Generalized tomographic maps

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We introduce several possible generalizations of tomography to curved surfaces. We analyze different types of elliptic, hyperbolic, and hybrid tomograms. In all cases it is possible to consistently define the inverse tomographic map. We find two different ways of introducing tomographic sections. The first method operates by deformations of the standard Radon transform. The second method proceeds by shifting a given quadric pattern. The most general tomographic transformation can be defined in terms of marginals over surfaces generated by deformations of complete families of hyperplanes or quadrics. We discuss practical and conceptual perspectives and possible applications.

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I. INTRODUCTION

Most of classical applications of tomography are based on light propagation along optic rays (implicitly assumed to be straight lines). Standard Radon transform theory guarantees that a measurement of the absorption of light beams traveling in dielectric media in straight lines allows the complete reconstruction of the matter density of these media. Indeed, the original Radon transform [1] maps functions of two variables in the plane onto functions of one real variable on a line and one variable on a circle. The crucial property is that the transform is invertible and continuous [2,3].

There exist several generalizations of the Radon transform. See, e.g., [4] and [5]. Further generalizations can be motivated by physical observations: for instance, if the function on the plane is a probability density, its Radon component is a family of probability densities of one random variable on the line, parametrized by a variable on a circle [6]. A tomographic approach in a similar framework was applied to a free classical particle moving on a circle [7], where the phase space is a two-dimensional cylinder.

In quantum mechanics, the Radon transform of the Wigner function [8] was considered in the tomographic approach to the study of quantum states [9,10] and experimentally realized with different particles and in diverse situations [11–13]. Other experiments have been proposed [14] and the whole field is in continuing evolution, also in view of its relevance in genuine quantum mechanical problems and quantum information related topics. Good reviews on recent tomographic applications can be found in Ref. [15], with emphasis on maximum likelihood estimations [16], which enable one to extract the maximum reliable information from the available data.

A further development, extending the analysis to incorporate more general symplectic transforms, was presented in [17], and the mathematical mechanism at the basis of the mapping of true density states onto tomographic probabilities was elucidated in [18]. There is an interesting relation between the Radon map of Wigner functions and the formalism of star product quantization [6,19]: symplectic tomograms are indeed the Radon components of the Wigner function, and this enables one to define a procedure aimed at determining the marginal probability densities along straight lines in phase space. The knowledge of all these marginals makes possible the reconstruction of the Wigner function in the quantum case and of the probability density in the classical case.

The generalization of tomographic maps to curved surfaces opens new perspectives in the applications of tomography to both quantum and classical systems. Some attempts to study marginals along curves other than straight lines were introduced in Ref. [20]. Very recently, optical "accelerating" Airy beams were observed [21]: these beams could be used to perform a tomographic map over parabolas in phase space. A generalization of tomography to this kind of application requires a generalization of the Radon transform.

The aim of this paper is to study generalizations of the Radon transform to multidimensional phase spaces and to marginals along curves or surfaces. Most of the generalizations of the Radon transform proceed by considering geodesic submanifolds of a given Riemannian manifold. We develop here a different approach, which can be applied to the Radon components of the probability densities of classical particles in phase space, and construct the corresponding tomographic maps.

This paper is organized as follows. In Sec. II we review the standard tomographic application of the Radon transform on the plane. In Sec. III we consider the generalization to arbitrary dimensions. A deformation of the Radon transform with applications to elliptic and hyperbolic problems is presented in Sec. IV. In Sec. V we introduce a transform involving hyperbolic, elliptic, and parabolic quadrics. The transform is defined by translations of a basic pattern. Finally in

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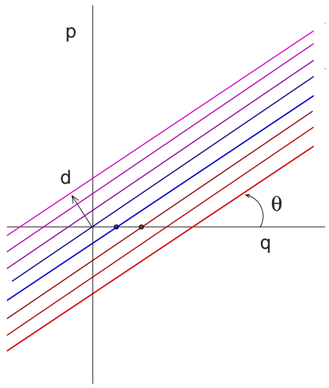

FIG. 1. (Color online) Tomography on the plane; $(q,p)\in\mathbb{R}^2$, $d\in\mathbb{R}$, $\theta\in\mathbb{S}$ (unit circle).

