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On the meaning and interpretation of tomography in abstract Hilbert spaces

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Abstract

The mechanism of describing quantum states by standard probabilities (tomograms) instead of wave function or density matrix is elucidated. Quantum tomography is formulated in an abstract Hilbert space framework, by means of decompositions of the identity in the Hilbert space of hermitian linear operators, with trace formula as scalar product of operators. Decompositions of the identity are considered with respect to over-complete families of projectors labeled by extra parameters and containing a measure, depending on these parameters. It plays the role of a Gram–Schmidt orthonormalization kernel. When the measure is equal to one, the decomposition of identity coincides with a positive operator-valued measure (POVM) decomposition. Examples of spin tomography, photon number tomography and symplectic tomography are reconsidered in this new framework.

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1. Introduction

Quantum states are described by vectors in a Hilbert space [1], or wave functions [2], in the case of pure states. In the case of mixed states, the density operators [3,4] are used instead to describe quantum states. On the other hand, different tools have been introduced to describe quantum states by means of functions on phase space. These functions, like the Wigner function [5], Husimi–Kano Q-quasidistribution [6,7], Sudarshan–Glauber diagonal (singular) P-quasidistribution [8,9], contain information about the quantum state which amount to the information carried by the density matrix in an arbitrary representation. In fact, these different quasidistributions are alternative, essentially equivalent, forms of representing the density operators. These quasidistributions have some properties similar to those of classical probability distributions on phase space, but they are not fair joint probability distributions since the uncertainty relation of position and momentum is incompatible with the existence of such probability density.

Recently, an approach to reconstruct Wigner functions from optical tomograms was suggested [10,11]. The optical tomography approach was generalized to provide symplectic tomography [12,13]. In the tomographic approach the quantum state is associated with a probability distribution depending on some extra parameters. This observation was used to develop a probability representation of quantum mechanics in which the tomographic probability distribution (tomogram) is considered as the primary object obeying an evolution equation of generalized Fokker–Planck type [14] and containing all the information of quantum state. Thus

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it is possible to formulate quantum mechanics by describing a quantum state by a fair probability distribution instead of wave functions or density matrices. By reading this chain of associations backwards, it is quite natural to ask if it is possible to provide an interpretation of tomograms directly at the level of the abstract Hilbert space. Although originally formulated for an infinite-dimensional Hilbert space, this interpretation should work equally well for finite level systems (spin tomography) and generic systems. The interpretation clarifies the mechanism for describing quantum states by fair probabilities instead of wave functions and density matrices.

It is the aim of this Letter to show how to provide such an interpretation without, however, indulging on more technical aspects (these will be considered elsewhere). The main idea consists of expressing the tomogram in terms of a scalar product in the space $\mathbb{H}$ of rank-one projectors, that is in the linear space of operators acting on the space of quantum states. These projectors are connected with special families of vectors in the Hilbert space $\mathcal{H}$ of quantum states. The vectors are eigenvectors of families of operators depending on some extra parameters. We will obtain a decomposition of the identity operator in terms of a weighted sum of the projectors depending on the extra parameter and determining the tomogram, so that any matrix, and in particular the density matrix, can be obtained as a weighted linear combination of the basis vectors in the space $\mathbb{H}$. This explains why the inversion formula (reconstruction formula) works for the tomographic maps. The tomograms can also be constructed for spin states [15,16]. The relation between tomographic maps and star-product quantization schemes was clarified in Refs. [17,18]. We develop our theory bearing in mind the case of spin tomography. However, that general picture of the tomographic map is applicable to other kinds of tomographies too, e.g., to photon number tomography [19–21] and symplectic tomography. To demonstrate the results we review the approach in which an $(n \times n)$-matrix is considered as an n^2-vector, used, for instance, in Ref. [22].

The Letter is organized as follows. In Section 2 we review the picture where a matrix is regarded as a vector. In Section 3 we define tomography for abstract finite-dimensional Hilbert spaces and give a new interpretation to the tomograms, both in terms of sets of rank-one projectors and of families of unitary, or Hermitian, operators. In Section 4 we discuss in the light of our picture some known examples as the spin tomography for the finite-dimensional case, the photon number and the symplectic tomographies for the infinite-dimensional case, deriving new identity decompositions in terms of rank-one projectors in Hilbert space of bounded Hermitian operators. Conclusions and perspectives are discussed in Section 5.

2. Matrices as vectors

In order to clarify how the tomographic approach provides relations connecting probability distribution with density matrix elements we start with a very elementary example. We consider two Hilbert spaces $\mathcal{H}$ and $\mathbb{H}$. For simplicity, we first identify the Hilbert spaces $\mathcal{H}$ with the qu-bit (i.e., spin-1/2) quantum state set, i.e., with vectors

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}. \tag{1}$$

Then the density matrix (density operator) for this pure state has the form

$$\hat{\rho}_\psi = |\psi\rangle\langle\psi| = \begin{bmatrix} \psi_1\psi_1^* & \psi_1\psi_2^* \\ \psi_2\psi_1^* & \psi_2\psi_2^* \end{bmatrix}. \tag{2}$$

It is well known (see, e.g., [23]) that the set of operators acting on the Hilbert space $\mathcal{H}$ is a linear space. This space is a Hilbert space, which we identify as $\mathbb{H}$, since one has the scalar product of two operators $\hat{A}$ and $\hat{B}$ acting on the space $\mathcal{H}$ given by the formula

$$\langle\hat{A}|\hat{B}\rangle = \mathrm{Tr}(\hat{A}^\dagger\hat{B}). \tag{3}$$

We use here the Dirac's notation for the scalar product. In fact, it is convenient to write an operator (a matrix) as a vector. For example the matrix of Eq. (2) is mapped to a 4-vector using the rule

$$\hat{\rho}_\psi \;\rightarrow\; |\rho_\psi\rangle = \begin{bmatrix} \psi_1\psi_1^* \\ \psi_1\psi_2^* \\ \psi_2\psi_1^* \\ \psi_2\psi_2^* \end{bmatrix}. \tag{4}$$

Conversely, this rule allows one to reconstruct a matrix if the corresponding 4-vector is given. For example, given a 4-vector $|A\rangle$, one obtains the matrix $\hat{A}$ as

$$|A\rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \;\rightarrow\; \hat{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}. \tag{5}$$

Of course, other "reconstructions" are possible, and indeed this possibility has been exploited to consider alternative associative products on the vector space of matrices [24].

