Role of preparation in quantum process tomography

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It was recently pointed out how differences in preparation procedures for quantum experiments can lead to nontrivial differences in the results of the experiment. The differences arise from the initial correlations between the system and the environment. Therefore, any quantum experiment that is prone to influences from the environment must be prepared carefully. We study quantum-process tomography in light of this. We suggest several experimental setups, where preparation of the initial state plays a role in the final outcome of the experiment. By studying the linearity and the positivity of the resulting maps, the experimenter can determine the nature of the initial correlations between the system and the environment.

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I. INTRODUCTION

Quantum-information processing promises powerful computational methods that surpass the methods of classical information processing [1,2]. These methods rely on taking advantage of quantum parallelism by using quantum superposition and quantum entanglement as resources. In order to implement such a device, one must have precise control over the system and isolate it from the surrounding environment to preserve coherence. Yet, realistically, it is nearly impossible to isolate the system of interest completely from its surroundings while having a great deal of control. With the rising interest in quantum computation and quantum-information processing, quantum-coherence experiments are performed readily these days, though with relatively small systems. One of the major problems with these experiments is the loss of coherence due to the interaction between the system of interest and the unknown environmental states. The methods for studying the interaction between the system and the environment are given by the quantum theory of open systems.

The quantum theory of open systems got its start almost 50 years ago with the introduction of dynamical maps [3,4] by Sudarshan, Mathews, Rau, and Jordan. Decades after its conception, the dynamical map formalism is finally being tested in the laboratory setting, and many questions are surfacing about the properties of dynamical maps [5–12]. The experimental determination of a dynamical map is achieved by a procedure called *quantum-process tomography*. Any experiment, including quantum-process-tomography experiments, requires a method to prepare the initial states of the system at the beginning of the experiment [13–18]. We study the effects of the preparation procedure on quantum systems that interact with an environment. The act of preparation has been neglected from the theory of quantum-process tomography (and for all quantum experiments that interact with a nontrivial environment). We investigate this issue for quantum-process tomography in detail in this paper. We present several simple examples to motivate our arguments.

In Sec. II, we review the concepts and the mathematics of quantum-process tomography and the preparation procedure. In Sec. III, we construct and discuss several simple examples of quantum-process tomography for different preparation procedures. In Sec. IV, we follow up these examples by analyzing some recent quantum-process-tomography experiments. The examples are in the same spirit as the ones in the previous papers, therefore allowing us to compare all of the results. We reproduce the examples from previous papers in later sections of this paper for ease. We have modified the language of these examples to fit the language of this paper. And finally, in Sec. V, we discuss how preparation procedures differentiate between the outcome of a quantum-process-tomography experiment and its theoretical analog, dynamical maps, along with our concluding remarks.

II. BRIEF REVIEW

A. Quantum-process tomography

Quantum-process tomography [19,20] is the experimental tool that determines the open evolution of a system that interacts with the surrounding environment. It is the tool that allows an experimenter to determine the unwanted action of a quantum process on the quantum bits going through it. It is an important tool for quantum-information processing. A state going through a quantum gate or a quantum channel will experience some interactions with the surrounding environment. Quantum-process tomography allows the experimentalist to distinguish the differences between the ideal process and the process found experimentally. Therefore, it is an important tool in quantum-control design and battling decoherence (loss of polarization).

The objective of quantum-process tomography is to determine how a quantum process acts on different states of the system. In very basic terms, a quantum process connects different quantum input states to different output states:

$$\text{input states} \rightarrow \text{process} \rightarrow \text{output states}. \tag{1}$$

The complete behavior of the quantum process is known if the output state for any given input state can be predicted.

The tomography aspect of quantum-process tomography is to use a finite number of input states, instead of all possible states, to determine the quantum process. For instance, to determine a dynamical map, $\mathcal{B}$, we only need to know the

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mapping of the elements of the density matrix from an initial time to a final time,

$$\mathcal{B} : \rho_{r's'}(t_0) \to \rho_{rs}(t). \tag{2}$$

The elements of the density matrix linearly span the whole state space [17].

Experimentally, we do not have access to the individual elements of the density matrix; we can only prepare physical states. Thus, a set of physical states that linearly span the state space is sufficient for the experiment. A state space of dimension d requires d^2 states to span the space. Once the evolution of each of these input states is known, by linearity the evolution of any input state is known (see [1] for a detailed discussion).

Using the set of linearly independent input states $P^{(m)}$, and measuring the corresponding output states $Q^{(m)}$, the evolution of an arbitrary input state can be determined. Let Λ be the map describing the process, which we call a *process map*, and an arbitrary input state be expressed (uniquely) as a linear combination $\sum_j p^{(m)} P^{(m)}$. The action of the map in terms of the matrix elements is as follows:

$$\sum_{r's'} \Lambda_{rr';ss'} \left(\sum_j p_j P^{(j)}_{r's'} \right) = \sum_j p_j Q^{(j)}_{rs}.$$

With the knowledge of the output states corresponding to the input state, we find the map via the following expression:

$$\Lambda_{rr';ss'} = \sum^{(m)} Q^{(m)}_{rs} \tilde{P}^{(m)^*}_{r's'}, \tag{3}$$

where $\tilde{P}^{(n)}$ are the duals of the input states satisfying the scalar product

$$\tilde{P}^{(m)^\dagger} P^{(n)} = \sum_{rs} \tilde{P}^{(m)^*}_{rs} P^{(n)}_{rs} = \delta_{mn}.$$

Today there are many variations of the quantum-process-tomography procedure described above, namely *ancilla (entanglement) assisted process tomography* [21–25], *direct characterization of quantum dynamics* [26,27], *selective efficient quantum-process tomography* [28], and *symmetrized characterization of noisy quantum processes* [29]. Some of these procedures have been experimentally tested [30–38].

1. Initially uncorrelated assumption

In every quantum-process-tomography procedure listed above, the input states are thought to be pure states. There are two advantages of using pure states as inputs. First, it is easier to span the space of the system with a set of pure states than it is with a set of mixed states. Second, pure states are always uncorrelated, which is one of the central assumptions in every procedure above. It is simply a matter of preparing the necessary pure states to perform a quantum-process-tomography experiment.