Sec. VI we discuss the relevance of our results for future applications.

II. TOMOGRAPHY ON THE PLANE

Let us consider a function $f(q,p)$ on the phase space $(q,p)\in\mathbb{R}^2$ of a particle moving on the line $q\in\mathbb{R}$. The Radon transform, in its original formulation, solves the following problem: reconstruct a function of two variables, say $f(p,q)$, from its integrals over arbitrary lines.

In the (q,p) plane, a line is given by the equation

$$X-\mu q-\nu p=0, \tag{1}$$

with $(\mu,\nu)\neq(0,0)$. Thus, the family of lines has the manifold structure $\mathbb{R}\times\mathbb{S}$, with $\mathbb{S}$ the unit circle, $d=X/\sqrt{\mu^2+\nu^2}\in\mathbb{R}$, and $\mu/\nu=\tan\theta$, $\theta\in\mathbb{S}$ (see Fig. 1).

It is possible to write the Radon transform in affine language (tomographic map) [1,5] as

$$\omega_f(X,\mu,\nu)=\langle\delta(X-\mu q-\nu p)\rangle$$
$$=\int_{\mathbb{R}^2}f(q,p)\,\delta(X-\mu q-\nu p)dq\,dp, \tag{2}$$

where δ is the Dirac function and the parameters $X,\mu,\nu\in\mathbb{R}$.

The inverse transform of (2) reads

$$f(q,p)=\int_{\mathbb{R}^3}\omega_f(X,\mu,\nu)e^{i(X-\mu q-\nu p)}\frac{dX\,d\mu\,d\nu}{(2\pi)^2}. \tag{3}$$

The positive homogeneity of $\omega_f(X,\mu,\nu)$,

$$\omega_f(\lambda X,\lambda\mu,\lambda\nu)=\frac{1}{|\lambda|}\omega_f(X,\mu,\nu), \tag{4}$$

$\forall\ \lambda\in\mathbb{R}$, $\lambda\neq 0$, is a direct consequence of (2). If the function $f(q,p)$ is a probability density function on the phase space of a classical particle, i.e.,

$$f(q,p)\geq 0,\qquad \int_{\mathbb{R}^2}f(q,p)dq\,dp=1, \tag{5}$$

the function $\omega_f(X,\mu,\nu)$ is also nonnegative and is called symplectic tomogram or Radon transform of the distribution function $f(q,p)$ (in analogy to the Fourier transform of a function). The Radon transform contains the same information on the state of the particle evolving on the phase space as the initial distribution function. In summary, the tomograms

$$\omega_f(X,\mu,\nu)\geq 0,\qquad \int_{\mathbb{R}}\omega_f(X,\mu,\nu)dX=1,\qquad \forall\ \mu,\nu, \tag{6}$$

form a family of density functions that depends on the two real parameters μ and ν.

III. TOMOGRAMS ON HYPERPLANES

The above construction can be generalized to higher-dimensional spaces in a straightforward way. Let us consider a function $f(q)$ on the n-dimensional space $q\in\mathbb{R}^n$. Is it possible to reconstruct the function f from its integrals over arbitrary $(n-1)$-dimensional linear submanifolds? The answer to this question is positive and provides a generalization of the original Radon transform.

A generic hyperplane is given by the equation

$$X-\mu\cdot q=0, \tag{7}$$

with $X\in\mathbb{R}$ and $\mu\in\mathbb{R}^n\backslash 0$. Due to homogeneity, this family of hyperplanes is an n-dimensional manifold diffeomorphic to $\mathbb{R}\times\mathbb{S}^{n-1}$, because any hyperplane can be characterized by its unit normal vector $\mu/|\mu|$ and its distance to the origin $|X|/|\mu|$. Note that this manifold is not diffeomorphic to $\mathbb{R}^n$ because the sphere $\mathbb{S}^{n-1}$ is compact.

