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Positive maps of density matrix and a tomographic criterion of entanglement

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Abstract

The positive and not completely positive maps of density matrices are discussed. Probability representation of spin states (spin tomography) is reviewed and $U(N)$-tomogram of spin states is presented. Unitary $U(\infty)$-group tomogram of photon state in Fock basis is constructed. Notion of tomographic purity of spin states is introduced. An entanglement criterion for multipartite spin-system is given in terms of a function depending on unitary group parameters and semigroup of positive map parameters. Some two-qubit and two-qutrit states are considered as examples of entangled states using depolarizing map semigroup.

Keywords: Unitary group; Entanglement; Tomogram; Positive map; Probability

1. Introduction

States in quantum mechanics are associated with vectors in Hilbert space [1] in the case of pure states. Mixed states are associated with density matrices [2,3]. In classical statistical mechanics, the states are associated with joint probability distributions in phase space. There is an essential difference in the concept of state in classical and quantum mechanics. This difference is clearly pointed out by the phenomenon of entanglement. The notion of entanglement [4] is related to the quantum superposition rule for the states of subsystems of a given multipartite system. For pure states, the notion of entanglement and separability can be given as follows.

If the wave function [5] of a state of a bipartite system is represented as the product of two wave functions depending on coordinates of the subsystems, the state is simply separable; otherwise, the state is simply entangled. An intrinsic approach to the entanglement measure was suggested in [6]. The measure was introduced as the distance between the system density matrix and the tensor product of the subsystem states. There are several other different characteristics and

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measures of entanglement considered by several authors [7–15]. Each of the entanglement measures describes some degree of correlation between the subsystems' properties.

The notion of entanglement is not an absolute notion for a given system but depends on the decomposition into subsystems. The same quantum state can be considered as entangled, if one kind of division of the system into subsystems is given, or as completely disentangled, if another decomposition of the system into subsystems is considered.

For instance, the state of two continuous photon quadratures can be entangled in Cartesian coordinates and disentangled in polar coordinates.

A review of different approaches to the entanglement notion and entanglement measures is given in [16] where the approach to describe entanglement and separability of composite systems is based, e.g., on entropy methods. Another detailed review of entanglement problem is given in [17]. In [18] a tomographic approach to entanglement of spin-system states was suggested.

The notion of entangled states has attracted a lot of efforts to find criterion to detect entanglement and to find quantitative characterization of entanglement. A criterion based on the use of the partial transpose transform [19] of subsystem density matrix (complex conjugation of the subsystem density matrix or its time reverse) provides the necessary and sufficient condition of separability of the system of two qubits and qubit-qutrit system [20]. Recently it was pointed out that the tomographic approach to reconstruct the Wigner function of a quantum state [21–23] can be developed to consider the positive probability distribution (tomogram) as an alternative to density matrix (or wave function) because the complete set of tomograms contains the complete information on the quantum state [24]. This representation (called probability representation) was constructed also for spin states including a bipartite system of two spins. Up to now the problem of entanglement was not discussed in the tomographic representation in detail. Some remarks on tomograms and entanglement of photon states in the process of Raman scattering were done in [25]. The tomographic approach has the advantage of dealing with positive probabilities and one deals with standard probability distributions which are positive and normalized. The positive and completely positive maps of density matrices [26,27] play an important role in studying the entanglement phenomenon and they induce specific transformation properties of the tomograms.

The aim of this work is to formulate (in view of the tomographic approach [18]) necessary and sufficient conditions of separability and entanglement of multipartite systems in terms of properties of the quantum tomogram. We apply the action of specific positive map (so-called depolarizing channel) to detect the entanglement. Then we focus on properties of entanglement and separability of a bipartite system using spin tomograms ($SU(2)$-tomograms) and tomograms of the $U(N)$-group for two qubit and two qutrit states.

The idea of the approach suggested can be summarized as following.

The positive but not completely positive linear maps of a subsystem density matrix do preserve the positivity of separable density matrices of the composite system. These maps contain also maps which do not preserve the positivity of density matrices of entangled states for the composite system. It means that the set of all linear positive maps of the subsystem density matrix will create from the initial entangled positive density matrix of composite system a set of Hermitian matrices including the matrices with negative eigenvalues. To detect the entanglement we use the tomograms of the obtained Hermitian matrices. The tomograms of state density matrices (state tomograms) are standard probabilities. In view of this the tomograms of the obtained Hermitian matrices corresponding to initial separable state preserve all the properties of the probability representation including positivity and normalization. But in case of entangled state the tomograms of the obtained Hermitian matrices can take negative values. The different behaviour of tomograms of separable and entangled states of composite systems under the action of positive maps of the subsystem density matrix provides the tomographic criterion of the separability.

