

Geometry of depolarizing channels

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Depolarizing maps acting on an N -dimensional system are completely positive maps resulting into compression of the Bloch “ball” along N^2-1 polarization directions. In the qubit case these maps are a convex sum of four extremal maps and form a simplex in the space of compression coefficients along the three polarization directions. We calculate the compression domain for three- and four-level systems. For a three-level system the region has curved surfaces, but it is a simplex for a four-level system. We conjecture that it is a simplex in the case of 2^n -level systems.

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I. INTRODUCTION

The dynamics of open quantum systems can be described in terms of completely positive (CP), trace preserving maps acting on the system [1–6]. In general these maps can rotate, compress and translate the Bloch ball, resulting into “Affine” maps [7]. These maps are used to simulate different noises that could be acting on the system [8,9].

A simple example of noise can be given by depolarizing maps, which result into shrinking of the Bloch ball. In general the compression can be anisotropic, giving a rich geometric structure to the problem. The requirement of complete positivity turns out to be stronger than just positivity. For example, for $N=2$ the map corresponding to shrinking of the Bloch sphere only along one direction and leaving the other two invariant (pancake-shaped map) is not CP. In $N=2$ case the general depolarizing map has a very simple structure and can be written as a convex sum of four fixed extremal maps. One would like to see if it true for higher-dimensional systems.

In what follows we answer the question for $N=3$ and $N=4$ cases. We also give a way to answer the question for higher level systems, although we must admit that the calculations becomes much more involved.

Our paper is organized as follows. In Sec. II we briefly describe the geometry of $N=2$ level depolarizing channels. In Sec. III we show how to calculate the region of positivity for an N -dimensional system. We carry out the calculations for $N=3$ and $N=4$ cases in Secs. IV and V, respectively. We conclude the paper with further discussions in Sec. VI.

II. THE QUBIT CASE

The qubit density matrix is given by

$$\rho = \frac{1}{2}(I + \vec{a} \cdot \vec{\sigma}). \quad (1)$$

Positivity of ρ requires that \vec{a} lies inside a sphere with unit radius. Let us assume that due to depolarization, the sphere shrinks along the three polarization directions by factors of

ν_1 , ν_2 , and ν_3 . The natural question arises—what are the allowed values of ν_1 , ν_2 , and ν_3 to ensure complete positivity of the map?

It turns out [10] that the allowed region in the space of ν_1 , ν_2 , and ν_3 forms a tetrahedron with vertices at $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, and $(-1,-1,1)$. These vertices correspond to four extremal maps given by $B_i(\rho) = \sigma_i \rho \sigma_i$, where $i=0,1,2,3$ and $\sigma_0=I_2$. The general depolarizing map can be written as a convex sum of these maps as

$$B(\rho) = p\sigma_1\rho\sigma_1 + q\sigma_2\rho\sigma_2 + r\sigma_3\rho\sigma_3 + (1-p-q-r)\rho, \quad (2)$$

where $0 \leq p, q, r \leq 1$.

Only the region inside the tetrahedron is CP. For example, partial transpose corresponds to the point $(1,-1,1)$ and lies outside the region (see Figs. 1 and 2).

III. THE N -LEVEL SYSTEM

For an N -level system the density matrix can be written in terms of N^2-1 traceless, Hermitian, and trace-orthogonal generators of $SU(N)$, denoted by J_i , and the polarizations along those directions (a_i) as

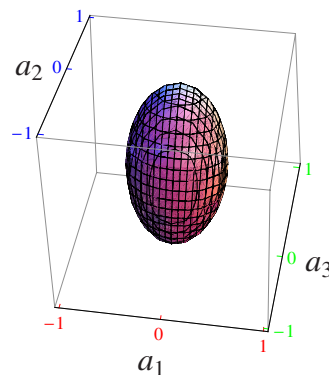


FIG. 1. (Color online) As a result of depolarization the Bloch sphere shrinks anisotropically.