The Radon transform is given by

$$\omega_f(X,\mu)=\langle\delta(X-\mu\cdot q)\rangle=\int_{\mathbb{R}^n}f(q)\,\delta(X-\mu\cdot q)d^nq. \tag{8}$$

When $n=2$ Eq. (2) is recovered.

The inverse transform of (8) reads

$$f(q)=\int_{\mathbb{R}^{n+1}}\omega_f(X,\mu)e^{i(X-\mu\cdot q)}\frac{dX\,d^n\mu}{(2\pi)^n}. \tag{9}$$

The homogeneity of $\omega_f(X,\mu)$

$$\omega_f(\lambda X,\lambda\mu)=\frac{1}{|\lambda|}\omega_f(X,\mu), \tag{10}$$

$\forall\lambda\in\mathbb{R}$, $\lambda\neq 0$, is a direct consequence of (8). If the function $f(q)$ is a probability density function on $\mathbb{R}^n$,

$$f(q)\geq 0,\qquad \int_{\mathbb{R}^n}f(q)d^nq=1, \tag{11}$$

the tomograms $\omega_f(X,\mu)$ are also probability densities,

$$\omega_f(X,\mu)\geq 0,\qquad \int_{\mathbb{R}}\omega_f(X,\mu)dX=1,\qquad \forall\ \mu\in\mathbb{R}^n, \tag{12}$$

and the family of tomograms depends on the n real parameters μ. In quantum mechanics this construction was applied to Wigner functions providing a center of mass tomography [22].

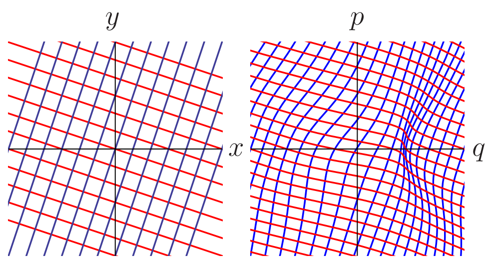

FIG. 2. (Color online) Diffeomorphism of the plane: $(q,p)\in\mathbb{R}^2\to(x,y)=\varphi(q,p)\in\mathbb{R}^2$.

IV. TOMOGRAMS ON HYPERSURFACES

A simple mechanism that allows nonlinear generalizations of the Radon transform is the combination of the standard transform with a diffeomorphism of the underlying $\mathbb{R}^n$ space. Let us consider a function $f(q)$ on the n-dimensional space $q\in\mathbb{R}^n$. The problem is to reconstruct f from its integrals over an n-parameter family of submanifolds of codimension 1.

We can construct such a family by diffeomorphic deformations of the hyperplanes (in the $x\in\mathbb{R}^n$ space)

$$X-\mu\cdot x=0, \tag{13}$$

with $X\in\mathbb{R}$ and $\mu\in\mathbb{R}^n\backslash 0$. Let us consider a diffeomorphism of $\mathbb{R}^n$

$$q\in\mathbb{R}^n\mapsto x=\varphi(q)\in\mathbb{R}^n. \tag{14}$$

The hyperplanes (13) are deformed by φ into a family of submanifolds (in the q space)

$$X-\mu\cdot\varphi(q)=0. \tag{15}$$

The case $n=2$ is displayed in Fig. 2: $(q,p)\in\mathbb{R}^2\to(x,y)=\varphi(q,p)\in\mathbb{R}^2$.

Given a probability density $\tilde{f}(x)$ on the x space, the Radon transform can be rewritten as

$$\omega_f(X,\mu)=\langle\delta(X-\mu\cdot x)\rangle=\int_{\mathbb{R}^n}\tilde{f}(x)\delta(X-\mu\cdot x)d^nx$$
$$=\int_{\mathbb{R}^n}\tilde{f}(\varphi(q))\delta(X-\mu\cdot\varphi(q))J(q)d^nq, \tag{16}$$

where

$$J(q)=\left|\frac{\partial x_i}{\partial q_j}\right|=\left|\frac{\partial\varphi_i(q)}{\partial q_j}\right| \tag{17}$$

is the Jacobian of the transformation.