2. Density operators as vectors and the Lie algebra of the unitary group

When matrices represent states it may be convenient to identify them with vectors. In this case, a

density matrix can be considered as a vector whose components satisfy additional properties. Linear transformations of the density matrices called superoperators can be interpreted as matrices acting on the vector space. It means that density matrices–vectors undergoing linear transformations are acted on by the matrices representing the action of the superoperators on the linear space. Given a rectangular matrix M with elements M_{id}, where $i = 1, 2, \ldots, n$ and $d = 1, 2, \ldots, m$, one can consider the matrix as a vector $\vec{\mathcal{M}}$ with $N = nm$ components constructed by the following rule:

$$\mathcal{M}_1 = M_{11}, \qquad \mathcal{M}_2 = M_{12}, \qquad \mathcal{M}_m = M_{1m},$$
$$\mathcal{M}_{m+1} = M_{21}, \quad \ldots, \quad \mathcal{M}_N = M_{nm}. \tag{1}$$

The $n \times n$ density matrix ρ has matrix elements

$$\rho_{ik} = \rho^*_{ki}, \qquad \operatorname{Tr}\rho = 1, \qquad \langle\psi|\rho|\psi\rangle \geqslant 0. \tag{2}$$

Since the density matrix is Hermitian, it can always be identified as an element of the convex subset of the linear space associated with the Lie algebra of the $U(n)$ group, on which the group $U(n)$ acts with the adjoint representation

$$\rho \to \rho_U = U\rho U^\dagger. \tag{3}$$

If we consider map of density matrix onto vector $\rho \to \vec{\rho}$ the above adjoint representation transform formula (3) yields

$$\vec{\rho} \to \vec{\rho}_U = (U \otimes U^*)\vec{\rho}.$$

It turns out that there exists interesting connection of quantum states (density matrices) with orbits of the unitary group. For finite n-dimensional system, the Hermitian states and the Hermitian observables may be mapped into the Lie algebra of the unitary group $U(n)$. The states correspond to nonnegative Hermitian operators, observables can be associated with both types of operators, including nonnegative and nonpositive ones. The space of states is therefore a convex-linear space which, in principle, is not equipped with an inner product structure. Due to this, transformations in the linear space of states are not required to preserve any product structure. In the set of observables, on the other hand, one has to be concerned with what is happening with the product of operators when some transformations are performed. State vectors can be transformed into other state vectors. Density operators also can be transformed. We will consider linear transformations of the density operators. The density operator has nonnegative eigenvalues. In any representation, diagonal elements of density matrix have physical meaning of probability distribution function.

Density operator can be decomposed as a sum of eigenprojectors with positive coefficients equal to its eigenvalues. Each one of the projectors defines a pure state if eigenvalues except the zero one, are not degenerate. There exists a basis in which every eigenprojector of rank one is represented by a diagonal matrix of rank one with only one matrix element equal to one and all other matrix elements equal to zero. Other density matrices with similar properties belong to the orbit of the unitary group acting on the starting eigenprojector. The number of distinct nonzero eigenvalues and their value determines the class of the orbit. Since density matrices of higher rank belong to an appropriate orbit of a convex sum of the different diagonal eigenprojectors (in special basis), we may say that generic density matrices belong to the orbits of the unitary group acting on the diagonal density matrices which belong to a Cartan subalgebra of the Lie algebra of the unitary group.

3. Semigroup of linear positive maps

If the initial density matrix ρ is diagonal, i.e., it belongs to a Cartan subalgebra of the Lie algebra of the unitary group, the diagonal elements of the obtained matrix ρ_U give a "smoother" probability distribution than the initial one. A set of transformations preserving previously stated properties may be given in the form (see [26,27])

$$\rho_0 \to \rho_V = L_V\rho_0 = \sum_k V_k\rho_0 V_k^\dagger,$$
$$\sum_k V_k^\dagger V_k = 1. \tag{4}$$

Here by definition $V_k^\dagger$ is Hermitian conjugate in terms of the standard scalar product. In the above equations, the operators V_k are not uniquely defined out of L_V. An orthogonal transformation on the index will leave the equations unchanged. For example, if V_k $(k = 1, 2, \ldots, N)$ are taken as square roots of orthogonal projectors onto complete set of N state,

the map provides the map of the density matrix ρ_0 onto diagonal density matrix ρ_{0d} which has the same diagonal elements as ρ_0 has. In this case, the matrices V_k have the only nonzero matrix element which is equal to one. Such a map may be called "decoherence map" because it removes all nondiagonal elements of the density matrix ρ_0 killing any phase relations. In quantum information terminology, one uses also the name "phase damping channel". More general map may be given if one takes V_k as N generic diagonal density matrices, in which eigenvalues are obtained by N circular permutations from the initial one. Due to this map, one has a new matrix with the same diagonal matrix elements but with changed nondiagonal elements. The purity of this matrix is smaller then the purity of the initial one. This means that the map is contractive. All diagonal matrices with the same diagonal elements up to permutations belong to a given orbit of the unitary group.

For a large number of terms with randomly chosen matrices V_k in the sum in (4), the above map gives the most stochastic density matrix

$$\rho_0 \to \rho_s = L_1\rho_0 = (n)^{-1}1.$$

For the qubit case, its four-dimensional superoperator matrix L_1 can be taken as matrix with four matrix elements different from zero. These matrix elements are equal to $1/2$. They have the labels L_{11}, L_{14}, L_{41}, L_{44}.