The scalar product of Eq. (3) expressed in terms of matrices $\hat{A}$ and $\hat{B}$ is nothing but the standard vector scalar product given by

$$\langle \hat{A}|\hat{B}\rangle = \langle A|B\rangle = \sum_{k=1}^{4} a_k^* b_k. \tag{6}$$

The set of 4-vectors equipped with this scalar product is the Hilbert space $\mathbb{H}$. Thus, having an initial Hilbert space of vectors $\mathcal{H}$ of two-dimensional qu-bit, we have also the four-dimensional Hilbert space of 4-vectors $\mathbb{H} = B(\mathcal{H})$, spanned by the linear operators acting on the Hilbert space $\mathcal{H}$. The orthogonal basis in the space $\mathcal{H}$ of spin up and down states (standard basis)

$$|e_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |e_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{7}$$

for instance, is complete. The completeness relation can be given in the form of an equality valid in the Hilbert space $\mathbb{H} = B(\mathcal{H})$, namely,

$$\hat{P}_1 + \hat{P}_2 = \mathbb{I}_{\mathcal{H}}. \tag{8}$$

Here the orthogonal projectors

$$\hat{P}_1 = |e_1\rangle\langle e_1| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{P}_2 = |e_2\rangle\langle e_2| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{9}$$

satisfying the relation (8), allow for the possibility to decompose each vector of $\mathcal{H}$ in terms of the basis vector $|e_1\rangle$ and $|e_2\rangle$, that is,

$$|\psi\rangle = \psi_1|e_1\rangle + \psi_2|e_2\rangle. \tag{10}$$

We consider some very simple properties of the linear spaces $\mathcal{H}$ and $\mathbb{H}$, because they are the basis for constructing tomographic probabilities and guaranteeing the existence of an inversion formula yielding the operator from its tomogram. In general, the completeness relation of a basis $|\mu\rangle$ of $\mathcal{H}$ can be represented in the form

$$\sum_{\mu} \hat{P}_\mu = \mathbb{I}_{\mathcal{H}}. \tag{11}$$

The rank-one projectors

$$\hat{P}_\mu = |\mu\rangle\langle\mu| \tag{12}$$

depend on a set of parameters μ (discrete or continuous, as well as finite or infinite) and, being in general non-orthogonal:

$$\hat{P}_\mu \hat{P}_{\mu'} \neq 0,$$

they form a positive operator-valued measure (POVM).

On the other hand, in the four-dimensional space $\mathbb{H}$ one has the standard basis

$$|B_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad |B_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad |B_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad |B_4\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{13}$$

so that each vector $|A\rangle$ may be decomposed as

$$|A\rangle = \sum_{k=1}^{4} a_k|B_k\rangle. \tag{14}$$

This decomposition of the vector $|A\rangle$ corresponds to the decomposition of the matrix $\hat{A}$ of Eq. (5) in the form

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

with respect to a basis of non-Hermitian operators. However, it is always possible to use a basis of Hermitian operators, for instance, the unit matrix $\hat{\sigma}_0$ together with the Pauli matrices $\hat{\sigma}_k$ $(k = 1, 2, 3)$. Then

$$|\sigma_0\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad |\sigma_1\rangle = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \qquad |\sigma_2\rangle = \frac{1}{2}\begin{bmatrix} 0 \\ -i \\ i \\ 0 \end{bmatrix}, \qquad |\sigma_3\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \tag{15}$$

so that

$$|A\rangle = \sum_{k=0}^{3} \alpha_k |\sigma_k\rangle. \tag{16}$$

The relation between old and new components is

$$\begin{bmatrix} \alpha_0 & \alpha_1 \\ \alpha_2 & \alpha_3 \end{bmatrix} = \begin{bmatrix} a_1 + a_4 & a_2 + a_3 \\ i\,(a_2 - a_3) & a_1 - a_4 \end{bmatrix}, \qquad \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{bmatrix}.$$

So, in general, if $\mathbb{H}$ were spanned by a Hermitian set of n basis matrices, vectors corresponding to Hermitian operators would be real and the Hermitian conjugation would act on them as the identity. In the case of a non-Hermitian basis, those vectors are not real, in general, and belong to a real linear n^2-dimensional submanifold of $\mathbb{C}^{n^2}$ which is invariant under Hermitian conjugation.

Now we transfer our discussion of the completeness relation of basis vectors for the two-dimensional Hilbert space $\mathcal{H}$ to the case of the four-dimensional Hilbert space $\mathbb{H}$. Thus, we get 4-projectors

$$\mathbb{P}_k = |B_k\rangle\langle B_k|, \quad k = 1, \ldots, 4. \tag{17}$$

The completeness relation in $\mathbb{H}$ reads

$$\sum_{k=1}^{4} \mathbb{P}_k = \mathbb{I}_{\mathbb{H}}, \tag{18}$$

where $\mathbb{I}_{\mathbb{H}}$ is the four-dimensional unit matrix. Alternatively, in the four-dimensional space, one can have a POVM $\mathbb{P}_\mu$ with a set of parameters such that

$$\sum_{\mu} \mathbb{P}_\mu = \mathbb{I}_{\mathbb{H}}. \tag{19}$$

This property means that one can decompose any (2×2)-matrix in terms of the (2×2)-matrices corresponding to the projectors $\mathbb{P}_\mu$. In matrix form Eq. (19) reads

$$\sum_{\mu} (\mathbb{P}_\mu)_{ij,mn} = \delta_{im}\delta_{jn}, \quad i, j, m, n = 1, 2. \tag{20}$$

That means that the index k in Eq. (18) is considered a double index in labelling the elements of two by two matrices. In principle, one can have more complicated conditions of completeness when the projectors $\mathbb{P}_\mu$ are not orthogonal and have trace different from unity. Then a Gram–Schmidt orthogonalization procedure can be encoded by an extra kernel in the relation (20). This is precisely the situation that takes place for some of the examples we consider later on.