Observe now that

$$\tilde{f}(x)d^nx=\tilde{f}(\varphi(q))J(q)d^nq, \tag{18}$$

whence $f(q)=\tilde{f}(\varphi(q))J(q)$ is a probability density. Therefore the tomograms are given by

$$\omega_f(X,\mu)=\langle\delta(X-\mu\cdot\varphi(q))\rangle=\int_{\mathbb{R}^n}f(q)\delta(X-\mu\cdot\varphi(q))d^nq, \tag{19}$$

with $X\in\mathbb{R}$ and $\mu\in\mathbb{R}^n$.

The inverse transform follows by (9):

$$f(q)=\tilde{f}(\varphi(q))J(q)=\int_{\mathbb{R}^{n+1}}\omega_f(X,\mu)J(q)e^{i[X-\mu\cdot\varphi(q)]}\frac{dX\,d^n\mu}{(2\pi)^n}, \tag{20}$$

with a modified kernel

$$K(q;X,\mu)=J(q)e^{i[X-\mu\cdot\varphi(q)]}=\left|\frac{\partial\varphi_i(q)}{\partial q_j}\right|e^{i[X-\mu\cdot\varphi(q)]}. \tag{21}$$

Therefore, a probability density distribution on $\mathbb{R}^n$,

$$f(q)\geq 0,\qquad\int_{\mathbb{R}^n}f(q)d^nq=1, \tag{22}$$

produces tomograms $\omega_f(X,\mu)$ that are probability densities,

$$\omega_f(X,\mu)\geq 0,\qquad\int_{\mathbb{R}}\omega_f(X,\mu)dX=1,\qquad\forall\,\mu\in\mathbb{R}^n. \tag{23}$$

The family of tomograms depends on the n real parameters μ. We can now consider different applications of these deformed generalizations of the Radon transform.

A. Circles in the plane

In the punctured (x,y) plane without the origin (0,0), the conformal inversion

$$(x,y)=\varphi(q,p)=\left(\frac{q}{q^2+p^2},\frac{p}{q^2+p^2}\right) \tag{24}$$

maps the family of lines

$$X-\mu x-\nu y=0 \tag{25}$$

into a family of circles

$$X(q^2+p^2)-\mu q-\nu p=0, \tag{26}$$

centered at

$$C=\left(\frac{\mu}{2X},\frac{\nu}{2X}\right) \tag{27}$$

and passing through the origin (see Fig. 3). When $X=0$ they degenerate into lines through the origin.

The Jacobian reads

$$J(q,p)=\left|\frac{\partial(x,y)}{\partial(q,p)}\right|=\frac{1}{(q^2+p^2)^2}, \tag{28}$$

whence the transformation is a diffeomorphism of the punctured plane. The singularity of the transformation at the origin (0, 0) is irrelevant for tomographic integral transformations of functions $f\in L^1(\mathbb{R}^2)$, because it affects only a zero-measure set .

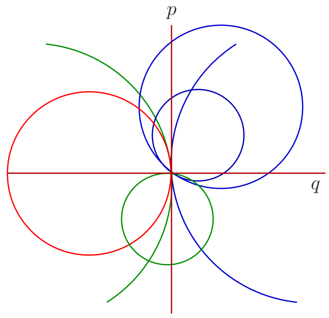

FIG. 3. (Color online) Deformed circular tomography. All circles pass through the origin.

A few comments on singularities are in order, in particular in view of possible quantum applications. The standard Radon transform, which is normally used in optical tomography schemes for measuring quantum states, has known singularities in the reconstruction formula of the Wigner function from the optical tomogram of the phase variable. The problems due to the singularities can be overcome in several interesting ways (see [23,24] and [15] for an overview). Notice that the symplectic tomography reconstruction formula has redundant information, which is contained in the extra parameter dependence of the tomogram, and does not display such singularities. In fact, it can be considered as one of the possible tools to avoid this difficulty, since it reconstructs the quantum state by making direct use of the experimental data. The nonlinear tomographic schemes we introduced generalize the standard symplectic tomographic scheme (that contain enough extra parameters, providing redundant information), but also avoid singularities in the reconstruction formula. A new type of singularity might appear in some cases, due to the presence of vanishing Jacobians, but these singularities are not generic, affect only sets of data with zero measure, and are therefore completely harmless from a physical viewpoint.