The positive map is called "not completely positive" if in superoperator form its matrix acting on vector $\vec{\rho}$ has the following expression:

$$\mathcal{L} = \sum_k V_k \otimes V_k^* - \sum_s v_s \otimes v_s^*,$$
$$\sum_k V_k^\dagger V_k - \sum_s v_s^\dagger v_s = 1.$$

An example of such a map is given by the positive map that connects the positive density matrix with its transpose (or complex conjugate). For example, in the qubit case this map can be presented as the connection of the matrix ρ with its transpose of the form

$$\rho \to \rho^{\mathrm{tr}} = \rho^* = \frac{1}{2}(1\cdot\rho\cdot 1 + \sigma_1\rho\sigma_1 - \sigma_2\rho\sigma_2 + \sigma_3\rho\sigma_3),$$

where σ_1, σ_2, and σ_3 are the standard Pauli matrices. There is no unitary transform connecting the matrices ρ and ρ^* for all ρ. (But for any fixed ρ, there is a unitary transformation depending on ρ.)

There is noncompletely positive map in the N-dimensional case, which is given by the formula (for some ε)

$$\rho \to \rho_s = -\varepsilon\rho + \frac{1+\varepsilon}{N}1_N.$$

In quantum information terminology, it is called "depolarizing channel".

For the qubit case, the matrix form of this map is

$$L_\varepsilon = \begin{pmatrix} \frac{1-\varepsilon}{2} & 0 & 0 & \frac{1+\varepsilon}{2} \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 \\ \frac{1+\varepsilon}{2} & 0 & 0 & \frac{1-\varepsilon}{2} \end{pmatrix}. \tag{5}$$

Thus, we have constructed the matrix representation of the positive map of density operators of the spin-1/2 system. This particular set of matrices provides a representation of the semigroup of real numbers $-1 \leqslant \varepsilon \leqslant 1$. If one considers the product $\varepsilon_1\varepsilon_2 = \varepsilon_3$, the result ε_3 belongs to the semigroup. This set of matrices is a subsemigroup of the semigroup of all linear positive maps. Only two elements 1 and -1 of the semigroup have the inverse. These two elements form the finite subgroup of the semigroup. The semigroup itself without element $\varepsilon = 0$ can be embedded into the group of real numbers with natural multiplication rule. Each matrix L_ε has an inverse element in this group but all the parameters of the inverse elements live out of the segment $-1, 1$. The group of the real numbers is commutative. The matrices of the nonunitary representation of this group commute too. It means that we have nonunitary reducible representation of the semigroup which is also commutative. To construct this representation, one needs to use the map of matrices on the vectors discussed in the previous section.

In some cases, the purity parameter $\mu = \mathrm{Tr}\,\rho^2$ after performing the positive map generically becomes smaller. There are maps for which the purity parameter is preserved, for example,

$$\rho \to \rho^{\mathrm{tr}}, \qquad \rho \to -\rho + \frac{2}{N}1. \tag{6}$$

The linear positive maps include also unitary transform of density matrices. There are maps which provide dilation. For qubit system, matrix of superopera-

tor

$$L = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

acting on arbitrary vector $\vec{\rho}_0$ corresponding to a density matrix ρ_0 gives the pure state with density matrix

$$\rho_f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{8}$$

This is a "pin map" [28] and it maps every density matrix onto a fixed density matrix. For the matrices L_ε the inverse matrices exist for $\varepsilon \neq 0$. But these inverse matrices do not provide positive trace preserving maps.

One has also for completely positive maps

$$\rho \to \rho' = \sum_k \rho'_k, \quad \rho'_k = V_k \rho V_k^\dagger,$$
$$\sum_k V_k^\dagger V_k = 1. \tag{9}$$

One can construct another positive map [29] called entanglement breaking map [30]

$$\rho \to \rho' = \sum_k r_k \operatorname{Tr}(R_k \rho), \tag{10}$$

where r_k are density matrices and R_k are positive operators satisfying the normalization condition $\sum_k R_k = 1$.

The entanglement breaking map is a positive map. There exist some special cases of completely positive maps. For example,

$$\rho \to -\varepsilon\rho + \frac{1+\varepsilon}{N}\rho \tag{11}$$

differs from the depolarizing map because the unity operator is replaced by the density operator. Another map reads

$$\rho \to \frac{1 - \operatorname{diag}\rho}{N}. \tag{12}$$

The decoherence map (phase damping map) of the kind

$$\rho_{ij} \to \begin{cases} \rho_{ij}, & i = j, \\ \lambda\rho_{ij}, & i \neq j, \end{cases} \tag{13}$$

where $|\lambda| < 1$ provides contractive map with uniform change of off-diagonal matrix elements of the density matrix. All the positive maps described can be combined with each other. The combined maps form a set which has the structure of a semigroup. The superoperator matrices describing all these combined maps yield the matrix representation of the semigroup.