3. The abstract Hilbert space definition of tomograms

3.1. Tomographic sets

In general, any kind of tomogram of a pure state $|\psi\rangle$ is the positive real number $\mathcal{W}_\psi(\alpha, \beta, \ldots)$, depending on a set of parameters $(\alpha, \beta, \ldots)$ which label a set of states $|\alpha, \beta, \ldots\rangle$, defined as

$$\mathcal{W}_\psi(\alpha, \beta, \ldots) = \left|\langle \alpha, \beta, \ldots |\psi\rangle\right|^2. \tag{21}$$

At a first glance, it seems quite difficult to read the tomogram as a scalar product, rather than a square modulus. Nevertheless, this is possible by thinking in terms of rank-one projectors rather than of (pure) states. In fact, by using the density operator $\hat{\rho} = |\psi\rangle\langle\psi|$ and the projectors $P_{\alpha,\beta,\ldots} = |\alpha, \beta, \ldots\rangle\langle\alpha, \beta, \ldots|$, we may interpret the tomogram as a scalar product on the space of rank-one projectors:

$$\mathcal{W}_\psi(\alpha, \beta, \ldots) = \left|\langle \alpha, \beta, \ldots |\psi\rangle\right|^2 = \mathrm{Tr}(\hat{\rho} P_{\alpha,\beta,\ldots}). \tag{22}$$

This definition may also be applied in the case of an arbitrary density operator $\hat{\rho}$.

In the following we wish to characterize the sets of vectors $|\alpha, \beta, \ldots\rangle$ which allow for a complete reconstruction of the state $|\psi\rangle$, or an arbitrary density operator $\hat{\rho}$, from the knowledge of its tomograms $\mathcal{W}_\psi(\alpha, \beta, \ldots)$. These sets will be called *tomographic sets*. In the light of our interpretation of the tomogram, the reconstruction is just a consequence of a decomposition of identity in the space of rank-one projectors in terms of the family $|P_{\alpha,\beta,\ldots}\rangle\langle P_{\alpha,\beta,\ldots}|$, after taking into account that the projectors $P_{\alpha,\beta,\ldots}$ in general are not orthogonal. We will discuss our interpretation for finite-dimensional Hilbert spaces. Our construction essentially goes along the following lines.

Suppose a set $\{|e_{\alpha\beta}\rangle\}_{\alpha,\beta=1}^{n}$ of n^2 vectors of C^n is found in such a way that the respective projectors $|e_{\alpha\beta}\rangle\langle e_{\alpha\beta}|$ are a basis $\{|P_k\rangle\}_{k=1}^{n^2}$ of $C^{n^2} = C^n \otimes C^n = B(C^n)$. We use here a collective index k instead of (α, β), e.g., $k = (\alpha - 1)n + \beta$. By means of the Gram–Schmidt procedure, for instance, we may convert the basis $\{|P_k\rangle\}_{k=1}^{n^2}$ into an orthonormal basis $\{|V_j\rangle\}_{j=1}^{n^2}$:

$$|V_j\rangle = \sum_{k=1}^{n^2} \gamma_{jk} |P_k\rangle, \qquad \langle V_i|V_j\rangle = \delta_{ij}. \tag{23}$$

In general, every element of the orthonormal basis $\{|V_j\rangle\}$ is a linear combination of projectors, rather than a single projector like $|P_k\rangle$ associated to a vector of $\mathbb{C}^n$.

There exists a decomposition of the identity on $\mathbb{C}^{n^2} = B(\mathbb{C}^n)$

$$\mathbb{I}_{n^2} = \sum_{j=1}^{n^2} |V_j\rangle\langle V_j| = \sum_{j,k,l=1}^{n^2} \gamma_{jk}^* \gamma_{jl} \hat{P}_l \operatorname{Tr}(\hat{P}_k \,\cdot\,) = \sum_{l=1}^{n^2} \hat{K}_l \operatorname{Tr}(\hat{P}_l \,\cdot\,), \tag{24}$$

where the Gram–Schmidt kernel $\hat{K}_l$ has been introduced

$$\hat{K}_l = \sum_{j,k=1}^{n^2} \gamma_{jl}^* \gamma_{jk} \hat{P}_k. \tag{25}$$

We observe that $\hat{K}_l$ is a nonlinear function of the projectors $\hat{P}_k$, because also the coefficients γ's depend on the projectors.

We define the set $\{|e_{\alpha\beta}\rangle\}_{\alpha,\beta=1}^{n}$ of n^2 vectors of $\mathbb{C}^n$ a *minimal tomographic set*. The tomogram of a density matrix $\hat{\rho}$ with respect to this minimal tomographic set is defined by

$$\mathcal{W}_\rho(\alpha, \beta) = \operatorname{Tr}\big(|e_{\alpha\beta}\rangle\langle e_{\alpha\beta}|\hat{\rho}\big) \quad (\alpha, \beta = 1, \ldots, n). \tag{26}$$

Then, from the decomposition of identity in terms of the tomographic projectors, we get an inversion formula for the density matrix $\hat{\rho}$, or any other operator on $\mathbb{C}^n$:

$$\hat{\rho} = \sum_{j,k,l=1}^{n^2} \gamma_{jk}^* \gamma_{jl} \hat{P}_l \operatorname{Tr}(\hat{P}_k \hat{\rho}) = \sum_{\mu,\nu=1}^{n} \left[\sum_{j,k,l=1}^{n^2} \gamma_{jk}^* \gamma_{jl} (\hat{P}_k)_{\mu\nu}^* \hat{P}_l \right] (\hat{\rho})_{\mu\nu}. \tag{27}$$

In other words, writing the previous equation in terms of matrix elements,

$$(\hat{\rho})_{\mu'\nu'} = \sum_{\mu,\nu=1}^{n} \left[\sum_{j,k,l=1}^{n^2} \gamma_{jk}^* \gamma_{jl} (\hat{P}_k)_{\mu\nu}^* (\hat{P}_l)_{\mu'\nu'} \right] (\hat{\rho})_{\mu\nu}, \tag{28}$$

we have the corresponding expression for the decomposition of the identity:

$$\sum_{j,k,l=1}^{n^2} \gamma_{jk}^* \gamma_{jl} (\hat{P}_k)_{\mu\nu}^* (\hat{P}_l)_{\mu'\nu'} = \sum_{k=1}^{n^2} (\hat{P}_k)_{\mu\nu}^* \left(\sum_{j,l=1}^{n^2} \gamma_{jk}^* \gamma_{jl} \hat{P}_l \right)_{\mu'\nu'} = \delta_{\mu\mu'} \delta_{\nu\nu'} = (\mathbb{I}_n \otimes \mathbb{I}_n)_{\mu\mu',\nu\nu'}. \tag{29}$$

A set containing more than n^2 vectors of $\mathbb{C}^n$ is a tomographic set when it contains a minimal set. In other words, a tomographic set is such that any vector belongs to a (minimal) tomographic subset of n^2 vectors. In particular, a set is tomographic if any subset of n^2 vectors is a (minimal) tomographic set.