Equations (19) and (20) become

$$\omega_f(X,\mu,\nu) = \left\langle \delta\left(X - \frac{\mu q}{q^2+p^2} - \frac{\nu p}{q^2+p^2}\right)\right\rangle = \int_{\mathbb{R}^2} f(q,p)\,\delta\left(X - \frac{\mu q}{q^2+p^2} - \frac{\nu p}{q^2+p^2}\right) dq\,dp \tag{29}$$

and

$$f(q,p) = \int_{\mathbb{R}^3} \omega_f(X,\mu) \frac{e^{i[X-\mu q/(q^2+p^2)-\upsilon p/(q^2+p^2)]}}{(2\pi)^2(q^2+p^2)^2} dX\,d\mu\,d\nu. \tag{30}$$

B. Hyperbolas in the plane

In the (x,y) plane the family of lines

$$X - \mu x - \nu y = 0 \tag{31}$$

is mapped into a family of hyperbolas

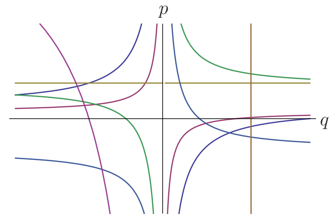

FIG. 4. (Color online) Hyperbolic tomography.

$$X - \frac{\mu}{q} - \nu p = 0, \tag{32}$$

with asymptotes

$$q = 0, \quad p = \frac{X}{\nu}, \tag{33}$$

by the transformation

$$(x,y) = \varphi(q,p) = \left(\frac{1}{q}, p\right). \tag{34}$$

For $\mu > 0$ the hyperbolas are in the second and fourth quadrants, while for $\mu < 0$ they are in the first and third quadrants (see Fig. 4). When $\mu=0$ or $\nu=0$ they degenerate into horizontal or vertical lines, respectively.

The Jacobian reads

$$J(q,p) = \left| \frac{\partial(x,y)}{\partial(q,p)} \right| = \frac{1}{q^2}, \tag{35}$$

whence the transformation is a diffeomorphism in the cut plane without the axis $(0,p)$.

Equations (19) and (20) become

$$\omega_f(X,\mu,\nu) = \left\langle \delta\left(X - \frac{\mu}{q} - \nu p\right)\right\rangle = \int_{\mathbb{R}^2} f(q,p)\,\delta\left(X - \frac{\mu}{q} - \nu p\right) dq\,dp \tag{36}$$

and

$$f(q,p) = \int_{\mathbb{R}^3} \omega_f(X,\mu) \frac{1}{q^2} e^{i(X-\mu/q-\nu p)} \frac{dX\,d\mu\,d\nu}{(2\pi)^2}. \tag{37}$$

The tomograms (19), (29), and (36) have the homogeneity property (10).

C. Hyperboloids in $\mathbb{R}^n$

The generalization to higher dimensions of tomographic maps that can be given in terms of quadratic expressions is straightforward. Let us consider for example the following tomographic map:

$$\omega_f(X,\mu,\nu) = \int_{\mathbb{R}^{2n}} \delta(X - \mu\cdot q - \nu(q,p)) f(q,p) d^n q\, d^n p, \tag{38}$$

where p and q are vectors in $\mathbb{R}^n$ and

$$\nu(q,p) = \sum_{j=1}^{n} \nu_j q_j p_j. \tag{39}$$

This map corresponds to a deformation of the standard multidimensional Radon transform by means of the following diffeomorphism of $\mathbb{R}^{2n}\setminus\cup_j\{(q,p):q_j=0\}$:

$$(q_i,p_j) \mapsto (x_i,y_j) = (q_i,q_jp_j), \tag{40}$$

whose Jacobian is

$$J(q,p) = \left|\frac{\partial(x,y)}{\partial(q,p)}\right| = \prod_{j=1}^{n} |q_j|. \tag{41}$$

The inverse map is given by

$$f(q,p) = \int_{\mathbb{R}^{2n+1}} \frac{dX\, d^n\mu\, d^n\nu}{(2\pi)^{2n}} \omega_f(X,\mu,\nu) \prod_{j=1}^{n} |q_j| e^{i[X-\mu\cdot q-\nu(q,p)]}. \tag{42}$$

This corresponds to the higher-dimensional generalization of the Bertrand-Bertrand tomography [9].