What if we depart from this assumption? It is well known that initially correlated states can lead to not-completely positive dynamics for the system [6,7,39]. In many recent experiments, the process maps that characterize the quantum operations have been plagued with negative eigenvalues and occasional nonlinear behavior. We now examine the how the two cases are related.

The quantum-process-tomography procedures we reviewed require the input states to be uncorrelated with the environment. Just before the experiment begins, in general the state of the system may be correlated with the environment. Thus, at the beginning of the experiment it is necessary to prepare the initially correlated total state into an uncorrelated state. The difference between the process maps found from a quantum-process-tomography experiment and the dynamical maps calculated theoretically is precisely the act of preparation of the input states. Let us investigate this issue by analyzing the steps involved in a quantum-process-tomography experiment.

B. Preparation procedure

In practice, preparation procedures can be very complicated. Instead of describing many different procedures, we follow the theory of preparations developed in [18]. Below is a brief review of the findings in that paper.

A preparation procedure is the mapping of a set of unknown states into a fixed, known input state. The most general transformation of a quantum state is described by a stochastic map [3]. In light of that, a preparation procedure that is very complicated due to the apparatus is simply denote by a stochastic map, $\mathcal{A}^{(m)}$. The procedure of preparing the mth input state is given as

$$\rho^{\mathcal{SE}} \longrightarrow [\mathcal{A}^{(m)} \otimes \mathcal{I}](\rho^{\mathcal{SE}}) = R^{\mathcal{SE}^{(m)}}, \tag{4}$$

where $\mathcal{I}$ is the identity map acting on the state of the environment. In other words, we are assuming that the preparation procedure acts only on the system.[1]

In general, the state of the system and the environment before preparation, $\rho^{\mathcal{SE}}$, will be correlated (see [40]). The goal of the preparation procedure is to eliminate these correlations, because for an ideal experiment the state of the system should be uncorrelated with the environment at the beginning of the experiment. Therefore, under the *perfect preparation procedure assumption*, the right-hand side of Eq. (4), or the post-preparation procedure state, is of product form.

In [18] two common preparation methods that achieve product states were discussed. In this paper we only focus on the method called *stochastic preparation*. The other method, *projective preparations*, is discussed elsewhere.

1. Stochastic preparation

Many quantum experiments begin by initializing the system into a specific state. For instance, in the simplest case, the system can be prepared to the ground state by cooling it to near absolute zero temperature [33,34,37,38]. Mathematically, this set of operations is written as a pin map [41],

$$\Theta = |\Phi\rangle \langle\Phi| \otimes \mathbb{1}, \tag{5}$$

[1]Generally we do not require the preparation map to be trace preserving. However, the preparation procedure used here, the stochastic procedure, is trace preserving; hence, we need to worry about this issue. See [18] for details.

where $\mathbb{1}$ is the identity matrix, acting as the "trace operator" ($\mathbb{1}_{rs}\rho_{rs} = \mathrm{Tr}[\rho]$) and $|\Phi\rangle$ is some fixed state (i.e., ground state) of the system. In this procedure, no matter what the initial state of the system, it is "pinned" to the final state $|\Phi\rangle\langle\Phi|$.

The action of the pin map Θ on a bipartite state of the system and the environment is

$$\Theta(\rho^{\mathcal{SE}}) = [|\Phi\rangle\langle\Phi| \otimes \mathbb{1}](\rho^{\mathcal{SE}}) = |\Phi\rangle\langle\Phi| \otimes \rho^{\mathcal{E}}. \tag{6}$$

The pin map fixes the system into a single pure state, which means that the state of the environment is fixed into a single state as well; the pin map decorrelates the system from the environment. Once the pin map Θ is applied, the system is prepared in the various different input states by applying local

$$\mathcal{A}^{(m)}(\rho^{\mathcal{SE}}) = [\Omega^{(m)} \circ \Theta](\rho^{\mathcal{SE}}) = P^{(m)} \otimes \rho^{\mathcal{E}}. \tag{7}$$

As seen the preceding equations, the advantage of the stochastic preparation method, as described here, is that the state of the environment is a constant for any input of the system. This constancy of the state of the environment is necessary to characterize properly the dynamical mechanisms [17].

It may seem that the stochastic preparation procedure alleviates the experiment from having an environment that depends on the preparation procedure. This is the main result of this paper, and we analyze this matter in much greater detail below to show how inconsistencies in the preparation procedure can lead to strange experimental results.

C. Quantum-process-tomography experiment in steps

The basic steps in a quantum process tomography experiment are broken down as follows.

1. Just before the experiment begins, the system and environment are in an unknown state $\rho^{\mathcal{SE}}$. The system and the environment in general are correlated; we are no longer making the initially uncorrelated assumption.

2. The system is altered to a known input state by a preparation procedure. The system and environment state after preparation is therefore given by

$$\mathcal{A}^{(m)}(\rho^{\mathcal{SE}}) \rightarrow P^{(m)} \otimes \rho^{\mathcal{E}}.$$

The input state is given by taking the trace with respect to the environment:

$$P^{(m)} = \mathrm{Tr}_{\mathcal{E}}[P^{(m)} \otimes \rho^{\mathcal{E}}].$$

3. The system is then sent through a quantum process. We consider the evolution to be a global unitary transformation in the space of the system *and* the environment:

$$U P^{(m)} \otimes \rho^{\mathcal{E}} U^{\dagger}.$$

4. Finally, the output state is observed. Mathematically it is the trace with respect to the environment,

$$Q^{(m)} = \mathrm{Tr}_{\mathcal{E}}[U P^{(m)} \otimes \rho^{\mathcal{E}} U^{\dagger}], \tag{8}$$

which we call the *stochastic process equation*.

5. Finally, using the input and the output states, we construct a map describing how the process is constructed.

The aforementioned procedure is identical to the procedure used to find a dynamical map, except for the preparation procedure. In the next two sections, we analyze quantum-process tomography, and the two preparation procedures are discussed in the last section. The differences between what we find here and dynamical maps is due to the preparation procedures (see Sec. V for a discussion). Our analysis in this section is within a general setting, but, for simplicity, our examples in the next section are of one-qubit systems and one-qubit environments. The examples should seen as toy models that serve to prove specific points.