4. Separable systems and separability criterion

For bipartite system, according to the definition, the system density matrix is called separable but not simply separable, if there is decomposition of the form

$$\rho_{AB} = \sum_k p_k \big(\rho_A^{(k)} \otimes \rho_B^{(k)}\big),$$
$$\sum_k p_k = 1, \quad 1 \geqslant p_k \geqslant 0. \tag{14}$$

This is Hilbert's problem of biquadrates. Is a positive biquadratic the positive sum of products of positive quadratics? In this formula, one may use also sum over two different indices. Using another labelling in such sum over two different indices, this sum can be always represented as the sum over only one index. The formula does not demand orthogonality of the density operators $\rho_A^{(k)}$ and $\rho_B^{(k)}$ for different k. Since every density matrix is a convex sum of pure density matrices, one could demand that $\rho_A^{(k)}$ and $\rho_B^{(k)}$ be pure.

There are several criteria for the state to be separable. We suggest in the next sections a new approach to the problem of separability and entanglement based on the tomographic probability description of quantum states. The states which cannot be represented in the form (14) by definition are called entangled states [16]. Thus the states are entangled if in formula (14) at least one coefficient (or more) p_k is negative which means that the positive ones can take values greater than unity.

As an example, we consider the generalized Werner state of bipartite spin-1/2 system. For the generalized Werner state with the density matrix

$$\rho_{AB} = \begin{pmatrix} \frac{1+p_3}{4} & 0 & 0 & \frac{p_1-p_2}{4} \\ 0 & \frac{1-p_3}{4} & \frac{p_1+p_2}{4} & 0 \\ 0 & \frac{p_1+p_2}{4} & \frac{1-p_3}{4} & 0 \\ \frac{p_1-p_2}{4} & 0 & 0 & \frac{1+p_3}{4} \end{pmatrix}, \tag{15}$$

there exists the known result. Namely, for the standard Werner state ($p_1 = 2p$, $p_2 = 0$, $p_3 = p$) and for $p < 1/3$ the state is separable and for $p > 1/3$ the state is entangled, since in the decomposition of the density operator in the form (14) the state

$$\rho_0 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{16}$$

has the weight $p_0 = (1 - 3p)/4$. For $p > 1/3$, the coefficient p_0 becomes negative.

One can show that for generalized Werner state the condition for separability becomes $|p_1| + |p_2| + |p_3| \leqslant 1$. According to the partial transpose criterion [19], the system is separable if the partial transpose of the matrix ρ_{AB} (14) gives a positive density matrix. In general, this condition is necessary but not sufficient. We will discuss the condition of separability using the tomographic description of spin states.

5. Spin tomography and unitary spin tomograms

Below we review the tomography of spin states for a given value of spin.

The tomographic probability (spin tomogram) completely determines the density matrix of a spin state. It has been introduced in [31–33]. The tomographic probability for the spin-j state is defined via the density matrix by the formula

$$\begin{aligned} &\langle jm|D^{\dagger}(g)\rho D(g)|jm\rangle = W^{(j)}\big(m, \vec{0}(g)\big), \\ &m = -j, -j+1, \ldots, j, \\ &\vec{0}(g) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \end{aligned} \tag{17}$$

where $D(g)$ is the matrix of $SU(2)$-group representation depending on the group element g determined by three Euler angles. The unit vector $\vec{O}(g)$ depends on the group element g. Below we will omit the explicit dependence on g. There are only a finite number of elements in the density matrix but there are an infinity of the spin tomograms. So there ought to be an infinite number of sum rules.

One can introduce unitary spin tomograms $w(m, u)$ by replacing in above formula (17) the matrix $D(g)$ by generic unitary matrix u. For example, in the case of higher spins $j = 1, 3/2, 2, \ldots$, the $n \times n$ projector matrix

$$\rho_1 = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad n = 2j + 1 \tag{18}$$

has the unitary spin tomogram denoted as

$$\begin{aligned} &w_1(j, u) = |u_{11}|^2, \qquad w_1(j-1, u) = |u_{12}|^2, \ \ldots \\ &w_1(-j, u) = |u_{1n}|^2. \end{aligned} \tag{19}$$

Other projectors

$$\rho_k = \begin{pmatrix} 0 & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & 0 \end{pmatrix}, \tag{20}$$

in which unity is located in the (k, k)-entry, have the tomogram $w_k(m, u)$ of the form

$$\begin{aligned} &w_k(j, u) = |u_{k1}|^2, \qquad w_k(j-1, u) = |u_{k2}|^2, \ \ldots \\ &w_k(-j, u) = |u_{kn}|^2. \end{aligned} \tag{21}$$

By using the Weyl trace of matrices, and matrices E_{jk} with (j, k) entries equal to one, all others being zero, any density matrix can be written in the form

$$\rho = \sum_{jk} \rho_{jk} E_{jk}, \tag{22}$$

the unitary spin tomogram can be presented in form of the decomposition

$$w_\rho(m, u) = \sum_{jk} \rho_{jk} w_{jk}(m, u), \tag{23}$$

where $w_{jk}(m, u)$ are basic unitary spin tomograms of transition operators E_{jk} of the form

$$w_{jk}(m, u) = \langle jm|u^{\dagger} E_{jk} u|jm\rangle. \tag{24}$$

If one uses a map

$$\rho \to \rho', \tag{25}$$

the unitary spin tomogram is transformed as

$$w_\rho(m, u) \to w'_\rho(m, u) = \sum_{jk} \rho'_{jk} w_{jk}(m, u). \tag{26}$$

If the transform (25) is linear

$$\rho_{jk} \to \rho'_{jk} = \sum_{ps} L_{jk,ps}\rho_{ps}, \tag{27}$$

the transform of the tomogram reads

$$w'_{\rho}(m,u) = \sum_{ps} \rho_{ps} w'_{ps}(m,u). \tag{28}$$

Here

$$w'_{ps}(m,u) = \sum_{jk} L_{jk,ps} w_{jk}(m,u) \tag{29}$$

is the linear transform of the basic tomograms of the operators E_{jk}.