Note that, when $n=2$, by interchanging the role of X and $-\mu$, one recovers the same distribution of hyperbolas in the plane analyzed in Sec. IV B.

Although the above generalizations might be very useful for light-ray tomograms, all of them involve integration over unbounded submanifolds. One would like to generalize the Radon transform to marginals defined over *compact* submanifolds, which are bounded on a compact domain around (p,q). This case will be investigated in the following section.

V. TOMOGRAMS ON QUADRICS

Let us now look for a different generalization of tomograms. We shall consider marginals along compact quadrics. This can be achieved by shifting and scaling a given quadric pattern

$$X = (q-\mu, B(q-\mu)), \tag{43}$$

where B is a nondegenerate symmetric operator with respect to the scalar product $(x,y)=x\cdot y$. A new generalization of the tomographic map can be defined by

$$\omega_f(X,\mu;B) = \int_{\mathbb{R}^n} f(q)\,\delta(X-(q-\mu,B(q-\mu)))d^nq. \tag{44}$$

Equation (44) defines a completely different type of transform, with support on the quadrics defined by Eq. (43). It is easy to show that the inverse map is defined by

$$f(q) = \frac{|\det B|}{\pi^n} \int_{\mathbb{R}^{n+1}} dX\, d^n\mu\, \omega_f(X,\mu;B) e^{i[X-(q-\mu,B(q-\mu))]}. \tag{45}$$

Indeed, by applying the definition of tomographic map (45),

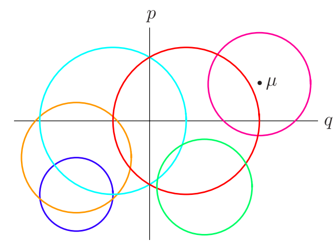

FIG. 5. (Color online) Tomography on circles of center μ and (squared) radius X/b^2 on the plane.

$$\begin{aligned}&\frac{|\det B|}{\pi^n}\int dX\, d^n\mu\, e^{i[X-(q-\mu,B(q-\mu))]}\omega_f(X,\mu;B)\\ &\quad = \frac{|\det B|}{\pi^n}\int dX\, d^n\mu\, e^{i[X-(q-\mu,B(q-\mu))]}\\ &\quad\quad \times \int d^n\xi\,\delta(X-(\xi-\mu,B(\xi-\mu)))f(\xi),\end{aligned} \tag{46}$$

which after integration over X yields

$$\begin{aligned}&\frac{|\det B|}{\pi^n}\int d^n\xi\, f(\xi)\int d^n\mu\, e^{i[(\xi-\mu,B(\xi-\mu))-(q-\mu,B(q-\mu))]}\\ &\quad = \frac{|\det B|}{\pi^n}\int d^n\xi\, d^n\mu\, f(\xi)e^{i[(\xi,B\xi)-(q,Bq)+2(q-\xi,B\mu)]}\\ &\quad = \int d^n\xi\, f(\xi)e^{i[(\xi,B\xi)-(q,Bq)]}\delta^n(q-\xi) = f(q).\end{aligned} \tag{47}$$

The meaning of the above tomographic map depends on the features of B. If we assume that B is strictly positive (elliptic case), this map corresponds to averages of f along the ellipsoids defined by Eq. (43). In particular, if all the eigenvalues of B are equal to b^2, it corresponds to integration over spheres centered at μ of (squared) radius X/b^2, namely,

$$b^2(q-\mu)^2 = X \quad (X>0). \tag{48}$$

Note that, in the two-dimensional case, the distribution of circles is different from that obtained by the transform defined by diffeomorphisms in Sec. IV. There, the family of tomograms was defined only on circles passing through the origin, including straight lines (circles of infinite radius). Here, we are taking into account all possible circles of finite radius in the plane (see Fig. 5). This corresponds to trajectories of particles moving in a plane under the action of a constant magnetic field. From a practical perspective, this new tomographic map would make possible a different practical implementation of tomography.