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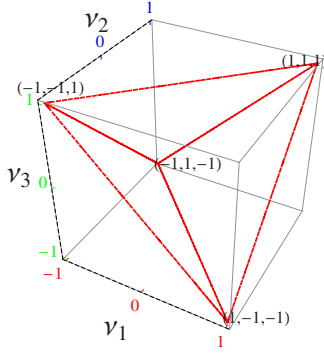


FIG. 2. (Color online) CP region in the space of v 's is a tetrahedron with vertices corresponding to maps $1\rho 1$, $\sigma_1\rho\sigma_1$, $\sigma_2\rho\sigma_2$ and $\sigma_3\rho\sigma_3$.

$$\rho = \frac{1}{N} \left(I_n + \sqrt{\frac{N(N-1)}{2}} \sum_{i=1}^{N^2-1} a_i J_i \right). \quad (3)$$

This normalization results into pure states having unit length [12], analogous to the qubit case.

Let us assume that a general map acting on ρ takes it into another density matrix given by ρ' . The map can be written in terms of the A matrix [1] as

$$\rho'_{rs} = A_{rs,r's'} \rho_{r's'}. \quad (4)$$

With change of indices, we get another matrix B which is Hermitian and has simpler conditions of complete positivity. It is obtained by

$$B_{rr',ss'} = A_{rs,r's'}. \quad (5)$$

The necessary and sufficient condition for the map to be CP is that B should be positive semidefinite (and thus have non-negative eigenvalues). In fact, eigenvectors of the B matrix constitute the C matrices that give the CP conditions in the well known form [5]

$$B(\rho) = \sum_i C_i^\dagger \rho C_i \quad (6)$$

with

$$\sum_i C_i C_i^\dagger = I. \quad (7)$$

Deriving A from compression of the coherence vector. The depolarizing map changes the decoherence vector as

$$a'_i = v_{ij} a_j. \quad (8)$$

In general the depolarizing map can combine both rotation and compression maps acting on the coherence vector. However, since rotations are completely positive maps, it's only the compression that gives physically interesting results. In other words, given

$$v = R_1 v_d R_2, \quad (9)$$

where R_1 and R_2 are rotations belonging to $SO(N^2-1)$ and v_d is diagonal, for complete positivity we need to consider only v_d .

We incorporate the $[(N^2-1) \otimes (N^2-1)]$ -dimensional v_d in $N^2 \otimes N^2$ matrix T which acts on the vector $(1, \vec{a})$ as

$$a'_i = T_{ij} a_j = (\delta_{ij} v_j) a_j. \quad (10)$$

Needless to say, $T_{11}=1$ as required by the trace preserving condition. Both the T and the A matrices contain all the information about the map and are thus related. The relation can be established in the following way.

Since the supermatrix A acts on all the entries in ρ , we can convert ρ into a vector (let us call it V^ρ) which is acted upon by A . The entries in V^ρ are linearly related to the coherence vector as

$$V_i^\rho = M_{ij} a_j. \quad (11)$$

The map takes V^ρ to $V^{\rho'}$ which is related to V^ρ as

$$V^{\rho'} = M a' = M T a = A V^\rho = A M a. \quad (12)$$

Thus

$$A = M T M^{-1}. \quad (13)$$

Once we get A and B we can find the eigenvalues of B and check for complete positivity. We solve the case of $N=3$ and $N=4$ in next two sections.

IV. $N=3$ CASE

The qutrit density matrix is written as

$$\rho = \frac{1}{3} \left(1 + \sqrt{3} \sum_{i=1}^8 a_i \lambda_i \right), \quad (14)$$

where λ 's are the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (15)$$

As shown in Eq. (10), T can be written as

$$T = \text{diag}(1, v_1, v_2, \dots, v_8). \quad (16)$$

M relates the density matrix to the coherence vector as

$$V_i^\rho = M_{ij} a_j. \quad (17)$$

Explicitly,

$$\frac{1}{3} \begin{pmatrix} 1 + \sqrt{3}a_3 + a_8 \\ \sqrt{3}(a_1 - ia_2) \\ \sqrt{3}(a_4 - ia_5) \\ \sqrt{3}(a_1 + ia_2) \\ 1 - \sqrt{3}a_3 + a_8 \\ \sqrt{3}(a_6 - ia_7) \\ \sqrt{3}(a_4 + ia_5) \\ \sqrt{3}(a_6 + ia_7) \\ 1 - 2a_8 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 1 \\ 0 & \sqrt{3} & -\sqrt{3}i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & -\sqrt{3}i & 0 & 0 & 0 \\ 0 & \sqrt{3} & \sqrt{3}i & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & -\sqrt{3}i & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & \sqrt{3}i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & \sqrt{3}i & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix}. \tag{18}$$