III. QUANTUM-PROCESS TOMOGRAPHY WITH STOCHASTIC PREPARATIONS

In this paper, we analyze quantum-process-tomography procedures when the stochastic preparation procedure is used to generate the input states. The stochastic preparation map's action on a bipartite state is given by Eq. (7), while the output states corresponding to the input are given by the stochastic process equation, Eq. (8). All of our examples that follow are based on these two equations. Furthermore, all of our examples have the same starting point, namely the unknown state before preparation,

$$\rho^{\mathcal{SE}} = \tfrac{1}{4}(\mathbb{I} \otimes \mathbb{I} + a_j\sigma_j \otimes \mathbb{I} + c_{23}\sigma_2 \otimes \sigma_3). \tag{9}$$

This is a correlated state, but it is not an entangled state. All examples also have the same global dynamics in common, given by

$$U = e^{-iHt} = \prod_j [\cos(\omega t)\mathbb{I} \otimes \mathbb{I} - i\sin(\omega t)\sigma_j \otimes \sigma_j], \tag{10}$$

where

$$H = \omega \sum_{j=1}^{3} \sigma_j \otimes \sigma_j. \tag{11}$$

Let us now delve into constructing our examples, starting with an ideal second quantum-process-tomography experiment.

A. Ideal stochastic preparation

Let the pin map Θ be

$$\Theta = |\phi\rangle\langle\phi| \otimes \mathbb{I}, \tag{12}$$

where $|\phi\rangle\langle\phi|$ is a pure state of the system. The preparation of $\rho^{\mathcal{SE}}$ in Eq. (9) with this pin map leads to

$$\Theta(\rho^{\mathcal{SE}}) = |\phi\rangle\langle\phi| \otimes \tfrac{1}{2}\mathbb{I}, \tag{13}$$

yielding the initial state $|\phi\rangle\langle\phi|$ for the system qubit and a completely mixed state for the environment qubit. The next step is to create the rest of the input states using maps $\Omega^{(m)}$. In this case, the fixed state $|\phi\rangle\langle\phi|$ can be locally rotated to get the desired input state $P^{(m)}$ [given in Eq. (15)]:

$$\begin{aligned}\Omega^{(m)}|\phi\rangle\langle\phi| \otimes \tfrac{1}{2}\mathbb{I} &= V^{(m)}|\phi\rangle\langle\phi|V^{(m)\dagger} \otimes \tfrac{1}{2}\mathbb{I} \\ &= P^{(m)} \otimes \tfrac{1}{2}\mathbb{I},\end{aligned} \tag{14}$$

where $m = \{(1,-),(1,+),(2,+),(3,+)\}$ and $V^{(m)}$ are local unitary operators acting on the space of the system. The inputs states are

$$\begin{aligned} P^{(1,-)} &= \tfrac{1}{2}(\mathbb{I} - \sigma_1), \quad P^{(1,+)} = \tfrac{1}{2}(\mathbb{I} + \sigma_1), \\ P^{(2,+)} &= \tfrac{1}{2}(\mathbb{I} + \sigma_2), \quad P^{(3,+)} = \tfrac{1}{2}(\mathbb{I} + \sigma_3). \end{aligned} \tag{15}$$

Now each input state is sent through the quantum process. We use the following unitary operator to develop the system and the environment. Noting that the duals for the input states in Eqs. (15) are

$$\begin{aligned} \tilde{P}^{(1,-)} &= \tfrac{1}{2}(1 - \sigma_1 - \sigma_2 - \sigma_3), \\ \tilde{P}^{(1,+)} &= \tfrac{1}{2}(1 + \sigma_1 - \sigma_2 - \sigma_3), \\ \tilde{P}^{(2,+)} &= \sigma_2, \quad \tilde{P}^{(3,+)} = \sigma_3, \end{aligned} \tag{16}$$

we can calculate the output states of Eq. (8). The output states are

$$\begin{aligned} Q^{(1,-)} &= \tfrac{1}{2}[\mathbb{I} - \cos^2(2\omega t)\sigma_1], \\ Q^{(1,+)} &= \tfrac{1}{2}[\mathbb{I} + \cos^2(2\omega t)\sigma_1], \\ Q^{(2,+)} &= \tfrac{1}{2}[\mathbb{I} + \cos^2(2\omega t)\sigma_2], \\ Q^{(3,+)} &= \tfrac{1}{2}[\mathbb{I} + \cos^2(2\omega t)\sigma_3]. \end{aligned} \tag{17}$$

The linear process map is constructed using Eq. (3), the duals in Eqs. (16), and the output states:

$$\Lambda_s = \frac{1}{2}\begin{pmatrix} 1+C^2 & 0 & 0 & 2C^2 \\ 0 & 1-C^2 & 0 & 0 \\ 0 & 0 & 1-C^2 & 0 \\ 2C^2 & 0 & 0 & 1+C^2 \end{pmatrix}, \tag{18}$$

where $C = \cos(2\omega t)$. The eigenvalues of the process map are plotted in Fig. 1. The map above is completely positive and can be tested for linearity. This would be the result of an ideal quantum-process-tomography experiment (see Sec. IV A for such an experiment). The key thing to note above is that the state of the environment is completely constant throughout the problem. Laterin this paper, we show that this need not be the case all the time.

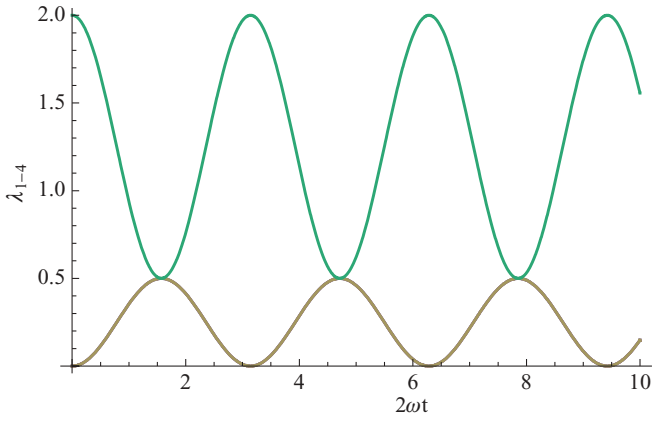

FIG. 1. (Color online) The four eigenvalues of the process map in Eq. (18) are plotted as a function of $2\omega t$. Eigenvalues are twofold degenerate. As expected, the eigenvalues are always positive. The stochastic preparation procedure allows the experimentalist to prepare any pure state for the system. Then, by convexity, all possible states of the system are in the compatibility domain of the process map in Eq. (18).