When B has both positive and negative eigenvalues this corresponds to hyperbolic tomography with averages of f along the hyperboloids defined by Eq. (43), e.g.,

$$b^2(q_1-\mu_1)^2 - c^2(q_2-\mu_2)^2 = X. \tag{49}$$

In the case of degenerate B forms we have to consider a hybrid transform. B can then be decomposed into a nondegenerate bilinear form and a linear form. In this case the tomography of the linear components should be treated as the standard Radon transform, whereas the nondegenerate variables should transform as above. Let us consider for example a simple three-dimensional case with

$$\bar{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (50)$$

In this case we can define the following tomographic map:

$$\omega_f(X,\mu;\bar{B}) = \int_{\mathbb{R}^3} d^3q\, f(q)\,\delta(X - (q_1-\mu_1)^2 - (q_2-\mu_2)^2 - \mu_3 q_3), \quad (51)$$

with inverse transform

$$f(q) = \int_{\mathbb{R}^4} \frac{dX\, d^3\mu}{2\pi^3}\omega_f(X,\mu;\bar{B})e^{i[X-(q_1-\mu_1)^2-(q_2-\mu_2)^2-\mu_3 q_3]}. \quad (52)$$

VI. CONCLUSIONS AND PERSPECTIVES

Let us discuss the main findings of this paper, from both mathematical and physical perspectives. From a mathematical viewpoint, the generalizations of the Radon transform introduced here enable one to consider marginals defined over submanifolds that are not necessarily geodesic submanifolds in Riemannian spaces or Lagrangian submanifolds of symplectic manifolds. These transforms define tomograms over compact submanifolds and can be more suitable for physical applications, because the practical implementation of a tomogram can be achieved only in local terms. In this framework, the recovery of a local value (of a probability distribution on phase space in the classical theory, or of a Wigner distribution in the quantum case) involves only integration over manifolds that do not reach infinity. In a previous paper [7], we considered the tomography of a classical particle moving on a circle, which required the definition of marginals over the helices on a cylinder. Now, in the light of the transforms just introduced, we have the possibility of performing tomography over compact submanifolds even for classical systems that evolve in unbounded domains. This is a significant conceptual step forward.

Physically, the reconstruction formulas enable one to generalize the measurement procedures of the matter density of an object. In a material medium with a strongly inhomogeneous refractive index, the radiation beams (light beams, sonic beams or matter waves) would propagate along curved lines and yet yield complete information on the matter distribution by means of generalized Radon transforms. For illustrative purposes, our examples focus on two-dimensional situations [see, e.g., Equations (24), (34), and (49)], but the approach we propose is more general and valid in $\mathbb{R}^n$.

In quantum optics these "nonlinear" Radon transforms can be easily extended to the quantum domain by using the Weyl-Wigner map. This will be discussed in a future presentation. The results of this paper show that the reconstruction of the Wigner function using optical or symplectic tomography based on a straight-line Radon transform can be extended to situations in which the marginals in phase space are measured for curved hyperbolas or ellipses. In particular, parabolic tomography could be implemented with the recently observed accelerated Airy beams [21].

Novel physical applications of tomography have attracted increasing attention during the last few years. Recent applications involving neutrinos, e.g., to get a mapping of the Earth inner density [25], do not require new concepts of tomographic maps. However, neutrino tomography of γ-ray bursts and massive stellar collapses [26] might require generalized tomography. In particular, γ-ray tomography that made possible the discovery of asphericity in supernovae explosions [27] or imaging of astrophysical sources [28] can involve nonlinear trajectories of γ rays due to strong gravitational lensing effects. In those cases, generalizations of tomographic maps like the ones considered in this paper are necessary.

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