Calculation for $A=MTM^{-1}$ yields

$$A = \begin{pmatrix} \frac{1}{3} + \frac{\nu_3}{2} + \frac{\nu_8}{6} & 0 & 0 & 0 & \frac{1}{3} - \frac{\nu_3}{2} + \frac{\nu_8}{6} & 0 & 0 & 0 & \frac{1}{3} - \frac{\nu_8}{3} \\ 0 & \frac{\nu_1}{2} + \frac{\nu_2}{2} & 0 & \frac{\nu_1}{2} - \frac{\nu_2}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\nu_4}{2} + \frac{\nu_5}{2} & 0 & 0 & 0 & \frac{\nu_4}{2} - \frac{\nu_5}{2} & 0 & 0 \\ 0 & \frac{\nu_1}{2} - \frac{\nu_2}{2} & 0 & \frac{\nu_1}{2} + \frac{\nu_2}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} - \frac{\nu_3}{2} + \frac{\nu_8}{6} & 0 & 0 & 0 & \frac{1}{3} + \frac{\nu_3}{2} + \frac{\nu_8}{6} & 0 & 0 & 0 & \frac{1}{3} - \frac{\nu_8}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{\nu_6}{2} + \frac{\nu_7}{2} & 0 & \frac{\nu_6}{2} - \frac{\nu_7}{2} & 0 \\ 0 & 0 & \frac{\nu_4}{2} - \frac{\nu_5}{2} & 0 & 0 & 0 & \frac{\nu_4}{2} + \frac{\nu_5}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\nu_6}{2} - \frac{\nu_7}{2} & 0 & \frac{\nu_6}{2} + \frac{\nu_7}{2} & 0 \\ \frac{1}{3} - \frac{\nu_8}{3} & 0 & 0 & 0 & \frac{1}{3} - \frac{\nu_8}{3} & 0 & 0 & 0 & \frac{1}{3} + \frac{2\nu_8}{3} \end{pmatrix}. \tag{19}$$

The B matrix is obtained by change of indices as

$$B_{rr',ss'} = A_{rs,r's'} \tag{20}$$

and is given by

$$B = \begin{pmatrix} \frac{2+3\nu_3+\nu_8}{6} & 0 & 0 & 0 & \frac{\nu_1+\nu_2}{2} & 0 & 0 & 0 & \frac{\nu_4+\nu_5}{2} \\ 0 & \frac{2-3\nu_3+\nu_8}{6} & 0 & \frac{\nu_1-\nu_2}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}-\frac{\nu_8}{3} & 0 & 0 & 0 & \frac{\nu_4-\nu_5}{2} & 0 & 0 \\ 0 & \frac{\nu_1-\nu_2}{2} & 0 & \frac{2-3\nu_3+\nu_8}{6} & 0 & 0 & 0 & 0 & 0 \\ \frac{\nu_1+\nu_2}{2} & 0 & 0 & 0 & \frac{2+3\nu_3+\nu_8}{6} & 0 & 0 & 0 & \frac{\nu_6+\nu_7}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3}-\frac{\nu_8}{3} & 0 & \frac{\nu_6-\nu_7}{2} & 0 \\ 0 & 0 & \frac{\nu_4-\nu_5}{2} & 0 & 0 & 0 & \frac{1}{3}-\frac{\nu_8}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\nu_6-\nu_7}{2} & 0 & \frac{1}{3}-\frac{\nu_8}{3} & 0 \\ \frac{\nu_4+\nu_5}{2} & 0 & 0 & 0 & \frac{\nu_6+\nu_7}{2} & 0 & 0 & 0 & \frac{1+2\nu_8}{3} \end{pmatrix}. \quad (21)$$