B. Negative maps due to control errors

Suppose the initial state is prepared well using a pin map

$$\Theta(\rho^{\mathcal{SE}}) = P^{(3,+)} \otimes \mathbb{I}. \tag{19}$$

After obtaining this state, the other input states are prepared by local rotations. Let us consider the case where one of the rotation is not perfect:

$$V^{(1,-)}|1\rangle \to \frac{1}{\sqrt{2}}(\sqrt{1-\epsilon}\,|1\rangle - \sqrt{1+\epsilon}\,|0\rangle), \tag{20}$$

where ϵ is taken to be a small positive real number. We introduced a small error for the preparation of $P^{(1,-)}$, but we have kept the error simple by not giving it an additional phase (i.e., keeping ϵ real). For simplicity, we have also assumed that the error of this sort occurs in the preparation of only one state.

Let us now pretend that we are not aware of this error. Then in reality we have the following set of input states:

$$\begin{aligned} P^{(1,-)} &= \tfrac{1}{2}[\mathbb{I} + \epsilon\sigma_3 - \sqrt{1-\epsilon^2}\sigma_1], \\ P^{(1,+)} &= \tfrac{1}{2}[\mathbb{I} + \sigma_1], \\ P^{(2,+)} &= \tfrac{1}{2}[\mathbb{I} + \sigma_2], \\ P^{(3,+)} &= \tfrac{1}{2}[\mathbb{I} + \sigma_3]. \end{aligned} \tag{21}$$

Let us use the same unitary evolution as given earlier in Eq. (10). The output states corresponding to the last three input states in Eqs. (21) are the same as before, given by Eqs. (17). For the input state $P^{(1,-)}$, the corresponding output state is as follows:

$$\begin{aligned} Q^{(1,-)} &= \tfrac{1}{2}[\mathbb{I} + (\epsilon\sigma_3 - \sqrt{1-\epsilon^2}\sigma_1)\cos^2(2\omega t)], \\ Q^{(1,+)} &= \tfrac{1}{2}[\mathbb{I} + \cos^2(2\omega t)\sigma_1], \\ Q^{(2,+)} &= \tfrac{1}{2}[\mathbb{I} + \cos^2(2\omega t)\sigma_2], \\ Q^{(3,+)} &= \tfrac{1}{2}[\mathbb{I} + \cos^2(2\omega t)\sigma_3]. \end{aligned} \tag{22}$$

Using these output states, the duals given by Eqs. (16), and Eq. (3), we can find the process map.

The process map turns out to be rather complicated, and its eigenvalues look even more complicated. Therefore, we do not write them out; instead, we have plotted the eigenvalues as a function of $2\omega t$. We take the value for the error to be $\epsilon = 0.1$ for the plot in Fig. 2. One of the eigenvalues in Fig. 2 is negative for certain times. This shows yet another cause for negative eigenvalues in a process map. The negative eigenvalues here have nothing to do with the initial correlations between the system and the environment. The negative eigenvalues are attributed to poor control in the preparation procedure.

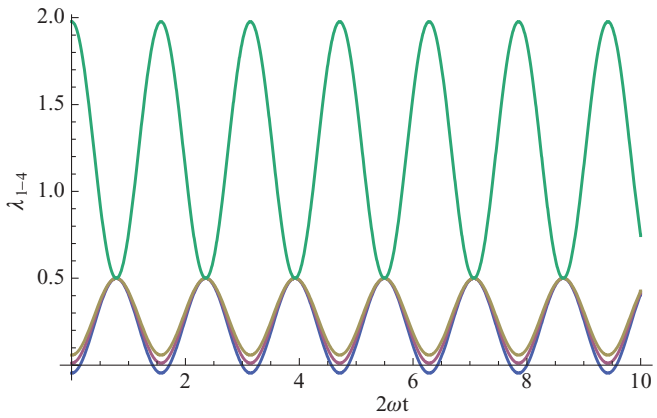

FIG. 2. (Color online) The eigenvalues of the process map found using the output states in Eqs. (22) and the duals in Eqs. (16). One of the eigenvalues is negative for certain times; we have taken $\epsilon = 0.1$. The negativity is due to the errors in the unitary operation implemented to prepare one of the input states.

Now we have discussed several scenarios that can lead to negative eigenvalues for a process map. Two comments are in order at this point. When the prepared input states are not pure and the process map has negative eigenvalues, one should be wary of initial correlations with the environment. As a further check, the process map should be tested for linearity by sending several additional input states through the process. If the process map predicts the output states properly, then one can be confident that the system is initially correlated with the environment. If the process map does not predict the output states correctly (nonlinear behavior), then there are additional problems with the experiment, including the possibility of poor control in the preparation procedure. In the case where the input states are pure and the process map has negative eigenvalues, the negativity can only come from either inconsistencies in the preparation procedure or poor preparation control.

C. Mixed input states

Demanding pure input states in a quantum-process-tomography experiment guarantees an initially uncorrelated state of the system. However, if it is not possible to prepare pure states, we can still determine the process. Below we discuss what happens when the input is not a pure state. We show that this creates an uncertainty about whether the system is correlated with the environment. Yet we show that, regardless of this uncertainty, the quantum-process-tomography experiment can be carried out consistently.

