U(1)_{SU(2)}, U(1)_{SU(3)}, \ldots, U(1)_{SU(N)}\}, \qquad (6.19)$$

where there is no necessary order in the sequential choice of $U(1)_{SU(Z_j)}$.

To begin we note the following using Eq. (3.4):

$$V\left(\prod_{i=1}^{x} U(P_i)\right) = \prod_{i=1}^{x} V_{U(P_i)} = \prod_{i=1}^{x}\left(2^{P_i/2}\pi^{P_i(P_i+1)/2}\sqrt{P_i+1}\prod_{k=1}^{P_i-1}\left(\frac{1}{k!}\right)\right)$$
$$= 2^{\sum_{i=1}^{x} P_i/2}\pi^{\sum_{i=1}^{x} P_i(P_i+1)/2}\prod_{i=1}^{x}\left(\sqrt{P_i+1}\prod_{k=1}^{P_i-1}\left(\frac{1}{k!}\right)\right). \tag{6.20}$$

We also can simplify the second volume of the three we need via generalizing Eq. (3.2):

$$V\left(\prod_{j=1}^{y} U(1)_{SU(Z_j)}\right) = \prod_{j=1}^{y} V_{U(1)_{SU(Z_j)}} = \prod_{j=1}^{y}\pi\sqrt{\frac{2Z_j}{Z_j-1}} = \pi^y 2^{y/2}\prod_{j=1}^{y}\sqrt{\frac{Z_j}{Z_j-1}}. \tag{6.21}$$

We are now in a position to write down the volume for $SU(N)/\prod_{i=1}^{x} U(P_i) \times \prod_{j=1}^{y} U(1)_{SU(Z_j)}$. Using Eqs. (3.1), (6.18), (6.20) and (6.21) we have

$$V\left(\frac{SU(N)}{\prod_{i=1}^{x} U(P_i) \times \prod_{j=1}^{y} U(1)_{SU(Z_j)}}\right)$$
$$= 2^{(N-(1+y+\sum_{i=1}^{x} P_i))/2}\pi^{((N-1)(N+2)-(2y+\sum_{i=1}^{x} P_i(P_i+1)))/2}$$
$$\times\sqrt{N}\frac{\prod_{k=1}^{N-1}(1/k!)\prod_{j=1}^{y}\sqrt{(Z_j-1)/Z_j}}{\prod_{i=1}^{x}\left(\sqrt{P_i+1}\prod_{k=1}^{P_i-1}(1/k!)\right)}. \tag{6.22}$$

For the special case when the "$\leq$" in Eq. (6.18) is replaced by "$=$" we have

$$V\left(\frac{SU(N)}{\prod_{i=1}^{x} U(P_i) \times \prod_{j=1}^{y} U(1)_{SU(Z_j)}}\right)$$
$$= \pi^{((N-1)(N+2)-(2y+\sum_{i=1}^{x} P_i(P_i+1)))/2} \times \sqrt{N}\frac{\prod_{k=1}^{N-1}(1/k!)\prod_{j=1}^{y}\sqrt{(Z_j-1)/Z_j}}{\prod_{i=1}^{x}\left(\sqrt{P_i+1}\prod_{k=1}^{P_i-1}(1/k!)\right)}. \tag{6.23}$$

One could continue simplifying Eq. (6.22) but it would be only worthwhile if additional knowledge concerning Z_j and P_i was available.

6.5. Grassmann volume

The general Grassmann manifolds, of which $\mathbb{C}P^N$ is a special case (see Eq. (4.2)), have the following definition for $N \geq M$:

$$G(N, M) = \frac{U(N)}{U(M) \times U(N-M)}. \tag{6.24}$$

Using Eq. (3.4) we can write down the general expression for the volume of *almost* any Grassmann manifold:

$$\begin{aligned}
V_{G(N,M)} &= \frac{V_{U(N)}}{V_{U(M)} \times V_{U(N-M)}} \\
&= \frac{2^{N/2}\pi^{N(N+1)/2}\sqrt{N+1}\prod_{k=1}^{N-1}(1/k!)}{2^{M/2}\pi^{M(M+1)/2}\sqrt{M+1}\prod_{k=1}^{M-1}(1/k!) \times 2^{(N-M)/2}\pi^{(N-M)(N-M+1)/2}\sqrt{N-M+1}\prod_{k=1}^{N-M-1}(1/k!)} \\
&= \pi^{M(N-M)}\sqrt{\frac{N+1}{(M+1)(N-M+1)}}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right)\prod_{k=1}^{M-1}k!\prod_{k=1}^{N-M-1}k! \\
&= \pi^{M(N-M)}\sqrt{\frac{N+1}{(M+1)(N-M+1)}}\prod_{k=M}^{N-1}\left(\frac{1}{k!}\right)\prod_{k=1}^{N-M-1}k!. \qquad (6.25)
\end{aligned}$$