For the map to be positive, all the eigenvalues should be positive semi-definite. Six of these eigenvalues are given by the ‘‘hyperplanes’’

$$\begin{aligned} h_1 = \lambda_1 &= \frac{1}{6}(2+3\nu_4-3\nu_5-2\nu_8), \\ h_2 = \lambda_2 &= \frac{1}{6}(2-3\nu_4+3\nu_5-2\nu_8), \\ h_3 = \lambda_3 &= \frac{1}{6}(2+3\nu_6-3\nu_7-2\nu_8), \\ h_4 = \lambda_4 &= \frac{1}{6}(2-3\nu_6+3\nu_7-2\nu_8), \\ h_5 = \lambda_5 &= \frac{1}{6}(2+3\nu_1-3\nu_2-3\nu_3+\nu_8), \\ h_6 = \lambda_6 &= \frac{1}{6}(2-3\nu_1+3\nu_2-3\nu_3+\nu_8). \end{aligned} \quad (22)$$

The remaining three eigenvalues are given by the eigenvalues of

$$H = \begin{pmatrix} \frac{2+3\nu_3+\nu_8}{6} & \frac{\nu_1+\nu_2}{2} & \frac{\nu_4+\nu_5}{2} \\ \frac{\nu_1+\nu_2}{2} & \frac{2+3\nu_3+\nu_8}{6} & \frac{\nu_6+\nu_7}{2} \\ \frac{\nu_4+\nu_5}{2} & \frac{\nu_6+\nu_7}{2} & \frac{1+2\nu_8}{3} \end{pmatrix}. \quad (23)$$

Since the matrix is Hermitian and thus has real roots, the condition of positive eigenvalues translates to $(\lambda_7+\lambda_8+\lambda_9)$, $(\lambda_7\lambda_8+\lambda_8\lambda_9+\lambda_9\lambda_7)$, and $\lambda_7\lambda_8\lambda_9$ being positive. These can be easily found from the characteristic equation as

$$\begin{aligned} h_7 = \lambda_7 + \lambda_8 + \lambda_9, \\ = 1 + \nu_3 + \nu_8, \end{aligned} \quad (24)$$

$$\begin{aligned} s_1 = \lambda_7\lambda_8 + \lambda_8\lambda_9 + \lambda_9\lambda_7, \\ = \frac{1}{12}[-3(\nu_1+\nu_2)^2 + 3\nu_3^2 - 3(\nu_4+\nu_5)^2 - 3(\nu_6+\nu_7)^2 \\ + (2+\nu_8)(2+3\nu_8) + 2\nu_3(4+5\nu_8)], \end{aligned} \quad (25)$$

$$\begin{aligned} s_2 = \lambda_7\lambda_8\lambda_9, &= \frac{1}{216}\{54\nu_2(\nu_4+\nu_5)(\nu_6+\nu_7) - 18\nu_1^2(1+2\nu_8) \\ &- 18\nu_2^2(1+2\nu_8) + 18\nu_1[3(\nu_4+\nu_5)(\nu_6+\nu_7) \\ &- 2\nu_2(1+2\nu_8)] + (2+3\nu_3+\nu_8)[4-9(\nu_4+\nu_5)^2 \\ &- 9(\nu_6+\nu_7)^2 + 10\nu_8 + 4\nu_8^2 + 6\nu_3(1+2\nu_8)]\}. \end{aligned} \quad (26)$$

Hence we see that the completely positive region is bounded by seven hyperplanes, a second degree surface s_1 and a third degree surface s_2 . This is quite different from the qubit case where the allowed region is bounded by four planes and is thus just a simplex. For $N=3$ the depolarizing map has an infinite number of extremal points corresponding to the curved surface.