1. Uncorrelated state

Suppose the initial state is prepared by a pin map leading to

$$\Theta(\rho^{\mathcal{SE}}) = [pP^{(3,+)} + (1-p)P^{(3,-)}] \otimes \mathbb{I} \tag{23}$$

$$= \tfrac{1}{2}(\mathbb{I} + p\sigma_3) \otimes \mathbb{I}, \tag{24}$$

where $0 \ll p < 1$. The rest of the input states are prepared by rotations. Then the input states are

$$P^{(1,-)} = \tfrac{1}{2}(\mathbb{I} - p\sigma_1), \quad P^{(1,+)} = \tfrac{1}{2}(\mathbb{I} + p\sigma_1), \tag{25}$$

$$P^{(2,+)} = \tfrac{1}{2}(\mathbb{I} + p\sigma_2), \quad P^{(3,+)} = \tfrac{1}{2}(\mathbb{I} + p\sigma_3). \tag{26}$$

For the unitary operator in Eq. (10), the corresponding output states are

$$\begin{aligned} Q^{(1,-)} &= \tfrac{1}{2}[\mathbb{I} - p\cos^2(2\omega t)\sigma_1], \\ Q^{(1,+)} &= \tfrac{1}{2}[\mathbb{I} + p\cos^2(2\omega t)\sigma_1], \\ Q^{(2,+)} &= \tfrac{1}{2}[\mathbb{I} + p\cos^2(2\omega t)\sigma_2], \\ Q^{(3,+)} &= \tfrac{1}{2}[\mathbb{I} + p\cos^2(2\omega t)\sigma_3]. \end{aligned} \tag{27}$$

The only change that we have to make to find the process map is to define a dual proper set, in this case

$$\begin{aligned} \tilde{P}^{(1,-)} &= \frac{1}{2p}(p\mathbb{I} - \sigma_1 - \sigma_2 - \sigma_3), \\ \tilde{P}^{(1,+)} &= \frac{1}{2p}(p\mathbb{I} + \sigma_1 - \sigma_2 - \sigma_3), \\ \tilde{P}^{(2,+)} &= \frac{1}{p}\sigma_2, \quad \tilde{P}^{(3,+)} = \frac{1}{p}\sigma_3. \end{aligned} \tag{28}$$

We can find the process map using Eq. (3):

$$\Lambda_{mx1} = \frac{1}{2}\begin{pmatrix} 1+C^2 & 0 & 0 & 2C^2 \\ 0 & 1-C^2 & 0 & 0 \\ 0 & 0 & 1-C^2 & 0 \\ 2C^2 & 0 & 0 & 1+C^2 \end{pmatrix}, \tag{29}$$

where $C = \cos(2\omega t)$. The process map in this case turns out to be the same as in Eq. (18), as expected; the process map for stochastic preparation is defined over the set of all states, including the input states above.

2. Correlated state

The downside, of course, is that there is no way to distinguish the states in Eq. (23) from the following state:

$$\Theta(\rho^{\mathcal{SE}}) = \tfrac{1}{2}(\mathbb{I} \otimes \mathbb{I} + p\sigma_3 + c_{23}\sigma_2 \otimes \sigma_3). \tag{30}$$

Unlike in Eq. (25), this is a correlated state, yet both total states have the same reduced state for the system part.

Even in this case, we can find the process map properly if we use the correct dual set given in Eqs. (28). The process map in that case is

$$\Lambda_{mx2} = \frac{1}{2}\begin{pmatrix} 1+C^2 & 0 & -c_{23}CS & 2C^2 \\ 0 & 1-C^2 & 0 & -c_{23}CS \\ -c_{23}CS & 0 & 1-C^2 & 0 \\ 2C^2 & -c_{23}CS & 0 & 1+C^2 \end{pmatrix}, \tag{31}$$

where $C = \cos(2\omega t)$ and $S = \sin(2\omega t)$, which is the same as the dynamical map calculated in Eq. (6) of [9] and its eigenvalues are plotted in Fig. 3. In both of the preceding examples, had we assumed that the input states were close enough to the pure states we desired and used the dual set given by Eqs. (16), then the process map would contain an error.

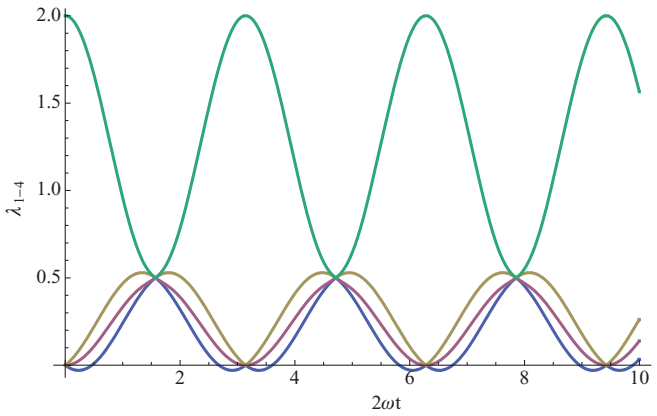

FIG. 3. (Color online) Eigenvalues of the dynamical map in Eq. (32) are plotted as a function of $2\omega t$. One of the eigenvalues is negative for certain values of ωt; we have taken $c_{23} = 0.5$. The negative eigenvalue is due to the initial correlations between the system and the environment.

These two examples illustrate an argument contrary to what some have suggested: that only the completely positive process maps should be considered physically valid. We showed in the second example that one can obtain a not-completely positive process map in a consistent fashion. The real issue is that the correlations with the environment are not convenient for experimental purposes. But that does not mean that one should fix a not-completely positive process map to a completely positive process map with numerical methods [42,43].

D. Multiple stochastic preparations

The pin map used in the stochastic preparation must be used consistently. Let us show an example of what happens when two stochastic preparation procedures are used. Consider a quantum-process-tomography experiment where the following linearly independent states are used to span the space of the system:

$$P^{\mathbb{I}} = \tfrac{1}{2}\mathbb{I}, \quad P^{(1+)}, \quad P^{(2+)}, \quad P^{(3+)}. \tag{32}$$

These states form the linearly independent set[2] $\{\mathbb{I},\sigma_j\}$, which is different from the set used in previous examples.