Now, a basis of n^2 rank-one projectors can always be found. In fact, an orthonormal basis of $B(\mathbb{C}^n)$ containing n^2 Hermitian operators is associated with the generators τ_k of the group $U(n)$, multiplying each element by the imaginary unit i. Each generator, using its spectral decomposition, can be written in terms of projectors. Moreover, each projector can be expressed by means of rank-one projectors. So, from the spectral decompositions of the generators of $U(n)$, we may extract a basis of n^2 rank-one projectors.

Alternatively, using an orthonormal basis $\{|\alpha\rangle\}_{\alpha=1}^{n}$ of n vectors of $\mathbb{C}^n$, we may define in $B(\mathbb{C}^n)$ an orthogonal basis of n^2 Hermitian operators given by

$$\big\{ \big(|\alpha\rangle\langle\beta| + |\beta\rangle\langle\alpha|\big),\ i\big(|\alpha\rangle\langle\beta| - |\beta\rangle\langle\alpha|\big) \big\} \quad (\alpha, \beta = 1, \ldots, n). \tag{30}$$

Then, from their spectral decompositions we may extract a basis of n^2 rank-one projectors.

For example, starting from a fiducial (orthonormal) basis of $\mathbb{C}^2$, two different suitable unitary operators

$$U_\alpha = \begin{vmatrix} a_\alpha & b_\alpha \\ -b_\alpha^* & a_\alpha^* \end{vmatrix}, \quad a_\alpha a_\alpha^* + b_\alpha b_\alpha^* = 1 \quad (\alpha = 1, 2)$$

are needed to generate another two bases of $\mathbb{C}^2$ and obtain a tomographic set of six vectors containing three different minimal tomographic sets. In fact, starting from the standard basis of $\mathbb{C}^2$, the matrix of 4-vectors

$$\begin{bmatrix} 1 & 0 & a_1a_1^* & b_1b_1^* & a_2a_2^* & b_2b_2^* \\ 0 & 0 & -a_1b_1 & a_1b_1 & -a_2b_2 & a_2b_2 \\ 0 & 0 & -a_1^*b_1^* & a_1^*b_1^* & -a_2^*b_2^* & a_2^*b_2^* \\ 0 & 1 & b_1b_1^* & a_1a_1^* & b_2b_2^* & a_2a_2^* \end{bmatrix}$$

has maximal rank, when the two different operators satisfy some extra condition such as, for instance,

$$\Im\bigl(a_1b_1a_2^*b_2^*\bigr) = \det\begin{vmatrix} \Im(a_1b_1) & \Re(a_1b_1) \\ \Im(a_2b_2) & \Re(a_2b_2) \end{vmatrix} \neq 0. \tag{31}$$

This condition shows that the two complex numbers a_1b_1 and a_2b_2 cannot be proportional on the reals or, equivalently, they have different phases.

Let us characterize the manifold of rank-one projectors in the real space $\mathbb{R}^{n^2}$ of Hermitian operators. Using the basis of the generators τ_k of $U(n)$, with $\tau_1 = \mathbb{I}$ and $\operatorname{Tr}\tau_k = 0$ $(k = 2, \ldots, n^2)$, we may express any Hermitian operator A as

$$A = \sum_{k=1}^{n^2} \alpha^k \tau_k. \tag{32}$$

The manifold of rank-one projectors is given by the vectors whose components $\{\alpha^k\} \in \mathbb{R}^{n^2}$ fulfill the conditions $A^2 = A$, $\operatorname{Tr} A = 1$ (which implies $\alpha^1 = 1/n$). In the dual space $u^*(n)$ of the Lie algebra of the generators τ_k, this manifold is an orbit $\mathcal{O}_\mathcal{P}$ of the co-adjoint action of the group $U(n)$. As the set of rank-one projectors may be identified with the projective space $\mathbb{P}(\mathbb{C}^n)$ with $2(n-1)$ real dimensions, the orbit $\mathcal{O}_\mathcal{P}$ is the only orbit with the same dimensions placed in the plane $\alpha^1 = 1/n$. Among the orbits, $\mathcal{O}_\mathcal{P}$ is the one with lowest dimensionality, as the stability group of its points is just $U(1) \times U(n-1)$. For instance, for $n = 2$, the rank-one projector set is the Bloch sphere S^2: $\{(\alpha^2)^2 + (\alpha^3)^2 + (\alpha^4)^2 = 1/4\}$ placed in the plane $\alpha^1 = 1/2$.

A minimal tomographic set therefore is a set of n^2 projectors $\{P(m_k)\}$, where $m_k \in \mathcal{O}_\mathcal{P}$, such that the linear span of $\{P(m_k)\}$ is the entire $\mathbb{R}^{n^2}$, that is $\det\{(\alpha_k)^l\} \neq 0$. Of course, $\mathcal{O}_\mathcal{P}$ is a maximal tomographic set.

3.2. Families of operators generating tomographic sets

An interesting question is how can one find a way to construct tomographic sets. This question may be answered in several equivalent ways. We consider some of them here and provide a few well-known examples to show how our proposal works.

The first way consists in taking a fiducial rank-one projector P_0 and acting on it with a suitable family of (at least n^2) unitary operators U_α, depending on some parameters α. The family has to be chosen in such a manner that the set of projectors

$$P_\alpha = U_\alpha P_0 U_\alpha^\dagger \tag{33}$$

results in a tomographic one. This is granted only if the family U_α is not contained in any proper subgroup of $U(n)$ or, equivalently, if the group generated by the family U_α is $U(n)$. This condition is also sufficient if, moreover, from the family of unitary operators it is possible to extract, *via* the Cayley map, for instance, a basis for the Lie algebra $u(n)$. Then, a family which is "skew" in the group $U(n)$ is a suitable *tomographic family of unitary operators*.

