Extremal Channels for a genaral Quantum system

Kuldeep Dixit* and E. C. G. Sudarshan

Department of Physics, University of Texas, Austin, TX 78712.

Quantum channels can be mathematically represented as completely positive trace-preserving maps that act on a density matrix. A general quantum channel can be written as a convex sum of 'extremal' channels. We show that for an N-level system, the extremal channel can be characterized in terms of N^2-N real parameters coupled with rotations. We give a representation for N= 2, 3, 4.

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I. INTRODUCTION

A completely positive map acting on an N-level open quantum system can be characterized in terms of C matrices as [1, 2, 3, 4]:

$$\rho' = B(\rho) = \sum_i C_i \, \rho \, C_i^\dagger. \tag{1}$$

The above form automatically satisfies the Hermiticity and positive semi-definiteness conditions required for the resulting density matrix. Since the trace of ρ must also be preserved, C matrices satisfy the additional condition given by:

$$\sum_i C_i^\dagger C_i = I. \tag{2}$$

In terms of open quantum system dynamics, Completely positive maps arise from the unitary evolution of a larger system that comprises the system and the environment starting out in a simply separable state, i. e.

$$\rho' = Tr_{env}[U \, (\rho_{sys} \otimes \rho_{env}) \, U^\dagger]. \tag{3}$$

The most general map acting on ρ can be expressed in terms of N^2 such C matrices. However, a general map can be written as a convex sum of 'extramal' maps [5, 6]. For example, Unitary evolution of a density matrix is equivalent to having only one C matrix. Though trivial, it turns out that such maps play an important part in center preserving (Unital) maps because in two and four dimensional quantum systems a general Unital map can be written as a convex sum of unitary maps [7, 8].

Physically, extremal maps arise if the environment starts out in a pure state in eq. (3). In addition to satisfying eq. (2), the C matrices for the extremal map can be written in a manifestly trace orthonormal form, i. e., for $i \neq j$

$$Tr[C_i^\dagger C_j] = 0. \tag{4}$$

*Electronic address: kuldeep@physics.utexas.edu

Based on this condition it can be easily seen that a general extremal map can only have a maximum of N such C matrices. Ruskai et. al. [7] have explored extremal maps for a qubit case in detail and have found that apart from rotations, the extremal map can be expressed in terms of two independent parameters. We generalize this result and give a representation of the map for quantum systems with up to four levels.

II. THEOREM 1

Apart from rotations, a general extremal map acting on an N level system can be expressed in terms of $N^2 - N$ real parameters.

Proof: Let's start with N C matrices and write the singular value decomposition of each one as

$$C_i = U_i D_i V_i. \tag{5}$$

Here U and V are unitary matrices and D is a real diagonal matrix with positive entries. First we show that the V's are just permutation matrices and can be dropped from parameter counting. We provide a proof by induction. Let's start with a two level system. Equation (2) imposes the condition:

$$V_1^\dagger D_1^2 V_1 + V_2^\dagger D_2^2 V_2 = I. \tag{6}$$

Multiplying the whole equation on the left with V_1 and on the right with $V_1^\dagger$ gives us

$$D_1^2 + V'^\dagger_2 D_2^2 V_2' = I \tag{7}$$

or

$$V_2'^\dagger D_2^2 V_2' = I - D_1^2. \tag{8}$$

Since the right hand side is diagonal and the operation on the left hand side does not change the eigenvalues of D_2^2, the only effect that V_2' has is to permute the eigenvalues of D_2^2. In other words we can absorb the $V's$ in D itself. For a three level system we have:

$$V_1^\dagger D_1^2 V_1 + V_2^\dagger D_2^2 V_2 + V_3^\dagger D_3^2 V_3 = I. \tag{9}$$

Once again, this results into

$$V_1'^\dagger D_1^2 V_1' + V_2'^\dagger D_2^2 V_2' = I - D_3^2. \tag{10}$$

Since $I - D_3^2$ is a positive matrix, we can define the square root of its multiplicative inverse. Let's call it M. Multiplication of M on both sides gives us

$$M V_1'^\dagger D_1^2 V_1' M + M V_2'^\dagger D_2^2 V_2' M = I. \tag{11}$$

Since $M V_1'^\dagger D_1^2 V_1' M$ is both positive and Hermitian, we can write it as $V_1''^\dagger D_1'^2 V_1''$. Using this the above equation is reduced to our already worked case of equation (6). Thus we have shown that qutrit (and thus by induction all higher level systems) can be reduced to the qubit case.

