# CHARACTERIZATION OF COMPLETELY POSITIVE TRACE-PRESERVING MAPS ON MN

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**Abstract**

Completely positive trace preserving maps of an N-level system are characterized and the external maps explicitly constructed. The parametrizations are natural and enable one to compute the volume measures on this set of maps. Completely positive trace preserving maps are shown to be constructions of unitary maps of extended systems. The inverse problem is posed and solved.

## 1 Introduction to Dynamical Maps

Linear quantum stochastic processes may be identified with the linear conntraction mappings of the set of density matrices into the set of density matrices. This is to be contrasted with the Hamiltonian evolution of such density matrices which are time dependent unitary transformations. While these form a group, the dynamical maps form only a semigroup. A unitary evolution is characterized by a unitary matrix with $N^2 - 1$ independent real parameters. Both such unitary evolutions and the stochastic evolutions can depend arbitrarily on time, it would be simpler to consider a time-independent law of evolution. In the unitary evolution

$$p(t) \rightarrow p(t') = U(t', t)p(t)U\dagger(t', t) \qquad (1)$$

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made steady by having

$$U(t',t) = U(t^1 - t) \tag{2}$$

which becomes

$$H(t^1 - t) = (t^1 - t)H. \tag{3}$$

In a similar fashion if we introduce the superoperator $A(t^1, t)$ which linearly map the density matrices into themselves:

$$p(t) \rightarrow p(t') = A(t',t)p(t) \tag{4}$$

which for a steady evolution becomes

$$\begin{aligned} p(t) &= A(t'-t) \cdot p(t) && (5) \\ &= exp(i(t/primet)\epsilon)p(t) && (6) \end{aligned}$$

where $\epsilon$ is the dissipative generator-superoperator.

Unlike $H$, which is hermitian, the superoperator $\epsilon$ is nonhermitian and has a negative "imaginary part" indicating dissipation and generates a semigroup.

In explicit matrix notation in an arbitrary basis we have, respectively,

$$\begin{aligned} \psi(t) \rightarrow \psi_r'(t') &= \sigma_r U_{rr'}(t't)\epsilon - r'(t) && (7) \\ \rho - rs(t) \rightarrow \rho_{rs}'(t') &= \sigma_{rs} A_{rs,r's'}(t't) \cdot \rho_{r's'}(t) && (8) \\ U_{rr'}(t't)U_{r'r''}t(tt'') &= U_{r'r''}(t't'') && (9) \\ A_{rs,r's'}(t't)A_{r's',r''s''}(t-t'') &= A_{rs,r''s''}(t't''). && (10) \end{aligned}$$

The super operator $A$ has the "hermiticity" and trace-preserving property:

$$\begin{aligned} A_{rs,r's'} &= (A^*_{r's',rs} \\ A_{rr,r's'} &= \delta_{r's'} && (11) \end{aligned}$$

These dynamical maps $\rho \rightarrow A\rho$ form a convex set for $A^I$ and $A^{II}$,

$$A^I cos^2\alpha + A^{II} sin^2\alpha = A(\alpha) \tag{12}$$

is also a dynamical (positivity and trace-preserving) map. Since they operator on $N \times N$ nonnegative matrices with unit trace, the set of superoperators is itself a compact convex set.

The "hermiticity" property of $A$ can be best exploited by defining a related superoperator $B$ using the correspondence

$$B_{rr',s's} \equiv A_{rs,r's'} \tag{13}$$

The superoperator $B$ so defined implements the dynamcial map

$$\rho_{rs} \rightarrow \rho'_{rs} = \sigma_{r',s'} B_{rr',s's} \rho_{r's'} \tag{14}$$

which preserves the positivity and trace of the density matrices and is a linear map. Thus $B_{rr',s',s}$ has the symmetry and trace properties

$$(B_{rr',s',s})\dagger = B_{ss',r',r} \tag{15}$$

$$B_{rr',r',s} = \delta_{rs} \tag{16}$$

$$\rho \geq 0 \longrightarrow \rho' \geq 0 \cdot . \tag{17}$$

Since the matrix $B$ so defined is a finite dimensional hermitian matrix, it can be diagonalized with real eigenvalues.