Consider the following two-qubit state as the state available to the experimenter at $t = 0_-$: This is the same state we used in Eq. (9) for the previous example. Let the pin map, Θ, for our example be

$$\Theta = |\phi\rangle\langle\phi| \otimes \mathbb{I}, \tag{33}$$

where $|\phi\rangle\langle\phi|$ is a pure state of the system. The preparation of $\rho^{\mathcal{SE}}$ with this pin map leads to

$$\Theta(\rho^{\mathcal{SE}}) = |\phi\rangle\langle\phi| \otimes \tfrac{1}{2}\mathbb{I}, \tag{34}$$

yielding the initial state $|\phi\rangle\langle\phi|$ for the system qubit and a completely mixed state for the environment qubit. The next step is to create the rest of the input states using maps $\Omega^{(m)}$. In this case, the fixed state $|\phi\rangle\langle\phi|$ can be locally rotated to get the desired input state $P^{(m)}$ [given in Eqs. (15)]

$$\begin{aligned}\Omega^{(m)}|\phi\rangle\langle\phi| \otimes \tfrac{1}{2}\mathbb{I} &= V^{(m)}|\phi\rangle\langle\phi|V^{(m)\dagger} \otimes \tfrac{1}{2}\mathbb{I} \\ &= P^{(m)} \otimes \tfrac{1}{2}\mathbb{I},\end{aligned} \tag{35}$$

where $m = \{(1,-),(1,+),(2,+),(3,+)\}$ and $V^{(m)}$ are local unitary operators acting on the space of the system.

Once again, we take the same initial state and unitary operators as in the last example. Now suppose the pin map given in Eq. (33) is used to prepare the state $|\phi\rangle\langle\phi|$, and then by local transformations $P^{(1,+)}$, $P^{(2,+)}$, and $P^{(3,+)}$ are prepared. Finally, the mixed state is prepared by letting $P^{(3+)}$ decohere. This is a different pin map than the one in Eq. (33), so we are using two pin maps to prepare two sets of input states. The unitary operator we are using is often called the SWAP gate, because it swaps the states of two qubits with a period of $t = \frac{\pi}{4\omega}$. Then at $t = \frac{\pi}{4\omega}$, the total state is

$$\rho^{\mathcal{SE}}\left(\frac{\pi}{4\omega}\right) = \frac{1}{4}(\mathbb{I}\otimes\mathbb{I} + a_j\mathbb{I}\otimes\sigma_j). \tag{36}$$

The state of the system has fully decohered.

The corresponding output states for the above-mentioned input states are found using Eq. (8). The state of environment in that equation for input $P^{\mathbb{I}} = \frac{1}{2}\mathbb{I}$ is $\rho^{\mathcal{E}} = \frac{1}{2}(\mathbb{I} + a_j\sigma_j)$, while for the other inputs the state of the environment is $\rho^{\mathcal{E}} = \frac{1}{2}\mathbb{I}$. Then the corresponding output states are

$$\begin{aligned}Q^{(\mathbb{I})} &= \tfrac{1}{2}[\mathbb{I} + \sin^2(2\omega t)\sigma_3], \\ Q^{(1,+)} &= \tfrac{1}{2}[\mathbb{I} + \cos^2(2\omega t)\sigma_1], \\ Q^{(2,+)} &= \tfrac{1}{2}[\mathbb{I} + \cos^2(2\omega t)\sigma_2], \\ Q^{(3,+)} &= \tfrac{1}{2}[\mathbb{I} + \cos^2(2\omega t)\sigma_3].\end{aligned} \tag{37}$$

The last three output states are the same as in the last example, but the fourth one is different.

Now suppose we calculate the output state corresponding to the input $P^{(-1)}$. By linearity, we have $P^{(-1)} = \mathbb{I} - P^{(+1)}$. If the process is linear, then the output state for this input state is given by

$$\begin{aligned}Q^{(-1)} &= 2Q^{(\mathbb{I})} - Q^{(+1)} \\ &= \tfrac{1}{2}[\mathbb{I} + 2\sin^2(2\omega t)\sigma_3 - \cos^2(2\omega t)\sigma_1].\end{aligned} \tag{38}$$

This state is not physical for certain times; therefore, the linearity and the positivity of the process is violated.

The process has not changed from the last example, only the method of determining the process. Thus, when the stochastic map is not used consistently, the process map can behave nonlinearly. For completeness, we find the process map for this example,

[2]The linear combination is not always convex. For example, $P^{(2,-)} = P^{(1,+)} + P^{(1,-)} - P^{(2,+)}$. Also notice that these four states form a linearly independent set, but they are not orthogonal to each other.

$$\Lambda_{ms} = \frac{1}{2}\begin{pmatrix} 1+C^2 & -(1+i)S^2 & 0 & 2C^2 \\ -(1-i)S^2 & 1-C^2+2S^2 & 0 & 0 \\ 0 & 0 & 1-C^2 & (1+i)S^2 \\ 2C^2 & 0 & (1-i)S^2 & 1+C^2-2S^2 \end{pmatrix}, \tag{39}$$

where $C = \cos(2\omega t)$ and $S = \sin(2\omega t)$. The eigenvalues of this process map are negative, as seen in Fig. 4. The fact that the state of the environment is not a constant for all of the input states leads to the negativity and nonlinearity of the process map.

The origin for the negative eigenvalues here is not due to the initial correlations between the system and the environment. Since none of the inputs were correlated with the environment here, negativity arises from the inconsistencies in the preparation procedure.

While this example may not seem realistic, the point regarding inconsistency arising from multiple pin maps still stands. In realistic cases, the trouble may not be seen so easily, due to complicated interactions with the environment. In Sec. IV D, we discuss an experiment where multiple stochastic preparation procedures are implemented.

IV. ANALYSIS OF EXPERIMENTS

In this section we analyze four quantum-process-tomography experiments that are analogs of the four examples constructed in the previous section. Each experiment is well performed in its own right. We analyze each experimental procedure and comment on why the result was not completely positive (when applicable).

A. Quantum-process tomography of Josephson-phase qubit and two-level state

In this experiment [35], a Josephson-phase qubit along with a more stable but less controllable two-level state acting as memory are examined. The Josephson phase is initially prepared in the ground state with high fidelity, which is then transformed into various excited states via unitary transformations. Finally it is "swapped" with the memory qubit and then some time later swapped back. Quantum-process tomography of this procedure is implemented. By doing so, it is observed that the decoherence of the Josephson-phase qubit can be hindered by transferring the information to the stable two-level state.