In the case of infinite-dimensional Hilbert space of photon states, the construction of the tomogram can be applied also. We construct the tomogram $w(m,u)$ using the same formula (17) where the state $|jm\rangle$ is replaced by the Fock state $|m\rangle$ $(m = 0,1,2,\ldots)$ and the matrix $D(j)$ is replaced by infinite-dimensional unitary matrix u.

For illustration, we can consider the tomogram of a specific spin state. If the state is pure state with density matrix

$$\rho_{+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{30}$$

the spin tomogram $W(m,\vec{0})$, where $m = \pm 1/2$, has the values

$$W_{+}\left(\frac{1}{2},\vec{0}\right) = \cos^2\frac{\theta}{2},$$
$$W_{+}\left(-\frac{1}{2},\vec{0}\right) = \sin^2\frac{\theta}{2}. \tag{31}$$

6. Tomogram of the group $U(n)$ for a bipartite system

In this section we discuss in more detail the separability criterion using the introduced notion of unitary spin tomogram. We focus on bipartite systems.

In order to formulate a criterion of separability for a bipartite spin system with spin j_1 and j_2, we introduce the tomogram for the group $U(n)$, where

$$n = n_1 n_2, \quad n_1 = 2j_1 + 1, \quad n_2 = 2j_2 + 1.$$

Let us denote $g^{(n)}$ as parameters of the group element. Let us define the $U(n)$ tomogram using the basis $|j_1, j_2, m_1, m_2\rangle$ namely for fundamental representation, i.e.,

$$w^{(j_1,j_2)}\big(m_1,m_2,g^{(n)}\big) = \langle j_1, j_2, m_1, m_2|U^{\dagger}\big(g^{(n)}\big)\rho U\big(g^{(n)}\big) \times |j_1, j_2, m_1, m_2\rangle. \tag{32}$$

This unitary spin tomogram becomes the spin-tomogram [34] for the $g^{(n)} \in U(2)\otimes U(2)$ subgroup of the group $U(n)$. The properties of this tomogram follow from its definition as the joint probability distribution of two random spin projections m_1, m_2 depending on $g^{(n)}$ parameters.

One has the normalization condition

$$\sum_{m_1,m_2} w^{(j_1,j_2)}\big(m_1,m_2,g^{(n)}\big) = 1. \tag{33}$$

Also all probabilities are nonnegative, i.e.,

$$w^{(j_1,j_2)}\big(m_l,m_2,g^{(n)}\big) \geqslant 0. \tag{34}$$

Due to this, one has

$$\sum_{m_1,m_2} \big|w^{(j_1,j_2)}\big(m_l,m_2,g^{(n)}\big)\big| = 1. \tag{35}$$

For the spin-tomogram, the group element is an ordered pair of orthogonal group elements

$$g^{(n)} \to (\vec{O}_1, \vec{O}_2) \tag{36}$$

and

$$w^{(j_1,j_2)}\big(m_l,m_2,g^{(n)}\big) \to W(m_1,m_2,\vec{O}_1,\vec{O}_2). \tag{37}$$

The separability and entanglement condition can be considered from the viewpoint of the properties of a $U(n)$ tomogram. If the two-spin $n\times n$ density matrix ρ is separable, it remains separable under the action of the generic positive map of the subsystem density matrices. This map can be described as follows.

Let ρ be mapped onto vector $\vec{\rho}$ with n^2 components. The components are simply ordered rows of the matrix ρ, i.e.,

$$\vec{\rho} = \big(\rho_{11},\rho_{12},\ldots,\rho_{1n},\rho_{21},\rho_{22},\ldots,\rho_{nn}\big). \tag{38}$$

Let the $n^2\times n^2$ matrix of superoperator L be taken in the form

$$L = \sum_s p_s L_s^{(j_1)} \otimes L_s^{(j_2)}, \quad p_s \geqslant 0, \quad \sum_s p_s = 1, \tag{39}$$

where the $n_1 \times n_1$ matrix $L_s^{(j_1)}$ and the $n_2 \times n_2$ matrix $L_s^{(j_2)}$ describe the positive maps of density matrices of spin-j_1 and spin-j_2 subsystems, respectively. We map vector $\vec{\rho}$ onto vector $\vec{\rho}_L$

$$\vec{\rho}_L = L\vec{\rho} \tag{40}$$

and construct the $n \times n$ matrix ρ_L, which corresponds to the vector $\vec{\rho}_L$. Then we consider the $U(n)$ tomogram of the matrix ρ_L, i.e.,

$$w_L^{(j_1,j_2)}\big(m_l, m_2, g^{(n)}\big) = \langle j_1, j_2, m_1, m_2|U^{\dagger}\big(g^{(n)}\big)\rho_L U\big(g^{(n)}\big) \times |j_1, j_2, m_l, m_2\rangle. \tag{41}$$