$$B_{rr',s's} = \sigma_{n=1}^{N^2} \beta(r) C_{n^{\iota}}(n) (C_{ss'}(n))^* \quad , \quad (\beta(n))^2 \geq 0 \cdot \tag{18}$$

In terms of the eigenmatrices $C(n)$ the mapping becomes

$$\rho \rightarrow \rho' = \sigma_n \beta(n) C(n) \rho C\dagger(n). \tag{19}$$

$$\sigma_{n=1}^{N^2} \beta(n) C\dagger(n) C(n) = \tag{20}$$

These are the generic dynamical maps. The positivity preserving property *does not* imply that $\beta(n)$ is positive. The simplest example is given for the map of $2 \times 2$ matrices:

$$\rho \rightarrow \rho' = \rho^* - \frac{1}{2}(1 \cdot \rho \cdot 1 + \sigma_1 \cdot \rho\sigma_1 - \sigma_2\rho sigma_2 + \sigma_3\rho\sigma_3) \tag{21}$$

with

$$\beta(n) = \{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\}. \tag{22}$$

When all the eigenvalues $\beta(n)$ are nonnegative, we call the dynamical map *completely positive.* A completely positive dynamical map is then implemented by

$$\rho \rightarrow \rho^1 = \sum_{n=1}^{N^2} D(n)\rho D^+(n) \tag{23}$$

with

$$D(n) = \sqrt{\beta(n)} \cdot C(n). \tag{24}$$

Then

$$\sum_{n=1}^{N^2} D^+(n)D(n) = \tag{25}$$

Our aim is to characterize those completely positive dynamical maps[2] and provide a natural parametrization.

## 2 Completely Positive Maps

We now study the set of completely positive dynamical maps in terms of the matrices $B$ and $D$. The completely positive dynamical map of $N \times N$ density matrices into themselves is a convex set:

$$B = B^I cos^2\alpha + B^{II} sin^2\alpha \tag{26}$$

is a completely positive dynamical map if $B^I$ and $BII$ are. Since these $N^2 \times N^2$ matrices are compact the convex set can be generated by convex combination of extremal maps.

A map is extremal if it has no nontrivial decomposition of this type. One such set of dynamical maps is the set of unitary maps with

$$B_{rr',s's} = U_{rr's} \quad , \quad U^{+}U = \tag{27}$$

We have already shown that antiunitary maps are one-to-one invertible maps, but they are not completely positive. Another set of extremal maps are the pin maps

$$\rho \rightarrow \rho^{o} = tr(\rho) \cdot \rho^{o} \tag{28}$$

$$\rho_{rs} \rightarrow \rho_{rs} = \Sigma^{\delta}_{r's'} \rho^{ors} \rho_{'s'} \tag{29}$$

where $\rho^{\circ}$ is a pure density matrix.

$$(\rho^{\circ})^{2} = \rho^{o}. \tag{30}$$

To study the generic extremal map we note that the set of matrices $D(n)$ define the completely positive map

$$\rho \rightarrow \sum_{n} D(n) \rho D^{+}(n). \tag{31}$$

This can be an extremal map only if the set of matrices $D^{+}(m)D(n)$ are linearly independent: that is

$$\sum_{m,n} f_{mn} D^{+}(m) D(n) = 0 \tag{32}$$

implies

$$f_{mn} = 0 \tag{33}$$

In that case

$$\rho \rightarrow \lambda \sum_{n} D^{I}(n) \rho D^{I} \dagger(n) + (1-\lambda) \sum_{n} D^{II} \dagger(n) \tag{34}$$

would imply that $\lambda = 0, 1$. To prove this we observe that the set of matrices $D(n)$ should span the matrices $D^{I}(n)$ and $D^{II}(r)$, so that

$$D^{I}(n) \quad = \quad \Sigma_{m} \mathcal{G}^{I}_{nm} D(m)$$

$$D^{II}(n) \;=\; \Sigma_m \mathcal{G}^{II}_{nm} D(m)$$

Then

$$\begin{aligned}
\Sigma_m D^{I\dagger}(n) D^I(m) &= \Sigma_{n,n'} \mathcal{G}^{*I}_{mn} \mathcal{G}^{I}_{mn'} D^\dagger(n) D(n') = \\
\Sigma_m D^{II\dagger}(m) &= \Sigma_{n,n'} \mathcal{G}^{*II}_{mn} \mathcal{G}^{II}_{mn'} D^\dagger(n) D(n') =
\end{aligned}$$