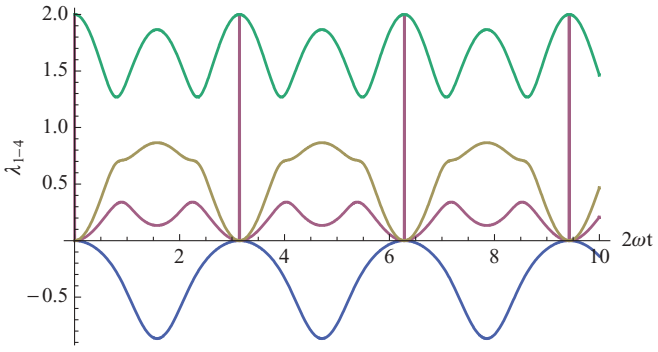

FIG. 4. (Color online) The eigenvalues of the process map in Eq. (39) plotted as function of $2\omega t$. One of the eigenvalues is negative for certain values of ωt. The negativity is due to the inconsistency in the preparation procedure, not the initial correlations with the environment.

The experimentalist found the resulting process map to be completely positive *without* any numerical corrections. This in principle is an ideal quantum-process-tomography experiment. The absence of negativity from the process map suggests that the system of interest was not in any way correlated with the surrounding degrees of freedom.

However, to be sure, one should check this process map for linearity by preparing arbitrary initial states and putting them through the SWAP procedure. If the experimental result matches the theoretical predictions of the process map, then the results of this experiment fit the analysis of the example constructed in Sec. III A.

B. Quantum-process-tomography solid-state qubit

In this experiment [36], a solid-state qubit is put through various gates. Quantum-process tomography of these gates is performed on three operations: identity, rotations along the x plane, and rotations along the y plane. The results are excellent, with very little error.

For the case where an identity operation is made, the process map is nearly identity, as expected. This means that the input states are well prepared and are nearly pure. For the other transformations, the errors are attributed to errors in preparation procedures and subsequent measurements.

The process map obtained in each case, however, has negative eigenvalues. Since the negativity is along the same magnitude for the case when no operation is made and when rotations along the x and y planes are implemented, we can presume that the errors must be due to the errors in unitary operations used to prepare the input states. This is along the same lines as the example in Sec. III B.

C. Quantum-process tomography in NMR

In this experiment [37], the system that is studied is an electron configuration formed in a nitrogen vacancy defect in a diamond lattice. The quantum state of the system is given by a spin triplet ($S = 1$). Again we write the initial state of the system and environment as $\rho^{S\mathcal{E}}$.

The system is prepared by optical pumping, which results in a strong spin polarization. The state of the system is said to have a 70% chance of being in a pure state $|\phi\rangle$ or, more mathematically, the probability of obtaining $|\phi\rangle$ is $\mathrm{Tr}(|\phi\rangle\langle\phi|\,\rho^{S\mathcal{E}}) = 0.7$.

Since the population probability is high, an assumption was made that the state of the system can be simply approximated as a pure state $|\phi\rangle\langle\phi|$. From this initial state, different input states can be prepared by suitably applying microwave pulses resonant with the transition levels. After preparation, the system is allowed to evolve, and the output states are determined by quantum-state tomography. With the knowledge of the input state and the measured output states, a linear process map is constructed.

It was found that the linear process map has negative eigenvalues, so the map was "corrected" using a least-squares fit between the experimentally determined map and a theoretical map based on Hermitian parametrization [42], while enforcing complete positivity.

However, if we do not regard the negative eigenvalues of the map as aberrations, then we should consider the assumptions about the preparation of the system more carefully. The assumption about the initial state of the system is

$$\rho^{\mathcal{SE}} \to |\phi\rangle\langle\phi| \otimes \tau, \tag{40}$$

which is, in effect, a pin map. Along with the pin map, the stochastic transformations are applied to the initial state to prepare the various input states; this is identical to the stochastic preparation method discussed in Sec. II B 1.

It is clear that the pure-initial-state assumption is unreasonable given our knowledge now of how the process is sensitive to the initial correlations between the system and the environment. In effect, the action of the pin map in this experiment is not perfect, and the pin map can be ignored. Then the process equation is

$$Q^{(m)} = \mathrm{Tr}_{\mathbb{B}}[U\Omega^{(m)}\rho^{\mathcal{SE}}U^\dagger], \tag{41}$$

where $\Omega^{(m)}$ is the stochastic transformation that prepares the mth input state. In this experiment, $\Omega^{(m)}$ is nothing more than a unitary transformation $V^{(m)}$ satisfying $V^{(m)}|\phi\rangle = |\psi^{(m)}\rangle$, where $|\psi^{(m)}\rangle$ is the desired pure mth input state.

We can write the unitary transformation for a two-level system as

$$V^{(m)} = |\psi^{(m)}\rangle\langle\phi| + |\psi_\perp^{(m)}\rangle\langle\phi_\perp|, \tag{42}$$

where $\langle\psi^{(m)}|\psi_\perp^{(m)}\rangle = \langle\phi|\phi_\perp\rangle = 0$. This defines $V^{(n)}$ as a transformation from the basis $\{|\phi\rangle\}$ to the basis $\{|\psi_i^{(n)}\rangle\}$. The equation for the process becomes:

$$\begin{aligned} Q^{(m)} = {} & \mathrm{Tr}_{\mathcal{E}}[U|\psi^{(m)}\rangle\langle\phi|\rho^{\mathcal{SE}}|\phi\rangle\langle\psi^{(m)}|U^\dagger] \\ & + \mathrm{Tr}_{\mathcal{E}}[U|\psi_\perp^{(m)}\rangle\langle\phi_\perp|\rho^{\mathcal{SE}}|\phi\rangle\langle\psi^{(m)}|U^\dagger] \\ & + \mathrm{Tr}_{\mathcal{E}}[U|\psi^{(m)}\rangle\langle\phi|\rho^{\mathcal{SE}}|\phi_\perp\rangle\langle\psi_\perp^{(m)}|U^\dagger] \\ & + \mathrm{Tr}_{\mathcal{E}}[U|\psi_\perp^{(m)}\rangle\langle\phi_\perp|\rho^{\mathcal{SE}}|\phi_\perp\rangle\langle\psi_\perp^{(m)}|U^\dagger]. \end{aligned}$$

Therefore, since $\langle\phi|\rho^{\mathcal{SE}}|\phi\rangle = 0.7$ to first approximation, the process is a linear mapping on the states $|\psi^{(m)}\rangle\langle\psi^{(m)}|$. However, it is clear that, if all terms are included, the process is not truly linear in the states $|\psi^{(m)}\rangle\langle\psi^{(m)}|$. The negative eigenvalues are therefore a result of fitting results into a linear map when the process is not truly represented by a linear map.