Using this tomogram we introduce the function

$$F\big(g^{(n)}, L\big) = \sum_{m_1,m_2} \big|w_L^{(j_1,j_2)}\big(m_1, m_2, g^{(n)}\big)\big|. \tag{42}$$

For separable states, this function does not depend on the $U(n)$-group parameter $g^{(n)}$ and on positive-map matrix elements of the matrix L.

For the normalized density matrix ρ of the bipartite spin-system, this function reads

$$F\big(g^{(n)}, L\big) = 1. \tag{43}$$

For entangled states, this function depends on $g^{(n)}$ and L and is not equal to unity. This property can be chosen as a necessary and sufficient condition for separability of bipartite spin-states.

We introduce also tomographic purity parameter μ_k of kth order by the formula

$$\mu_k\big(g^{(n)}, L\big) = \sum_{m_1 m_2} \big|w_L^{(j_1,j_2)}\big(m_1, m_2, g^{(h)}\big)\big|^k, \quad k > 1.$$

The tomographic purity parameters can take values larger than unity for entangled states for some semigroup parameters L. For identity semigroup element and specific $g_0^{(n)}$ unitary transform diagonalizing the density matrix, the tomographic purity μ_2 is identical to purity parameter of the state ρ. The parameters for $k = 2, 3, \ldots$, correspond to $\operatorname{Tr}\rho^{k+1}$.

In fact, the formulated approach can be extended to multipartite systems too. The generalization is as follows.

Given N spin-systems with spins $j_1, j_2, \ldots, j_N$, let us consider the group $U(n)$ with

$$n = \prod_{k=1}^{N} n_k, \quad n_k = 2j_k + 1. \tag{44}$$

Let us introduce the basis

$$|\vec{m}\rangle = \prod_{k=1}^{N} |j_k m_k\rangle \tag{45}$$

in the linear space of the fundamental representation of the group $U(n)$. We define now the $U(n)$ tomogram of a state with the $n \times n$ matrix ρ:

$$w_\rho\big(\vec{m}, g^{(n)}\big) = \langle\vec{m}|U^{\dagger}\big(g^{(n)}\big)\rho U\big(g^{(n)}\big)|\vec{m}\rangle. \tag{46}$$

For a positive Hermitian matrix ρ with $\operatorname{Tr}\rho = 1$, we formulate the criterion of separability as follows.

Let the map matrix L be of the form

$$L = \sum_s p_s \left(\prod_{k=1}^{N} \otimes L_s^{(k)}\right), \quad p_s \geqslant 0, \quad \sum_s p_s = 1, \tag{47}$$

where $L_s^{(k)}$ is the positive-map matrix of the density matrix of the kth spin subsystem. We construct the matrix ρ_L as in the case of the bipartite system using the matrix L. The function

$$F\big(g^{(n)}, L\big) = \sum_{\vec{m}} \big|w_{\rho_L}\big(\vec{m}, g^{(n)}\big)\big| \geqslant 1 \tag{48}$$

is equal to unity for separable state and depends on the matrix L and $U(n)$-parameters $g^{(n)}$ for entangled states. Thus, for all separable states, one has equality in relation (48). If the state is nonseparable, one can conjecture that there exist some elements of semigroup L and some unitary group elements for which sum (48) is bigger than unity.

This criterion can be applied also in the case of continuous variables, e.g., for Gaussian states of photons. Function (48) can provide the measure of entanglement. Thus one can use the maximum value (or a mean value) of this function as a characteristic of entanglement. In the previous section, we considered the generalized Werner states. Using the above criterion, one can get the domain of values of the parameters of the states for which one has separability or entanglement. In fact, the separability criterion is related to the following positivity criterion of finite or infinite (trace class) matrix A. The matrix A is positive iff the sum of moduli of diagonal matrix elements of the matrix $UAU^{\dagger}$ is equal to a positive trace of the matrix A for an arbitrary unitary matrix U.