and

$$\sum_{n,n'} (g^\dagger g - h^\dagger(n) D(n') \;=\; 0 \tag{35}$$

Thus the map $\rho \to \Sigma_n D(n)$ is not extremal. But if $\nu_{mn}$ is a unitary matrix $D'(n)$ defined by

$$D(n') = \sum_n \nu_{mn} D(n) \tag{36}$$

generates the same dynamical map as $D(n)$. If $G, H$ have the double polar factorizations

$$\mathcal{G} = u_1 \mathcal{G} u_2^\dagger h = \nu + 1 \tag{37}$$

then

$$\mathcal{G} g = h^\dagger h \tag{38}$$

implies

$$u_2 \mathcal{G}^2 u_2^\dagger = \nu_2 \tag{39}$$

or

$$\mathcal{G}^2 = u_2^\dagger \nu_2 \tag{40}$$

Since $G^2$ and $H^2$ are both nonnegative real matrices, $u_2^+ \nu_2$ must be a permutation matrix in the generic case; in case $G^2$ and $??^2$ have degnerate eigenvalues there can be a nontrivial matrix. But is easy to verify that in neither case does it alter the dynamical maps. So it follows that the two maps are the same; and the apparent linear combination is trivial.

In case we have a nontrivial linear relation between the matrices $D^\dagger(m)D(n)$ the map is not extremal. We see that the map

$$\rho \rightarrow \sum_m D(m)\rho D^\dagger(m) = \rho' \tag{41}$$

is the convex sum of

$$\rho' \pm \epsilon F_{mn} D(m)\rho D^\dagger(n) \tag{42}$$

for a sufficiently small $\epsilon$. Then the map $\rho \rightarrow \rho'$ is not extremal.

Thus if and only if condition of linear independence of $D^\dagger(m)D(n)$ hold do we have an extremal map. But clearly if $m, n > N$ this linear independence is not possible. So $M \leq N$ are the individual matrices for extremal maps.

# 3 Parametrization of Extremal Maps

With this understanding that $M \leq N$ for extremal maps we may extend the unitary and pin maps (which correspond to $M = 1$ and $M = N$ respectively)[4] to intermediate values of $M$. In particular for $M = 2$ we have

$$\rho \rightarrow \rho^1 \quad = \quad D(1)\rho D^\dagger(1) + D(2)\rho D^\dagger(2). \tag{43}$$

we could write

$$\tag{44}$$

$$D(1) \quad = \quad U(1)D^\circ(1)V \tag{45}$$

$$D(2) \quad = \quad U(2)D^\circ(2)V \tag{46}$$

$$\tag{47}$$

and make $D^\circ(1)$ and $D^\circ(2)$ diagonal. Let the eigenvalues of $D(1)$ be $\cos\,\theta_1, cos\theta_2$:

$$D(1) \quad = \quad U(1)\begin{pmatrix} \cos \quad \theta_1 & 0 \\ 0 & \cos \quad \theta_2 \end{pmatrix} V$$

Then

$$D^{\dagger}(2)D(2) \;\; = \tag{48}$$

so the matrix $D(2)$ has the form

$$D(2) = U(2)\begin{pmatrix} sm & \theta_1 & 0 \\ 0 & \sin & \theta_2 \end{pmatrix} \equiv U(2)SV \tag{49}$$

for an arbitrary unitary matrix $U(2)$:

One could generalize this for higher values of $M$.

For the generic case of $M$ matrices[5] satisfying

$$\sum_{n1}^{M} D^{\dagger}(n)D(n) = \tag{50}$$

we have the polar decompositions $D(n) = U(n)H(n)$ with $H(n)$ hermitian. We can choose to diagonalize $H(1)$ using a unitary matrix $V$:

$$\triangle(1) = C(1) = diag(\cos, \theta, \ldots) \tag{51}$$

Let the diagonal eigenvalues be $\cos_{11}, \ldots, \cos\theta_{1M}$. Then we have

$$H^2(1) + H^2(2) + \ldots + H^2(M) = \tag{52}$$

Now define

$$H(n) = S(l)H_{(1)}(n) \quad 2 \leq n \leq M \tag{53}$$

Then

$$\Sigma_2^M H_{(1)}^2(n) = \tag{54}$$

We can diagonalize $H_1(2)$:

$$H_{(1)}(2) = V_{(1)}\triangle(2)V_{(1)}^{\dagger} \quad \triangle(2) \equiv C(2) = diag(cost\theta_{II}) \tag{55}$$