Alternatively, it is possible to start with a fiducial Hermitian operator A_0 and to act on it with a "skew" family of unitary operators U_α, generating a family of (iso-spectral) Hermitian operators

$$A_\alpha = U_\alpha A_0 U_\alpha^\dagger. \tag{34}$$

Choosing A_0 to be generic, i.e., with simple eigenvalues, the action of U_α on the rank-one projectors associated with the eigenstates of A_0 gives rise to a tomographic set of projectors. In other words, we may obtain a tomographic set from a suitable family of Hermitian operators.

Taking $U(n)$ as a tomographic family we obtain a (maximal) decomposition of the identity, analogous to Eq. (24). However, integrating all projectors P over the symplectic orbit $\mathcal{O}_\mathcal{P}$ and using the volume $\Omega = \omega^{n-1}$ constructed with the canonical symplectic form ω, we have to use a Hermitian kernel $\hat{K}(m)$, which is an operator-valued function of the point m on the orbit $\mathcal{O}_\mathcal{P}$, that plays the same role of the Gram–Schmidt kernel in the minimal case:

$$\mathbb{I}_{n^2} = \int_{\mathcal{O}_\mathcal{P}} \hat{K}(m)\operatorname{Tr}\bigl(P(m)\,\cdot\bigr)\Omega. \tag{35}$$

For instance, in the $U(2)$ case, we have

$$\hat{K}(\theta,\phi)=\frac{1}{4\pi}\begin{bmatrix}1+3\cos\theta & 3e^{-i\phi}\sin\theta\\ 3e^{i\phi}\sin\theta & 1-3\cos\theta\end{bmatrix} \tag{36}$$

so that, for any operator A, it results

$$A=\int_0^{2\pi}\int_0^{\pi}\hat{K}(\theta,\phi)\,\mathrm{Tr}\big(P(\theta,\phi)A\big)\sin\theta\,d\theta\,d\phi. \tag{37}$$

By using Eq. (35) in the scalar product of any pair of operators A, B, we obtain

$$\mathrm{Tr}(AB)=\mathrm{Tr}\bigg(A\int_{\mathcal{O}_{\mathcal{P}}}\hat{K}(m)\,\mathrm{Tr}\big(P(m)B\big)\Omega\bigg)=\int_{\mathcal{O}_{\mathcal{P}}}\mathrm{Tr}\big(\hat{K}(m)A\big)\,\mathrm{Tr}\big(P(m)B\big)\Omega$$
$$=\int_{\mathcal{O}_{\mathcal{P}}}\mathrm{Tr}\big(\hat{K}(m)B\big)\,\mathrm{Tr}\big(P(m)A\big)\Omega=\mathrm{Tr}\bigg(A\int_{\mathcal{O}_{\mathcal{P}}}P(m)\,\mathrm{Tr}\big(\hat{K}(m)B\big)\Omega\bigg).$$

So, at least in a "weak sense",

$$\mathbb{I}_{n^2}=\int_{\mathcal{O}_{\mathcal{P}}}P(m)\,\mathrm{Tr}\big(\hat{K}(m)\,\cdot\big)\Omega\;\Rightarrow\;\hat{K}(m)=\int_{\mathcal{O}_{\mathcal{P}}}P(m')\,\mathrm{Tr}\big(\hat{K}(m')\hat{K}(m)\big)\Omega'.$$

Finally, substituting the previous expression of $\hat{K}(m)$ in Eq. (35), we obtain

$$\mathbb{I}_{n^2}=\int_{\mathcal{O}_{\mathcal{P}}}\bigg[\int_{\mathcal{O}_{\mathcal{P}}}\mathrm{Tr}\big(\hat{K}(m')\hat{K}(m)\big)P(m')\Omega'\bigg]\mathrm{Tr}\big(P(m)\,\cdot\big)\Omega \tag{38}$$

in full analogy with Eq. (24). Hence, $\hat{K}(m)$ is just a Gram–Schmidt kernel, at least in a "weak sense".

In general, a tomographic family of unitary operators ranges between a minimal family and a maximal family, and is representative of the whole group $U(n)$, so that necessarily the commutant of the family must be the identity. Then, a decomposition of unity must be available by integrating on the space of parameters of the family with a suitable kernel $\hat{K}(m)$.

4. Examples

4.1. Spin tomography

For a qu-bit (i.e., a spin 1/2 particle), a tomographic iso-spectral two-parameter family of Hermitian operators is

$$A(\theta,\phi):=\begin{bmatrix}\cos\theta & e^{-i\phi}\sin\theta\\ e^{i\phi}\sin\theta & -\cos\theta\end{bmatrix}. \tag{39}$$

A given operator of the family corresponds to the component of the spin (up to a factor $\hbar/2$) in the direction

$$\vec{n}=(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta).$$

In fact,

$$\vec{n}\cdot\vec{\sigma}=A(\theta,\phi). \tag{40}$$

The spectrum of $A(\theta,\phi)$ and corresponding orthonormal eigenvectors are

$$m_\pm=\pm1,\qquad |m_+\theta\phi\rangle=\begin{bmatrix}e^{-i\phi/2}\cos\frac{\theta}{2}\\ e^{i\phi/2}\sin\frac{\theta}{2}\end{bmatrix},\qquad |m_-\theta\phi\rangle=\begin{bmatrix}e^{-i\phi/2}\sin\frac{\theta}{2}\\ -e^{i\phi/2}\cos\frac{\theta}{2}\end{bmatrix}, \tag{41}$$

while the respective projectors are

$$|m_+\theta\phi\rangle\langle m_+\theta\phi|=\begin{bmatrix}\cos^2\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2}\end{bmatrix},\qquad |m_-\theta\phi\rangle\langle m_-\theta\phi|=\begin{bmatrix}\sin^2\frac{\theta}{2} & -e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\\ -e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2}\end{bmatrix}.$$

They may be collected in the general form of a rank-one projector:

$$P(\theta,\phi)=\frac{1}{2}[\mathbb{I}+\vec{n}\cdot\vec{\sigma}]=\frac{1}{2}\begin{bmatrix}1+\cos\theta & e^{-i\phi}\sin\theta\\ e^{i\phi}\sin\theta & 1-\cos\theta\end{bmatrix}, \tag{42}$$

as

$$P(\theta,\phi)=|m_+\theta\phi\rangle\langle m_+\theta\phi|,\qquad P(\pi-\theta,\pi+\phi)=|m_-\theta\phi\rangle\langle m_-\theta\phi|.$$