7. Bipartite system and depolarizing map

Let us discuss the property of tomogram of bipartite system with density matrix ρ_{12}. We consider now the tomogram of density matrix using depolarizing map to detect the entanglement. If the density matrix is separable, than the depolarizing map of the second subsystem provides the following density matrix

$$\rho_{12} \to \rho_\varepsilon = -\varepsilon\rho_{12} + \frac{1+\varepsilon}{N_2}\underline{\rho^{(1)}} \otimes 1_2, \tag{49}$$

where

$$\underline{\rho^{(1)}} = \mathrm{Tr}_2(\rho_{12}) \tag{50}$$

and 1_2 is the N_2-dimensional unity matrix. Then one has the property of unitary spin tomogram

$$w_\varepsilon\big(m_1, m_2, g^{(n)}\big) = -\varepsilon w_{12}\big(m_1, m_2, g^{(n)}\big) + \frac{1+\varepsilon}{N_2}\underline{w}\big(m_1, m_2, g^{(n)}\big), \tag{51}$$

where $g^{(n)}$ is matrix of $U((2j_1+1)(2j_2+1))$ unitary transform; $w_\varepsilon(m_1, m_2, g^{(n)})$ is the tomogram of transformed density matrix of bipartite system; $\underline{w}(m_1, m_2, g^{(n)})$ is the unitary spin tomogram of tensor product of partial trace $\underline{\rho^{(1)}}$ over the second subsystem's coordinates of the density matrix ρ_{12} and unity operator 1_2; $w_{12}(m_1, m_2, g^{(n)})$ is the unitary spin tomogram of the state with density matrix ρ_{12}.

The generic criterion of separability for the partial case of depolarizing map means

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\left|\frac{1+\varepsilon}{2j_2+1}\underline{w}\big(m_1, m_2, g^{(n)}\big) - \varepsilon w_{12}\big(m_1, m_2, g^{(n)}\big)\right| = 1 \tag{52}$$

for arbitrary $g^{(n)}$ and ε.

For Werner states, the tomogram of transformed state (in this case, it means that $n=4$ and $p \to -\varepsilon p$) is related to the initial-state tomogram w_W

$$w_\varepsilon\big(m_1, m_2, g^{(n)}\big) = -\varepsilon w_W\big(m_1, m_2, g^{(n)}\big) + \frac{1+\varepsilon}{4}. \tag{53}$$

The criterion of separability yields

$$\sum_{m_1,m_2=-1/2}^{1/2}\left|\frac{1+\varepsilon}{4} - \varepsilon w_W\big(m_1, m_2, g^{(n)}\big)\right| = 1. \tag{54}$$

Equality (54) takes place for arbitrary $g^{(n)}$ and ε only for $|p| \leqslant 1/3$. For $p > 1/3$, the above sum depends on $g^{(n)}$ and ε and it is larger than one.

It is obvious if one calculates the tomogram using the element of the unitary group of the form

$$g_0^{(n)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{55}$$

At this point, the sum (54) reads

$$\sum_{m_1,m_2=-1/2}^{1/2}\left|\frac{1+\varepsilon}{4} - \varepsilon w_W\big(m_1, m_2, g^{(n)}\big)\right| = 3\left|\frac{1+\varepsilon p}{4}\right| + \left|\frac{1-3p\varepsilon}{4}\right|. \tag{56}$$

One can see that this sum equals to one independently on the value of parameter $|\varepsilon| \leqslant 1$ only for values $|p| \leqslant 1/3$. For $p = 1$, the maximum value of the sum equals $2 = (1+3\varepsilon)/2$ $(\varepsilon = 1)$. This value can characterize the degree of entanglement of Werner state. For generalized Werner state, sum (54) reads

$$\frac{1}{4}\big\{\big|1+\varepsilon(p_1-p_2-p_3)\big| + \big|1+\varepsilon(p_2-p_1-p_3)\big| + \big|1+\varepsilon(p_3-p_1-p_2)\big| + \big|1+\varepsilon(p_1+p_2+p_3)\big|\big\}.$$

This sum is equal to one for any ε, if $|p_1| + |p_2| + |p_3| \leqslant 1$. For $\varepsilon = 1$, we have a measure of entanglement.

8. Qutrit states

One can also check the criterion using example of two-qutrit pure entangled state with wave function

$$|\psi\rangle = \frac{1}{\sqrt{3}}\sum_{m=-1}^{1}|u_m\rangle|v_m\rangle. \tag{57}$$

The sum defining the criterion of separability for specific $U(9)$ transform $g_0^{(n)}$ which is diagonalizing

the Hermitian matrix $L_\varepsilon|\psi\rangle\langle\psi|$ reads

$$F\big(\varepsilon, g_0^{(n)}\big) = 8\left|\frac{1+\varepsilon}{9}\right| + \left|\frac{1-8\varepsilon}{9}\right|. \tag{58}$$

For $1/2 > \varepsilon > 1/8$, this sum is larger than one, that means that the state is entangled. For $\varepsilon = 1/2$, the function has maximum and it is equal to $5/3$.

The entanglement of the considered state can be detected using partial transposition criterion too.

For the case of pure entangled state of two-qutrit system with the wave function

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\big(|u_1\rangle|v_1\rangle + |u_0\rangle|v_0\rangle\big), \tag{59}$$

in which the states with spin projections $m = -1$ do not participate, the partial transpose criterion does also detect entanglement. Our criterion yields for specific $U(9)$ transform $g_0^{(n)}$, which diagonalizes the Hermitian matrix $L_\varepsilon|\Phi\rangle\langle\Phi|$ the following expression for the function $F(\varepsilon, g_0^{(n)})$, which reads

$$F\big(\varepsilon, g_0^{(n)}\big) = 5\frac{|1+\varepsilon|}{6} + \frac{|1-5\varepsilon|}{6}. \tag{60}$$

The function takes maximum value for $\varepsilon = 1/2$ that equals to $3/2$. This value is smaller than $5/3$ of the previous case. It corresponds to our intuition that the superposition of three product states of two qutrit system is more entangled than the superposition of only two such product states.