Then

$$H_{(1)}(n) = S(2)H_{(2)}(n); M > n \geq 2 \tag{56}$$

where $S(2)$ is diagonal with eigenvalues for $\theta_{2n}$.

These $H_{(2)}(n)$ satisfy

$$\Sigma_3^M H_2^2(n) = 1. \tag{57}$$

Diagonalize $H^2(3)$:

$$H2(3) = V(2)\triangle(3)V_{(2)}^\dagger \quad \triangle(3) = C(3)diag(\cos\theta_{3n}) \tag{58}$$

In this manner we find the generic formula

$$\begin{aligned}
D(1) &= U(1)H(1) = U(l)HV\Delta^{(1)}V^\dagger; \Delta^{(1)} = C(1) = diag(\cos\theta_{11}) \\
D(2) &= U(2)H(2) = U(2)S(1)V_{(1)}\triangle(2)V_{(2)}^\dagger; \triangle(2) = C(2) = diag(\cos\theta_{2n}) \\
D(3) &= U(3)S(l)V_(1)S(2)W_{(2)}\triangle(3)V_{(2)}^\dagger V_{(1)}^\dagger, \triangle(3) = C(3) = diag(\cos\theta_{3n})
\end{aligned} \tag{59}$$

We still have the choice of forming orthogonal transformations in the matrices:

$$D(n) \to \sum_m R_{nm}D(m) \cdot \quad R^\dagger R = \tag{60}$$

By such transformations we can make

$$\cos\theta_{II} = 1 \quad , \quad sin\theta_{II} = 0 \tag{61}$$

and sequentially

$$equation?? \tag{62}$$

Thus there need be only $\frac{1}{2}M(M-1)$ angle parameters; the unitary matrices $U(1), V(2)\cdots, U(M-1), V(1), V(2), \cdots, V(M-1)$ are additional parameters that characterize the dynamical map. The rotation matrix $R_{Pmn}$, are irrevalant parameters. Since we know how to compute the volumes for unitary matrices and we may choose

$$equation?? \tag{63}$$

we can determine the volume of the set of dynamical maps, or the set of extremal dynamical maps.

# 4 Completely Positive Maps Realized as Contractions

To complete the discussion of completely positive dynamical maps of $N \times N$ matrices into themselves we recognize that such a map of rank $M$ can be realized by considering an extended system which is the Kronecker product of the states. So the composite density matrices $P$ are spanned by the products of density matrices:

$$P = \rho \times \chi \text{with} \chi_{mn} = \Sigma_\alpha \lambda_\alpha \varsigma^{*\alpha}_{m \varsigma^\alpha_n} \quad \text{and} \tag{64}$$

where $\chi$ are $M \times M$ density matrices. A unitary evolution of the composite system will entangle the two sets of states:

$$\rho \times \chi \to \rho \times \chi W \dagger \tag{65}$$

where $W$ is a unitary matrix in the Kronecker product states in dimension $MN \times MN$. If we contract this evolution by taking the $\chi$-trace we get

$$\rho \to tr_\chi(W \rho \times \chi W \dagger) = A_{rs,r's'} \rho_{r',s'}$$

with

$$equation?$$

This is a completely positive dynamical map

$$\rho \to \sum_n^\dagger D(n) \rho D \dagger (n)$$

with $D(n)$ defined by

$$equation?$$

Thus the contraction of a unitary evolution of an $MN \times MN$ density matrix yields a compleely positive map which is not extremal if the density matrix $\chi$ is not pure.