Even though we have used the Gell-Mann basis for the calculation, this result is independent of a specific choice of basis. To see that let's use another basis of traceless, Hermitian and trace-orthogonal matrices which is obtained from the Gell-Mann basis as

$$\pi_i = k_{ij}\lambda_j. \tag{27}$$

Hermiticity requires that k_{ij} 's should be real. Trace orthogonality condition

$$\text{Tr}(\pi_i^\dagger \pi_l) = \text{Tr}(\lambda_j k_{ji} k_{lm} \lambda_m) = \delta_{il} \tag{28}$$

is satisfied if

$$k_{ij}k_{jk} = \delta_{ik}. \tag{29}$$

In other words, k is just a rotation matrix. As a result of the rotation the geometric properties of the region (for example, whether it is a simplex) remain unchanged.

V. $N=4$ CASE

From the previous section we might expect that the CP region will be a curved surface for $N=4$, but quite analogous to the qubit case it turns out to be a simplex.

Let us start with taking the generators to be $J = \{I \otimes \sigma_i, \sigma_i \otimes I, \sigma_1 \otimes \sigma_i, \sigma_2 \otimes \sigma_i, \sigma_3 \otimes \sigma_i\}$ for $i=1,2,3$. The density matrix is written as [11–13]

$$\rho = \frac{1}{4}(I + \sqrt{6}\vec{a} \cdot \vec{J}). \tag{30}$$

The M matrix is given by

$$M = \frac{\sqrt{6}}{4} \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 & 1 & -i & 0 & 0 & 0 & 1 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i & 0 & -i & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & i & 0 \\ \frac{1}{\sqrt{6}} & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & i & 0 & -i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -i & 0 & 0 & 0 & -1 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & i & 0 & 0 & 0 & 1 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i & 0 & i & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & i & 0 & i & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & i & 0 & 0 & 0 & -1 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -i & 0 \\ \frac{1}{\sqrt{6}} & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{31}$$