9. Multipartite system

The criterion can be extended to multipartite spin system.

We have to apply for n-partite system the transform of the density matrix ρ with superoperator of the form

$$L_{\vec{\varepsilon}} = L_{\varepsilon_1}^{(1)} \otimes L_{\varepsilon_2}^{(2)} \otimes \cdots \otimes L_{\varepsilon_n}^{(n)}, \tag{61}$$

where the transform $L_{\varepsilon_k}^{(k)}$ acts as depolarizing map on the kth subsystem. If the state is separable

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \cdots \otimes \rho_k^{(n)},$$
$$\sum_k p_k = 1, \quad p_k \geqslant 0, \tag{62}$$

each of the terms $\rho_k^{(j)}$ $(j = 1, 2, \ldots, n)$ in the tensor product is replaced by the term

$$\rho_k^{(j)} \to -\varepsilon_j \rho_k^{(j)} + \frac{1+\varepsilon_j}{N_j} 1_j. \tag{63}$$

This means that the transformed density matrix reads

$$\rho \to L_{\vec{\varepsilon}}\rho = \sum_k p_k \left[\prod_{j=1}^{n} \otimes \left(-\varepsilon\rho_k^{(j)} + \frac{1+\varepsilon_j}{N_j} 1_j\right)\right]. \tag{64}$$

The unitary spin tomogram of the transformed density matrix takes the form $(\vec{\varepsilon} = \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$

$$w_{\vec{\varepsilon}}\big(m_1, m_2, \ldots, m_n, g^{(N)}\big) = \sum_k p_k w_{pr}^{(k)}\big(m_1, m_2, \ldots, m_n, g^{(N)}, \vec{\varepsilon}\big), \tag{65}$$

where $N = \prod_{s=1}^{n}(2j_s + 1)$ and element $g^{(N)}$ is the unitary matrix in N-dimensional space. The tomogram $w_{pr}^{(k)}(m_1, m_2, \ldots, m_n, g^{(N)}, \vec{\varepsilon})$ is the joint probability distribution of spin projections $m_s = -j_s, -j_s + 1, \ldots, j_s$, which depends on the unitary transform $g^{(N)}$ in the state with density matrix

$$\rho_k = \prod_{s=1}^{n} \otimes \left(-\varepsilon_s \rho_k^{(s)} + \frac{1+\varepsilon_s}{N_s} 1_s\right). \tag{66}$$

For the elements

$$g^{(N)} = \prod_{s=1}^{n} \otimes u_s(2j_s + 1),$$

where $u_s(2j_s + 1)$ is unitary matrix, the tomogram (65) takes the form of sum of the products

$$w_{\vec{\varepsilon}}\big(m_1, m_2, \ldots, m_n, g_{pr}^{(N)}\big) = \sum_k p_k \prod_{s=1}^{n} w_k\big(m_s, u_s(2j_s + 1), \varepsilon_s\big), \tag{67}$$

with $w_k(m_s, u_s(2j_s + 1), \varepsilon_s)$ being the unitary spin tomograms of the sth spin subsystem with transformed density matrix $L_{\varepsilon_s}\rho_k^{(s)}$.

For a separable state of the multipartite system, one has

$$\sum_{m_1,\ldots,m_n} \big|w_{\vec{\varepsilon}}\big(m_1, m_2, \ldots, m_n, g^{(N)}\big)\big| = 1 \tag{68}$$

for all elements $g^{(N)}$ and all parameters $\vec{\varepsilon}$.

For entangled state, there can be some values of parameters $\vec{\varepsilon}$ and group elements $g^{(N)}$ for which the sum is larger than one.

10. Conclusions

We summarize the main results of the Letter.

The new type of tomographic probability distributions describing state of multipartite spin systems, which depend on unitary group elements, is introduced. For a partial case of the unitary group element expressed in terms of tensor product of matrices of irreducible representation of rotation group, the introduced tomographic probability distribution coincides with spin tomogram of a spin-system state.

We have given a criterion for the separability of multipartite spin system. The criterion is called "tomographic criterion" of separability. The suggested criterion is connected with properties of the unitary spin tomogram given by function (48). For entangled density matrix, the function depends on unitary group parameters g and the parameters of positive map semigroup L. For separable density matrix, the function equals to unity. We applied the particular case of the positive map (depolarizing map) to detect entanglement of some qubit and qutrit states. Suggested criterion differs from available ones. To apply this criterion, one needs to calculate the sum of moduli of diagonal matrix elements of product of three matrices. One of the matrices is Hermitian and two others are unitary ones. This procedure does *not* need the calculation of the eigenvalues of a matrix. We also introduced the concept of tomographic purity which in a limit case coincides with usual purity parameter of mixed quantum state. The structure of positive (including not completely positive) map semigroup with elements L needs further investigation.

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