The reason for the "almost" above is that for $M = 1$ we do not regain the volume for $\mathbb{C}P^{N-1}$ that we originally calculated in Eq. (5.3):

$$\begin{aligned}
V_{\mathbb{C}P^{N-1}} \equiv V_{G(N,1)} &= \pi^{N-1}\sqrt{\frac{N+1}{(1+1)(N-1+1)}}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right)\prod_{k=1}^{N-1-1}k! \\
&= \pi^{N-1}\sqrt{\frac{N+1}{2N}}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right)\prod_{k=1}^{N-2}k! = \frac{\pi^{N-1}}{(N-1)!}\sqrt{\frac{N+1}{2N}} \neq \frac{\pi^{N-1}}{(N-1)!}.
\end{aligned} \qquad (6.26)$$

We are "off" by a factor of $\sqrt{(N+1)/2N}$ which occurs because of the following reason: Eq. (3.4), for $N = 1$, yields 2π which is correct *if* one is looking for the volume of the $SU(2)$ variant of $U(1)$ (see Eq. (3.2) for $N = 1$), but that is *not* the case for the $U(1)$ components of greater $SU(N)$ groups (again see Eq. (3.1) for $N \geq 2$), which is the case here. In Eq. (6.26) we get the factor of 2π from the $U(1)$ component, but without the additional contraction term due to the λ_{N^2-1} Cartan subalgebra component of $U(N)$ from which the $U(1)$ term is defined!

The flaw in Eq. (6.26) can also be seen from Eq. (4.2):

$$\mathbb{C}P^{N-1} \equiv G(N,1) = \frac{U(N)}{U(1)\times U(N-1)} = \frac{SU(N)\times U(1)}{U(1)\times U(N-1)} = \frac{SU(N)}{U(N-1)}. \qquad (6.27)$$

The $U(1)$ term in the numerator is the same as the $U(1)$ term in the denominator and as such, in any representation, cancels out, thus leaving the standard coset relationship for $\mathbb{C}P^{N-1}$ which, from Eq. (5.3), does yield the correct volume for $\mathbb{C}P^{N-1}$. Therefore, if we use Eq. (3.2) for the $U(1)$ term in Eq. (6.25) (combined with Eq. (3.4) for the other two terms) when $M = 1$, we will generate the correct volume for $\mathbb{C}P^{N-1}$:

$$
\begin{aligned}
V_{G(N,1)} &= \frac{V_{U(N)}}{V_{U(1)} \times V_{U(N-1)}} \\
&= \frac{2^{N/2}\pi^{N(N+1)/2}\sqrt{N+1}\prod_{k=1}^{N-1}(1/k!)}{\pi\sqrt{2(N+1)/N} \times 2^{(N-1)/2}\pi^{(N-1)(N)/2}\sqrt{N}\prod_{k=1}^{N-2}(1/k!)} \\
&= \frac{\pi^{N-1}}{(N-1)!}.
\end{aligned} \tag{6.28}
$$

Therefore, in general, if we demand that $M \neq 1$ then Eq. (6.25) will correctly produce the Grassmann manifold volumes (see [3] and references within).

7. Conclusion

Using the volume equations given herein, we are now in a position to explicitly write down the measures and volumes for the whole range of manifolds which occur in discussions concerning separability and entanglement of multi-particle systems. Therefore, this work allows us to explicitly write down the volumes of the manifolds of the local orbits of a given state $|\psi\rangle$ with respect to some transformation $U \in SU(N)$ (or more generally $U(N)$), in a manner that we hypothesize also elucidates the topology of the manifolds as well [3,31]. Applications beyond quantum information theory are also possible [3].

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