Let us take the T matrix as in the previous case

$$T = \text{diag}\{1, \nu_1, \dots, \nu_{15}\}. \tag{32}$$

The B matrix is given by

$$B = \frac{1}{4} \begin{pmatrix} 1 + \nu_3 + \nu_6 + \nu_{15} & 0 & 0 & 0 & 0 & \nu_1 + \nu_2 + \nu_{13} + \nu_{14} & 0 \\ 0 & 1 - \nu_3 + \nu_6 - \nu_{15} & 0 & 0 & \nu_1 - \nu_2 + \nu_{13} - \nu_{14} & 0 & 0 \\ 0 & 0 & 1 + \nu_3 - \nu_6 - \nu_{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 - \nu_3 - \nu_6 + \nu_{15} & 0 & 0 & \nu_1 - \nu_2 - \nu_{13} + \nu_{14} \\ 0 & \nu_1 - \nu_2 + \nu_{13} - \nu_{14} & 0 & 0 & 1 - \nu_3 + \nu_6 - \nu_{15} & 0 & 0 \\ \nu_1 + \nu_2 + \nu_{13} + \nu_{14} & 0 & 0 & 0 & 0 & 1 + \nu_3 + \nu_6 + \nu_{15} & 0 \\ 0 & 0 & 0 & \nu_1 - \nu_2 - \nu_{13} + \nu_{14} & 0 & 0 & 1 - \nu_3 - \nu_6 + \nu_{15} \\ 0 & 0 & \nu_1 + \nu_2 - \nu_{13} - \nu_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu_4 - \nu_5 + \nu_9 - \nu_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu_4 - \nu_5 - \nu_9 + \nu_{12} & 0 & 0 & \nu_7 - \nu_8 - \nu_{10} + \nu_{11} \\ \nu_4 + \nu_5 + \nu_9 + \nu_{12} & 0 & 0 & 0 & 0 & \nu_7 + \nu_8 + \nu_{10} + \nu_{11} & 0 \\ 0 & \nu_4 + \nu_5 - \nu_9 - \nu_{12} & 0 & 0 & \nu_7 - \nu_8 + \nu_{10} - \nu_{11} & 0 & 0 \\ 0 & 0 & 0 & \nu_7 - \nu_8 - \nu_{10} + \nu_{11} & 0 & 0 & \nu_4 - \nu_5 - \nu_9 + \nu_{12} \\ 0 & 0 & \nu_7 + \nu_8 - \nu_{10} - \nu_{11} & 0 & 0 & 0 & 0 \\ 0 & \nu_7 - \nu_8 + \nu_{10} - \nu_{11} & 0 & 0 & \nu_4 + \nu_5 - \nu_9 - \nu_{12} & 0 & 0 \\ \nu_7 + \nu_8 + \nu_{10} + \nu_{11} & 0 & 0 & 0 & 0 & \nu_4 + \nu_5 + \nu_9 + \nu_{12} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & \nu_4 + \nu_5 + \nu_9 + \nu_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \nu_4 + \nu_5 - \nu_9 - \nu_{12} & 0 & 0 \\ \nu_1 + \nu_2 - \nu_{13} - \nu_{14} & \nu_4 - \nu_5 + \nu_9 - \nu_{12} & 0 & 0 & 0 & 0 & \nu_7 + \nu_8 - \nu_{10} - \nu_{11} \\ 0 & 0 & \nu_4 - \nu_5 - \nu_9 + \nu_{12} & 0 & 0 & \nu_7 - \nu_8 - \nu_{10} + \nu_{11} & 0 \\ 0 & 0 & 0 & 0 & \nu_7 - \nu_8 + \nu_{10} - \nu_{11} & 0 & 0 \\ 0 & 0 & 0 & \nu_7 + \nu_8 + \nu_{10} + \nu_{11} & 0 & 0 & 0 \\ 0 & 0 & \nu_7 - \nu_8 - \nu_{10} + \nu_{11} & 0 & 0 & \nu_4 - \nu_5 - \nu_9 + \nu_{12} & 0 \\ 1 + \nu_3 - \nu_6 - \nu_{15} & \nu_7 + \nu_8 - \nu_{10} - \nu_{11} & 0 & 0 & 0 & 0 & \nu_4 - \nu_5 + \nu_9 - \nu_{12} \\ \nu_7 + \nu_8 - \nu_{10} - \nu_{11} & 1 + \nu_3 - \nu_6 - \nu_{15} & 0 & 0 & 0 & 0 & \nu_1 + \nu_2 - \nu_{13} - \nu_{14} \\ 0 & 0 & 1 - \nu_3 - \nu_6 + \nu_{15} & 0 & 0 & \nu_1 - \nu_2 - \nu_{13} + \nu_{14} & 0 \\ 0 & 0 & 0 & 1 + \nu_3 + \nu_6 + \nu_{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - \nu_3 + \nu_6 - \nu_{15} & 0 & 0 \\ 0 & 0 & \nu_1 - \nu_2 - \nu_{13} + \nu_{14} & 0 & 0 & 1 - \nu_3 - \nu_6 + \nu_{15} & 0 \\ \nu_4 - \nu_5 + \nu_9 - \nu_{12} & \nu_1 + \nu_2 - \nu_{13} - \nu_{14} & 0 & 0 & 0 & 0 & 1 + \nu_3 - \nu_6 - \nu_{15} \\ 0 & 0 & 0 & 0 & \nu_1 - \nu_2 + \nu_{13} - \nu_{14} & 0 & 0 \\ 0 & 0 & 0 & \nu_1 + \nu_2 + \nu_{13} + \nu_{14} & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \nu_7 + \nu_8 + \nu_{10} + \nu_{11} \\ \nu_7 - \nu_8 + \nu_{10} - \nu_{11} & 0 \\ 0 & 0 \\ 0 & 0 \\ \nu_4 + \nu_5 - \nu_9 - \nu_{12} & 0 \\ 0 & \nu_4 + \nu_5 + \nu_9 + \nu_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \nu_1 + \nu_2 + \nu_{13} + \nu_{14} \\ \nu_1 - \nu_2 + \nu_{13} - \nu_{14} & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 - \nu_3 + \nu_6 - \nu_{15} & 0 \\ 0 & 1 + \nu_3 + \nu_6 + \nu_{15} \end{pmatrix} \quad (33)$$