Due to their form, two pairs of eigenvectors $|m_\pm\theta\phi\rangle$, $|m_\pm\theta'\phi'\rangle$ are not sufficient to yield a basis of projectors, so that at least three different operators of the family $A(\theta,\phi)$ are needed to construct a minimal tomographic set, as the previous example has shown. In the spin case, starting from the fiducial basis associated with $A(0,0)$

$$|m_+\rangle=\begin{bmatrix}1\\0\end{bmatrix},\qquad |m_-\rangle=\begin{bmatrix}0\\1\end{bmatrix},$$

any $A(\theta,\phi)$-basis is related to the fiducial basis *via* the unitary transformation $U(\theta,\phi)$:

$$U(\theta,\phi)|m_\pm\rangle=|m_\pm\theta\phi\rangle,$$

where

$$U(\theta,\phi):=\begin{bmatrix}e^{-i\phi/2}\cos\frac{\theta}{2} & e^{-i\phi/2}\sin\frac{\theta}{2}\\ e^{i\phi/2}\sin\frac{\theta}{2} & -e^{i\phi/2}\cos\frac{\theta}{2}\end{bmatrix}. \tag{43}$$

Previous analysis has shown that two unitary operators $U(\theta,\phi),U(\theta',\phi')$ are a suitably "skew" set when $\Im(e^{i(\phi'-\phi)}\sin\theta'\sin\theta)\neq 0$. Thus, in the qu-bit case, a minimal set is obtained by taking a projector from each pair of eigenvectors and completing the basis with any other of the remaining projectors. So, as well known [27,28], only three independent directions of $\vec{n}$ are needed to reconstruct a spin 1/2 state, because the fourth projector is along any direction orthogonal to one of the first three directions.

However, a decomposition of identity involving the whole family exists and is given by (see the previous Eqs. (36), (37))

$$\mathbb{I}=\int_0^{2\pi}\int_0^{\pi}\hat{K}(\theta,\phi)\,\mathrm{Tr}\big(P(\theta,\phi)\,\cdot\big)\sin\theta\,d\theta\,d\phi. \tag{44}$$

We observe that the kernel $\hat{K}(\theta,\phi)$ is the simplest, the only one containing the same few spherical functions which appear in the projectors $P(\theta,\phi)$, but it is not unique. In fact, a family of equivalent kernels can be obtained by adding to $\hat{K}(\theta,\phi)$ any other kernel $\hat{K}_0(\theta,\phi)$, containing only superpositions of spherical functions orthogonal to those of $P(\theta,\phi)$.

As a matter of fact, in the qu-bit case the equation

$$\mathbb{I}=\int_0^{2\pi}\int_0^{\pi}P(\theta,\phi)\,\mathrm{Tr}\big(\hat{K}(\theta,\phi)\,\cdot\big)\sin\theta\,d\theta\,d\phi \tag{45}$$

holds in a strong sense. Hence $\hat{K}(\theta,\phi)$ is just a Gram–Schmidt kernel in a strong sense.

For the general case of spin $-j\leqslant m\leqslant j$, a decomposition of identity involving the whole family exists [18] and reads

$$\sum_{m=-j}^{j}\int d\Omega\,\big(R(\theta,\phi)|m\rangle\langle m|R^\dagger(\theta,\phi)\big)_{m'm''}\big(\hat{K}(m,\theta,\phi)\big)_{s's''}=\delta(m'-s')\delta(m''-s''),$$

where $R(\theta,\phi)$ is a rotation through (θ,ϕ) angles and $d\Omega=\sin\theta\,d\theta\,d\phi$, while

$$\big(\hat{K}(m,\theta,\phi)\big)_{s's''}=\sum_{j_3=0}^{2j}\sum_{m_3=-j_3}^{j_3}(2j_3+1)^2\int(-1)^m D^{(j_3)}_{0m_3}(\phi,\theta,\gamma)\begin{pmatrix}j & j & j_3\\ m & -m & 0\end{pmatrix}\begin{pmatrix}j & j & j_3\\ s' & -s'' & m_3\end{pmatrix}\frac{d\gamma}{8\pi^2}. \tag{46}$$

The problem of a minimal reconstruction formula for spin states was discussed in Refs. [27,28].

4.2. Photon number tomography

This is an infinite-dimensional case. However, the iso-spectral tomographic family of operators has a countable discrete spectrum $n=1,2,3,\dots,\infty$, so that a generalization of our definitions is straightforward. We assume the fiducial basis $\{|n\rangle\}$ of the harmonic oscillator number operator $\hat{a}^\dagger\hat{a}$, and with the unitary family of displacement operators

$$\mathcal{D}(\alpha)=\exp\big(\alpha\hat{a}^\dagger-\alpha^*\hat{a}\big) \tag{47}$$

we generate the tomographic family $A(\alpha)$ depending on the complex parameter α:

$$A(\alpha)=\mathcal{D}(\alpha)\hat{a}^\dagger\hat{a}\mathcal{D}^\dagger(\alpha)=\big(\hat{a}^\dagger-\alpha^*\big)(\hat{a}-\alpha), \tag{48}$$

and the respective basis of eigenvectors $\{|n\alpha\rangle\}=\{\mathcal{D}(\alpha)|n\rangle\}$.

The photon number tomogram of a density operator $\hat{\rho}$ is

$$\mathcal{W}_\rho(n,\alpha) = \mathrm{Tr}\big(|n\alpha\rangle\langle n\alpha|\hat{\rho}\big) \tag{49}$$

while the inversion formula reads

$$\hat{\rho} = \sum_{n=0}^{\infty}\int \frac{d^2\alpha}{\pi}\mathcal{W}_\rho(n,\alpha)K^{(s)}(n,\alpha). \tag{50}$$

The operator-valued kernel $K^{(s)}$ is given by

$$K^{(s)}(n,\alpha) = \frac{2}{1-s}\left(\frac{s+1}{s-1}\right)^n T(-\alpha,-s), \tag{51}$$

where the operator T is

$$T(\alpha,s) = \mathcal{D}(\alpha)\left(\frac{s+1}{s-1}\right)^{\hat{a}^\dagger\hat{a}}\mathcal{D}^\dagger(\alpha). \tag{52}$$

Here s is a real parameter, $-1 < s < 1$, which labels the family of equivalent kernels $K^{(s)}(n,\alpha)$.