Conversely given a completely positive map we can always cast it in the standard form

$$\rho \to \sum_n D(n)\rho D\dagger(n).$$

Now construct an $MN \times MN$ unitary matrix $W_{rm,sm'}$ with

$$W_{rnr'} = D^{(n)}_{rr'}$$

and the other matrix elements arbitary but such as to make $W$ unitary. Then the completely positive map $\rho \to \sum_n D(n)\rho D\dagger(n)$ can be extended to the unitary map

$$\rho \times \chi \to W\rho \times \chi W\dagger. \tag{66}$$

If the map is extremal, $\chi$ may be chosen to be the project

$$\chi_{mm'} = \delta_{m'}\delta_{m'}$$

For a nonexternal completely positive map of the form

$$equation?$$

we could expand the map to the unitary map $W_{rn,\gamma' n'}$ with an impure density matrix with eigenvalues ??. For the nonexternal maps $M$ may range from 1 to $N^2$.

Since the convex set of completely positive maps is compact and locally compact, it is generated by the extremal maps. Except for the "edges" which are on the boundary of the compact set, the decomposition in terms of convex combinations of external elements is not unique.

This is not unlike the decomposition of an impure density matrix as a sum of projections; there are infinitely many ways of doing this. If we demand that the decomposition be in terms of orthogonal projections it becomes unique in the generic case. (If the density matrix has degenerate eigenvalues then there will be corresponding ambiguities.)

We can use this observation to decompose the dynamical maps. The completely positive map corresponds to $B$ matrix:

$$equation?$$

By a simple change of basis for $\chi$ we can cast this in the form

$$equation?$$

which corresponds to the decomposition of the map as a convex sum:

$$equation?$$

Each of the $B^{(\alpha)}$ furnish a completely positive map. In terms of the $D$ matrice, this corresponds to the decomposition

$$D(n)\rho D\dagger(n) = \sum \mu_\alpha D(n,\alpha)\rho D\dagger(n,\alpha) \tag{67}$$

with the understanding that the $D(n,\alpha)$ can be spanned by $D(n)$:

$$D(n,\alpha) = \sum_m \quad , D(m') =$$

The indecomposable (extremal) maps have the $D(m)$

$$\sum_n D\dagger(n,\alpha)D(n,\alpha) = \sum b^*_{nm}(\alpha)D\dagger(m)b_{nm'}(\alpha)D(m') =$$

for more than one $\alpha$. This can be shown to be equivalent to the condition

$$C_{m,n}D\dagger(m)D(n) = 0 \Longrightarrow c_{mn} = 0$$

The external maps correspond to $\chi$ being a projection while the generic maps correspond to $\chi$ being the convex sum of orthogonal projection.

# 5 Summary

In this paper we have analyzed completely positive maps of $N \times N$ density matrices into density matrices. They can all be defined in terms of the eigenvector decomposition of the $N^2 \times N^2$ dynamical matrix $B_{rr',ss'}$ The rank of this hermitian nonnegative matrix determines the rank of the dynamical map. This class)fication is natural and unique.

A generic rank $M$ completely positive map is displayed in parametrized form[5] with $1/2M(M)$ angle parameters lying in the interval $0 \leq \theta \leq \pi/2$, and a set of unitary matrices of dimensions $M, M-1, \cdots$ to dimension 1, modulo permutation of the $M$ indices.

The extremal elements of the convex set of completely positive dynamical maps have the matrices $D(n)$ such that the set $D\dagger(m)D(n)$ are linearly independent.

Any completely positive dynamical map is the contraction of a unitary map of an extended system with density matrix $\rho \times \chi$ with dimension $MN \times MN$. Conversely given a completely positive dynamical map we can expand it into a unitary map of an extended system. If the map is external the density matrix $\chi$ should be a projection. External maps have rank $M \leq N$. For $M = 1$ they are unitary maps contracted; for $M = N$ they are the pin maps.[4]

Many of these results have been known for decades, yet they continue being rediscovered and presented without reference to existing literature.[6] It is hoped that this need not happen because of unfamiliarity with the field, especially for newcomers to the field from quantum computing and coherent atomic physics.

The dynamical maps from a semigroup; since the generic maps are contractive. If we consider a continuous semigroup labeled by a time parameter which is additive, one can extract the generator of the semigroup. Complete positivity of the semigroup maps impose nontrivial. Conditions on the generators of the semigroups, some of which can be immediately applied. For example there is an inequality between the longitudinal and transverse relaxation times in the Bloch equations for spin 1/2 particles.

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