The eigenvalues of B are

$$\{\lambda_{ij}\} = \left[\begin{aligned} &\frac{1}{4}(1 + \nu_1 + \nu_2 + \nu_3 + \nu_4 - \nu_5 - \nu_6 + \nu_7 + \nu_8 + \nu_9 - \nu_{10} - \nu_{11} - \nu_{12} - \nu_{13} - \nu_{14} - \nu_{15}), \\ &\frac{1}{4}(1 + \nu_1 + \nu_2 + \nu_3 - \nu_4 + \nu_5 - \nu_6 - \nu_7 - \nu_8 - \nu_9 + \nu_{10} + \nu_{11} + \nu_{12} - \nu_{13} - \nu_{14} - \nu_{15}), \\ &\frac{1}{4}(1 + \nu_1 - \nu_2 - \nu_3 + \nu_4 + \nu_5 + \nu_6 + \nu_7 - \nu_8 - \nu_9 + \nu_{10} - \nu_{11} - \nu_{12} + \nu_{13} - \nu_{14} - \nu_{15}), \\ &\frac{1}{4}(1 + \nu_1 - \nu_2 - \nu_3 - \nu_4 - \nu_5 + \nu_6 - \nu_7 + \nu_8 + \nu_9 - \nu_{10} + \nu_{11} + \nu_{12} + \nu_{13} - \nu_{14} - \nu_{15}), \\ &\frac{1}{4}(1 - \nu_1 + \nu_2 - \nu_3 + \nu_4 + \nu_5 + \nu_6 - \nu_7 + \nu_8 - \nu_9 - \nu_{10} + \nu_{11} - \nu_{12} - \nu_{13} + \nu_{14} - \nu_{15}), \\ &\frac{1}{4}(1 - \nu_1 + \nu_2 - \nu_3 - \nu_4 - \nu_5 + \nu_6 + \nu_7 - \nu_8 + \nu_9 + \nu_{10} - \nu_{11} + \nu_{12} - \nu_{13} + \nu_{14} - \nu_{15}), \\ &\frac{1}{4}(1 - \nu_1 - \nu_2 + \nu_3 + \nu_4 - \nu_5 - \nu_6 - \nu_7 - \nu_8 + \nu_9 + \nu_{10} + \nu_{11} - \nu_{12} + \nu_{13} + \nu_{14} - \nu_{15}), \\ &\frac{1}{4}(1 - \nu_1 - \nu_2 + \nu_3 - \nu_4 + \nu_5 - \nu_6 + \nu_7 + \nu_8 - \nu_9 - \nu_{10} - \nu_{11} + \nu_{12} + \nu_{13} + \nu_{14} - \nu_{15}), \\ &\frac{1}{4}(1 - \nu_1 - \nu_2 + \nu_3 - \nu_4 - \nu_5 + \nu_6 + \nu_7 + \nu_8 - \nu_9 + \nu_{10} + \nu_{11} - \nu_{12} - \nu_{13} - \nu_{14} + \nu_{15}), \\ &\frac{1}{4}(1 - \nu_1 - \nu_2 + \nu_3 + \nu_4 + \nu_5 + \nu_6 - \nu_7 - \nu_8 + \nu_9 - \nu_{10} - \nu_{11} + \nu_{12} - \nu_{13} - \nu_{14} + \nu_{15}), \\ &\frac{1}{4}(1 - \nu_1 + \nu_2 - \nu_3 - \nu_4 + \nu_5 - \nu_6 + \nu_7 - \nu_8 + \nu_9 - \nu_{10} + \nu_{11} - \nu_{12} + \nu_{13} - \nu_{14} + \nu_{15}), \\ &\frac{1}{4}(1 - \nu_1 + \nu_2 - \nu_3 + \nu_4 - \nu_5 - \nu_6 - \nu_7 + \nu_8 - \nu_9 + \nu_{10} - \nu_{11} + \nu_{12} + \nu_{13} - \nu_{14} + \nu_{15}), \\ &\frac{1}{4}(1 + \nu_1 - \nu_2 - \nu_3 - \nu_4 + \nu_5 - \nu_6 - \nu_7 + \nu_8 + \nu_9 + \nu_{10} - \nu_{11} - \nu_{12} - \nu_{13} + \nu_{14} + \nu_{15}), \\ &\frac{1}{4}(1 + \nu_1 - \nu_2 - \nu_3 + \nu_4 - \nu_5 - \nu_6 + \nu_7 - \nu_8 - \nu_9 - \nu_{10} + \nu_{11} + \nu_{12} - \nu_{13} + \nu_{14} + \nu_{15}), \\ &\frac{1}{4}(1 + \nu_1 + \nu_2 + \nu_3 - \nu_4 - \nu_5 + \nu_6 - \nu_7 - \nu_8 - \nu_9 - \nu_{10} - \nu_{11} - \nu_{12} + \nu_{13} + \nu_{14} + \nu_{15}), \\ &\frac{1}{4}(1 + \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 + \nu_6 + \nu_7 + \nu_8 + \nu_9 + \nu_{10} + \nu_{11} + \nu_{12} + \nu_{13} + \nu_{14} + \nu_{15}) \end{aligned} \right]. \tag{34}$$