The matrix form of Eq. (50) in the position representation is

$$\rho(x,y) = \int dx'\,dy'\left[\sum_{n=0}^{\infty}\int\frac{d^2\alpha}{\pi}\langle y'|n\alpha\rangle\langle n\alpha|x'\rangle\langle x|K^{(s)}(n,\alpha)|y\rangle\right]\rho(x',y'),$$

where $|x\rangle$, $|x'\rangle$, $|y\rangle$, $|y'\rangle$ are position eigenstates. To evaluate the matrix elements of the Gram–Schmidt kernel operator $K^{(s)}$, we first calculate those of the displacement operator $\mathcal{D}(\alpha)$. Remembering that $\hat{a} = (Q+iP)/\sqrt{2}$, we have

$$\mathcal{D}(\alpha) = \exp\left((\alpha-\alpha^*)\frac{Q}{\sqrt{2}} - i(\alpha+\alpha^*)\frac{P}{\sqrt{2}}\right)$$

and putting

$$\alpha = (\nu - i\mu)/\sqrt{2},$$

we get

$$\langle y|\exp(-i\mu Q - i\nu P)|y'\rangle = \delta(y-y'-\nu)\exp\big[i(-\mu y' - \mu\nu/2)\big].$$

Now the following map is useful

$$\left(\frac{s+1}{s-1}\right)^{\hat{a}^\dagger\hat{a}} = \exp\left[-i\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\tau_s + i\frac{\tau_s}{2}\right], \quad \tau_s := i\ln\left(\frac{s+1}{s-1}\right),$$

where a determination of ln has been chosen in such a way that $\tau_s > 0$ for $s = 0$. Then one readily obtains

$$\langle x|\exp\left[-i\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\tau_s + i\frac{\tau_s}{2}\right]|y\rangle = \frac{1}{\sqrt{2\pi i\sin\tau_s}}\exp\left(i\left[\left(x^2+y^2\right)\frac{\cot\tau_s}{2} - \frac{xy}{\sin\tau_s} + \frac{\tau_s}{2}\right]\right),$$

and the matrix element $\langle x|T(\alpha,s)|y\rangle$ results as

$$\int dx'\,dy'\,\langle x|\mathcal{D}(\alpha)|x'\rangle\langle x'|\left(\frac{s+1}{s-1}\right)^{\hat{a}^\dagger\hat{a}}|y'\rangle\langle y'|\mathcal{D}(-\alpha)|y\rangle$$
$$= \frac{1}{\sqrt{2\pi i\sin\tau_s}}\exp\left(i\left[\left((x-\nu)^2+(y-\nu)^2\right)\frac{\cot\tau_s}{2} - \frac{(x-\nu)(y-\nu)}{\sin\tau_s} - \mu(x-y) + \frac{\tau_s}{2}\right]\right),$$

so that eventually the matrix element $\langle x|K^{(s)}(n,\alpha)|y\rangle$ reads

$$\langle x|K^{(s)}(n,\alpha)|y\rangle = \frac{2}{1-s}\left(\frac{s+1}{s-1}\right)^n\frac{1}{\sqrt{2\pi i\sin\tau_{-s}}}$$
$$\times\exp\left(i\left[\left((x+\nu)^2+(y+\nu)^2\right)\frac{\cot\tau_{-s}}{2} - \frac{(x+\nu)(y+\nu)}{\sin\tau_{-s}} + \mu(x-y) + \frac{\tau_{-s}}{2}\right]\right). \tag{53}$$

4.3. Symplectic tomography

In the symplectic case ($\hbar = 1$), we start from the fiducial basis $\{|X\rangle\}$ of (improper) eigenvectors of the position operator Q: $Q|X\rangle = X|X\rangle$, whose spectrum is the whole real axis: $X \in \mathbb{R}$. The two-real parameter family of unitary canonical operators $S(\mu,\nu)$:

$$S(\mu,\nu) = \exp\left[i\frac{\lambda}{2}(QP+PQ)\right]\exp\left[i\frac{\theta}{2}\big(Q^2+P^2\big)\right] \quad \big(\mu = e^{\lambda}\cos\theta, \nu = e^{-\lambda}\sin\theta\big),$$

generates both an iso-spectral family $A(\mu,\nu)$ of Hermitian operators

$$A(\mu,\nu) = S(\mu,\nu)QS^{\dagger}(\mu,\nu) = \mu Q + \nu P$$

and a tomographic set of (improper) eigenvectors $|X\mu\nu\rangle = S(\mu,\nu)|X\rangle$, such that $\langle X'\mu\nu|X\mu\nu\rangle = \delta(X-X')$. In the position representation $\{|q\rangle\}$ it is, for $\nu \neq 0$:

$$\langle q|X\mu\nu\rangle = \langle q|S(\mu,\nu)|X\rangle = \frac{1}{\sqrt{2\pi|\nu|}}\exp\left[-i\left(\frac{\mu}{2\nu}q^2 - \frac{X}{\nu}q\right)\right]. \tag{54}$$

Now, substituting the explicit expression of the symplectic tomogram:

$$\mathcal{W}_\rho(X,\mu,\nu) = \mathrm{Tr}\big(|X\mu\nu\rangle\langle X\mu\nu|\hat\rho\big) = \int \langle q|X\mu\nu\rangle\langle X\mu\nu|q'\rangle\langle q'|\hat\rho|q\rangle\,dq\,dq',$$

in the well-known inversion formula for $\langle y|\hat\rho|y'\rangle$:

$$\rho(y,y') = \frac{1}{2\pi}\int \mathcal{W}_\rho(X,\mu,\nu)\langle y|\exp\big[i(X-\mu Q-\nu P)\big]|y'\rangle\,dX\,d\mu\,d\nu, \tag{55}$$

we obtain

$$\rho(y,y') = \frac{1}{2\pi}\int\left\{\int \langle y|\exp\big[i(X-\mu Q-\nu P)\big]|y'\rangle\langle q'|X\mu\nu\rangle\langle X\mu\nu|q\rangle\,dX\,d\mu\,d\nu\right\}\rho(q,q')\,dq\,dq'. \tag{56}$$

In different terms

$$I(y,y';q,q') = \int \frac{dX}{2\pi}\,d\mu\,d\nu\,\langle y|\exp\big[i(X-\mu Q-\nu P)\big]|y'\rangle\langle q'|X\mu\nu\rangle\langle X\mu\nu|q\rangle = \delta(q-y)\delta(q'-y'),$$

so that a partition of identity generated by the tomographic set of rank-one projectors $|X\mu\nu\rangle\langle X\mu\nu|$ appears in the inversion formula.