We have shown that V's appear trivially in C. As the next step let's examine U's. The trace orthogonality equation (4) results into:

$$Tr[D_j D_i U_i^\dagger U_j] = 0. \tag{12}$$

Since $D_j D_i$ is a diagonal matrix, the above equation is satisfied if we let $U \equiv U_i^\dagger U_j$ have only non-diagonal entries. Such a U can be trivially constructed and we give examples for up to four dimensional quantum systems.

As seen from above each of the C matrices can be characterized in terms of just D matrices. Since for the most general case there are N C matrices and thus N of these diagonal matrices (requiring N^2 real parameters) and there is only one constraint on them, namely the completeness condition (which imposes N constraints), the extremal map will have only $N^2 - N$ independent parameters. In the following section we give some simple examples.

III.   EXAMPLES OF EXTREMAL MAPS

For the N=2 case the simple choice of matrices is given by:

$$D_1 = diag(a, b), \quad D_2 = \sqrt{I - D_1^2}, \quad U_1 = I, \quad U_2 = \sigma_1. \tag{13}$$

The geometry of the map can be made more transparent by putting

$$a = \mu_0 + \mu_3$$
$$b = \mu_0 - \mu_3$$
$$\mu_0 = \frac{1}{4}\sqrt{1 + \nu_1 + \nu_2 + \nu_3}$$
$$\mu_3 = \frac{1}{4}\sqrt{1 + \nu_3 - \nu_1 - \nu_2}. \tag{14}$$
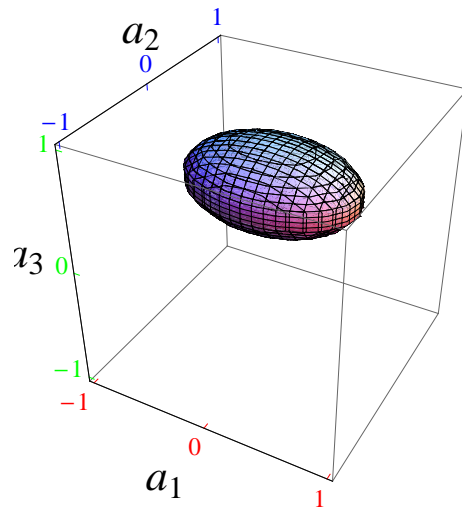

FIG. 1: "(color online)" Extremal map acting on a qubit compresses the Bloch sphere along the three polarizations by ν_1, ν_2, $\nu_1\nu_2$ and translates the resulting ellipsoid by $\sqrt{(1 - \nu_1\nu_2)^2 - (\nu_1 - \nu_2)^2}$ along the σ_3 direction.

Geometrically the map is a depolorizing map (see [8]) coupled with translation, i. e. the map shrinks σ_i polarization by a factor of ν_i and translates the resulting Bloch ellipsoid along σ_3 by an amount

$$t_3 = \sqrt{(1 - \nu_3)^2 - (\nu_1 - \nu_2)^2}. \tag{15}$$

The completeness condition gives the relation

$$\nu_3 = \nu_1 \nu_2. \tag{16}$$

Thus the shrinking along σ_3 is a product of the shrinkages along the other two directions. The independent parameters of the map are ν_1 and ν_2.

For $N = 3$ case the geometry of the map is much more complicated. However the C matrices can be easily constructed as

$$D_1 = diag(a, b, c), \quad D_2 = diag(d, e, f), \quad D_3 = \sqrt{I - D_1^2 - D_2^2}$$

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{17}$$

Finally, for $N = 4$ the U matrices are given by:

$$U_1 = I, \quad U_2 = 1 \otimes \sigma_1, \quad U_3 = \sigma_1 \otimes 1, \quad U_4 = \sigma_1 \otimes \sigma_1. \tag{18}$$

IV.   CONCLUSION

We have shown how to represent an extremal map for dimensions N = 2, 3, 4. N=4 level case is particularly

important since it allows for entanglement between two qubit subsystems. The generalization to higher level systems is easily accomplished. In each case the map can be written in terms of $N^2 - N$ real parameters. For N=2 case the parameters present a very intuitive geometrical picture, namely that the two parameters are the compression coefficients along two axes of the block sphere. The compression along the third direction is just the product of these two compressions. For higher level systems the geometry becomes more difficult, however it would be very insightful to see if similar relations exist between the compression coefficients.

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