Since all of the eigenvalues are hyperplanes, CP region is a simplex. Also, the vertices of this polyhedron correspond to the maps $1\rho 1$, $J_i\rho J_i$, identical to the qubit maps. The extremal map $J_i\rho J_i$ is just a rotation of the Bloch ball about the J_i axis. The general depolarizing map can be written as a convex sum of these sixteen extremal maps.

VI. DISCUSSION

We found that the general depolarizing map in $N=2$ and $N=4$ cases has a simple geometric structure, while for

$N=3$ it is much more complicated. We conjecture that for any 2^n -dimensional case the CP region will be a simplex, with vertices corresponding to a unitary rotation of the Bloch ball. It would be interesting to explore the extent to which the $N=2$ and $N=4$ cases are similar for other maps.

We would also like to point out that our method of calculating B , and thus finding out the region where a map is CP is general and is not restricted to depolarizing maps. The most general map (Affine map) can be easily obtained if we

let T also contain translations (see Ref. [10]). In that case T becomes

$$T = \begin{pmatrix} 1 & 0 \\ \vec{t} & T' \end{pmatrix}, \quad (35)$$

where T' contains only the unital part.

The calculations presented here were carried out in Mathematica. Even though our method is general it might become impractical for higher dimensions. However, in the depolar-

izing case the T matrix is diagonal and the calculations involve partially filled matrices. Also, the method is still useful for $N=3$ and 4 cases where one is concerned with nondepolarizing maps.

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- [1] E. C. G. Sudarshan, P. M. Mathews, and J. Rau, *Phys. Rev.* **121**, 920 (1961).
 [2] T. F. Jordan and E. C. G. Sudarshan, *J. Math. Phys.* **2**, 772 (1961).
 [3] T. F. Jordan, M. A. Pinsky, and E. C. G. Sudarshan, *J. Math. Phys.* **3**, 848 (1962).
 [4] E. Størmer, *Acta Math.* **110**, 233 (1963).
 [5] M. D. Choi, *Can. J. Math.* **24**, 520 (1972).
 [6] E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, New York, 1976).
 [7] T. F. Jordan, A. Shaji, and E. C. G. Sudarshan, *Phys. Rev. A* **73**, 012106 (2006).
 [8] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
 [9] J. Preskill, quantum computation lecture notes at www.theory.caltech.edu/epreskill/ph219/index.html.
 [10] Mary Beth Ruskai, Stanislaw Szarek, and Elisabeth Werner, *Linear Algebr. Appl.* **347**, 159 (2002).
 [11] L. Jakóbczyk and M. Siennicki, *Phys. Lett. A* **286**, 383 (2001).
 [12] M. S. Byrd and N. Khaneja, *Phys. Rev. A* **68**, 062322 (2003).
 [13] Gen Kimura, *Phys. Lett. A* **314**, 339 (2003).