There is no way to tell if the system in the preceding experiment was initially correlated with the environment or not, as shown in Sec. III C. However, the negativity in the obtained process map indicates that the system may have been correlated initially. If this was the case, the obtained map may be linear and well behaved. Fixing it numerically to make it completely positive may in reality make it behave in nonlinear fashion.

D. Quantum-process tomography of motional states

In this experiment [38], quantum-process tomography of the motional states of trapped ^{85}Rb atoms in the potential wells of a one-dimensional optical lattice is performed. Only two bound bands are considered, which are labeled as states $|0\rangle$ and $|1\rangle$. The states are prepared stochastically. An initial state of the system is the ground state $|g\rangle\langle g|$, and from it states $|r\rangle\langle r|$, $|i\rangle\langle i|$, and the fully mixed state $\frac{1}{2}\mathbb{I}$ are prepared. The states $|r\rangle$ and $|i\rangle$ stand for the real and imaginary coherence states, which are prepared by applying appropriate unitary transformations to the ground state. This is achieved by displacing the lattice for the real coherence state and for the imaginary coherence state; a quarter-period delay is added after the displacement. The identity state is prepared by letting a superposition state decohere. The input states are allowed to evolve and the output states are determined by quantum-state tomography.

The process map is found by following the usual procedure laid out in Sec. II. Since there are particles lost to the neighboring cells, the map is not required to be trace preserving. Based on this loss, it is also argued that the map can pick nonphysical behavior (not completely positive). The map is forced to be "physical" (completely positive) by using the maximum-likelihood method [43].

In our terminology, the states prepared are $P^{(1,+)}$, $P^{(2,+)}$, $P^{(3,-)}$, and $\frac{1}{2}\mathbb{I}$. The three projective states are prepared by a single consistent stochastic preparation, while the fully mixed state is prepared by letting the state $P^{(1,+)}$ decohere, which means an additional pin map is applied to prepare one of the input states. As we saw in the example in Sec. III D, this can lead to a not-completely positive and nonlinear process map.

Furthermore, the input states have varying values for polarization. Their data are listed in the following table:

	ρ_g	$\rho_{\mathbb{I}}$	ρ_r	ρ_i
$P^{(3,-)}$	0.90	0.60	0.69	0.69
$P^{(3,+)}$	0.10	0.40	0.31	0.31
$P^{(1,+)}$	0.82	0.59	0.85	0.63
$P^{(2,+)}$	0.84	0.58	0.64	0.37

where ρ_j are the experimentally prepared states projected onto projectors $P^{(m)}$. The polarization of the imaginary state in the σ_2 direction is very low. This could mean that it is correlated with the environment, while the polarization of the ground state along the negative σ_3 direction is almost unity, meaning it is only weakly correlated at best.

Here we have two potential causes for the process map to have negative eigenvalues. The first problem is with the experimental procedure; which applies multiple stochastic preparations that may affect the state of the environment differently. The second problem may be unavoidable, since it is extremely difficult to prepare pure states in this type of setup. Even if the negative eigenvalues are due to the initial correlations, we do not have a prescription to obtain a process map in a consistent fashion. For the last example shown in Sec. III, we assumed that the initial correlations were constant

throughout the process. This does not seem to be the case here, since the ground state is almost pure while the imaginary state is clearly not pure. Therefore, the correlation with the environment for these two inputs must be different, as in the case of the example in Sec. III D.

V. CONCLUSIONS

The examples with multiple stochastic preparations (in Sec. III D) and projective preparations (in Sec. V A in [15]) look very similar. For multiple stochastic preparations, the state of the environment depends on the stochastic map. If we think of each projective preparation as an independent stochastic preparation then the state of the environment, in the example (in Sec. V A) in [15], depends on the preparation map. Are the two situations the same? We show below that the two situations are fundamentally different.

In the initially uncorrelated limit, the projective preparation procedure does not play an important role. As we saw in the previous section, when the system and the environment are initially uncorrelated, the projective preparation procedure has no effect on the state of the environment. This is because the state of the environment is affected only indirectly due to the initial correlations between the system and the environment. Thus, the projective preparations, in the initially uncorrelated limit, yield a linear process map.

This is not the case in the multiple stochastic preparation example. When multiple stochastic maps are used to prepare different input states, the inconsistencies do not stem from the initial correlations between the system and the environment. In fact, in our example, all input states are initially uncorrelated from the environment. The inconsistencies arise from the preparation procedures themselves, leading to different states of the environment for different input states. These inconsistencies are absent in the case where the system develops in a closed form, since the system does not feel the presence of the environment during the quantum process. Hence, in the "weak interaction limit," multiple stochastic preparations yield a linear process map.

Finally, note that each of the process maps found in this section, given by Eq. (39), and (in Sec. V A) in [15] are different from each other. This clearly shows that the preparation procedures play a nontrivial role in open quantum system experiments. The preparation procedure is the only thing that distinguished each case. For the case where no preparation procedure is applied, that is, the case of the dynamical map (see Eq. (6) in [9]), the situation is still different. The dynamical map, which has negative eigenvalues, is linear and has a valid interpretation within the compatibility domain. The process maps in Eq. (39) and (in Sec. V A) in [15] do not have any consistent interpretation.

When negative eigenvalues are found in a process map, they hint at some problem in the preparation procedure, although this statement is a bit premature at this point. We analyze the negative eigenvalues in a process map in more detail in the next section.

Our study of quantum-process tomography started by noting that the dynamical map acting on a system can have negative eigenvalues. The dynamical map has negative eigenvalues when the system is initially correlated with the environment. In the course of our studies of quantum-process tomography, we showed that the preparation procedure cannot be neglected for any quantum system that interacts with an environment. These are the two major themes discussed in this paper. However, along the way, we presented a method of quantum-process tomography that is independent of the preparation procedure. The map arising from this procedure leads us to an expression that quantifies the memory effect on the dynamics of the system due to the initial correlations with the environment. Determining the memory effect is an important task in coherence control.

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