Let us check the previous equation. From

$$\begin{aligned}\langle y|\exp\big[i(X-\mu Q-\nu P)\big]|y'\rangle &= e^{iX}\langle y|\exp(-i\nu P)\exp(-i\mu Q)\exp(-i\mu\nu/2)|y'\rangle \\ &= \delta(y-y'-\nu)\exp\big[i(X-\mu y'-\mu\nu/2)\big],\end{aligned} \tag{57}$$

we have the following expression of $I(y,y';q,q')$:

$$I(y,y';q,q') = \int \frac{dX\,d\mu\,d\nu}{(2\pi)^2|\nu|}\delta(y-y'-\nu)\exp\left[i\left(X-\mu y'-\frac{\mu\nu}{2}\right)\right]\exp\left[i\frac{\mu}{2\nu}\big(q^2-q'^2\big)-i\frac{X}{\nu}(q-q')\right]. \tag{58}$$

Integrating over X we get $|\nu|\delta(q-q'-\nu)$, which can be used to linearize the quadratic term:

$$\exp\left[i\frac{\mu}{2\nu}\big(q^2-q'^2\big)\right]\delta(q-q'-\nu) = \exp\left[i\frac{\mu}{2}(q+q')\right]\delta(q-q'-\nu) \tag{59}$$

and we may write $I(y,y';q,q')$ as

$$\int \frac{d\mu\,d\nu}{2\pi}\delta(y-y'-\nu)\delta(q-q'-\nu)\exp\left[i\frac{\mu}{2}(q+q'-2y'-\nu)\right]. \tag{60}$$

Integration over μ yields

$$I(y,y';q,q') = \int d\nu\,\delta(y-y'-\nu)\delta(q-q'-\nu)2\delta(q+q'-2y'-\nu). \tag{61}$$

Eventually, we get the expected result:

$$\begin{aligned}I(y,y';q,q') &= 2\delta\big(q-q'-(y-y')\big)\delta\big(q+q'-2y'-(y-y')\big) \\ &= 2\delta\big(q-y-(q'-y')\big)\delta\big(q-y+(q'-y')\big) = \delta(q-y)\delta(q'-y').\end{aligned} \tag{62}$$

We conclude this subsection recalling a problem posed by Pauli [25], whether it is possible to recover the state vector of a quantum system from the marginal probability distributions of the physical observables (e.g., position and momentum) of that system. The answer to the question is obviously negative [26]. The example of two squeezed states described by Gaussian wave functions,

$$\psi_1(x) = N \exp\left[-\alpha x^2 + i\beta x\right], \qquad \psi_2(x) = N \exp\left[-\alpha^* x^2 + i\beta x\right],$$

where $\Re\alpha > 0$ and $\beta^* = \beta$, demonstrates readily this negative answer. The moduli of the functions are equal, as are the moduli of their Fourier transforms. But the states are different since the scalar product of the two wave functions gives the fidelity which is not equal to one. So, different wave functions have the same marginal probability distributions of position and momentum. Now we are able to understand why the answer must be negative: in fact, the family containing only the operators position and momentum is not tomographic because it is too small and cannot generate an inversion formula.

4.4. *Squeeze tomography*

Finally, we discuss an example where an inversion formula is still lacking: the squeeze tomography [29]. The tomogram is defined using the same unitary operators $S(\mu, \nu)$ of the symplectic tomography, which acting on the fiducial basis $\{|n\rangle\}$ of the photon number tomography generate a basis of squeezed eigenvectors $\{|n\mu\nu\rangle\} = \{S(\mu, \nu)|n\rangle\}$ of the squeezed tomographic family $A_{sq}(\mu, \nu)$

$$A_{sq}(\mu, \nu) = S(\mu, \nu)\hat{a}^\dagger \hat{a} S^\dagger(\mu, \nu). \tag{63}$$

In this case, however, the commutant of the family $A_{sq}(\mu, \nu)$ contains the parity operator and is nontrivial. Then the family is not a tomographic family *strictu senso*. Nevertheless, we get a true tomographic family by a restriction to the subspace of even wave functions. Then the existence of an inversion formula is granted.

5. Conclusions

We summarize the main results of the Letter. The mechanism by which quantum states are describable as fair tomographic probabilities instead of wave functions or density matrices was clarified. The mathematical reason why the pure state projector $|\psi\rangle\langle\psi|$ (or density operator $\hat{\rho}$) may be expressed in terms of tomograms is based on the simple observation that any kind of tomogram is just a scalar product of the projector, treated as a vector in the Hilbert space of operators, and a basis vector in this Hilbert space. The only property to be fulfilled is that the basis vectors in that Hilbert space are a complete (or even an overcomplete) set. For known examples of tomographies we have shown that it is always so. In view of this very elementary property it is even mysterious why the finding of the tomographic probability description of quantum states was done only relatively recently.

Another result of the Letter concerns finding explicit Gram–Schmidt orthogonalizator kernels for symplectic and photon number tomographies.

Consideration of tomograms of quantum states can be conceptually extended to the case of many degrees of freedom and even to quantum field theory. For this one only needs a pair of Hilbert spaces, $\mathcal{H}$ and $B(\mathcal{H})$, and the construction of a tomographic basis in $B(\mathcal{H})$. To extend to field theory, one, in principle, only has to add some extra ingredients to take into account the existence of different non-unique representations of the infinite Heisenberg–Weyl algebra and to use extra topological arguments. The tomographic approach can also be applied to quantum groups. For this one can construct a tomographic basis from the eigenstates of quantum group operators. For the $su_q(2)$ case it only needs the introduction of operators dependent on the Casimir.

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