Six lectures on Quantum Mechanics

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Prof. E.C.G. Sudarshan visited the School of Physics, University of Hyderabad, Hyderabad - 500 134, India between 12th Dec. 1994 and 27th Dec. 1994 as a Jawaharlal Nehru Professor for the Sciences. During this stay he delivered six delightful lectures on Quantum Mechanics. Quantum Mechanics has undergone a sea of change - quantum mechanics on multiply connected manifolds and new statistics are some of the new and exciting developments. One of the leading theoretical physicists of our times, who learnt the subject from the founding father P.A.M. Dirac, Professor Sudarshan describes this fascinating theme with modern tools. He points out some of the fallacies which we are fed from text books and folk lore: The new mathematics is presented in a simple lucid style. These lecture notes are no substitute for his lectures. However, in the lecturer's absence these set of notes may stimulate us when reread, sending us back into the memory lane so that we relive the good times. Those who have not heard him, we are sure, will profit by reading these lecture notes.

I am very happy to be invited to come here again. Many years ago when I was a young student, I had my first lectures from Dirac. Prof. Dirac came and then said, "My name is Dirac. I have done some work on quantum mechanics; I have even written a book!", and he showed us the book that he had written. And then of course he went on to teach us quantum mechanics. I cannot claim the same ability, but on the other hand I can say that I took my first lessons from him and imitation is the most transparent homage. As Prof. Srinivasan quoted Walter Thirring, there are things in the text books which are not very useful. It was useful for the people who wrote it and probably for the people for whom it was intended, but those tools may not be adequate at the present time. And in some cases some things are just wrong.

Yesterday I was reading a book written by Victor Weisskopf. Prof. Weisskopf was recalling his youth and he said that he wrote to Max Planck when Weisskopf was of 14 years and had asked him "How can mass increase with velocity so that when you approach the speed of light mass becomes infinite and yet light quanta travel with the velocity of light?" Apparently Professor Planck did not know that this was written by a fourteen-year old boy; Planck wrote a very formal letter in which he explained that "... you must distinguish between corpuscular and wave motions. Waves can travel at any speed but corpuscles cannot travel at a great speed." Weisskopf said that this letter from Planck was striking in many ways: one of them was of course that to get a letter from a world famous physicist addressing him as "Dear Sir," and signing it "Yours respectfully", but the second thing that was very interesting was that his answer was wrong! In fact, it was before de Broglie's

are wrong in some of the text books: one of the assertions, which many of you know, that if you have identical particles then the wave function should be symmetric or anti-symmetric as we interchange the identical particles. The argument goes as follows: if you interchange the particles then the wave function goes into something else; if you interchange it again, then the wave function should go back to itself. Therefore, if the linear operator P realized the permutation, then $P^2 = 1$ and therefore the wave function must either be symmetric or anti-symmetric. $P = \pm 1$. You see this in these very nice books, yet the statement is wrong! Now-a-days we know that there are particles whose statistics is not of this kind. So, I want to talk to you about, amongst put it into a proper setting I might as well start right at the begining.

Quantum mechanics is patterned after classical mechanics with the further unification between dynamical variables and dynamical operations. This entails non-commutativity, superposition and the non-specifiability of all dynamical variables at any time. A complete description is obtained by specifying a maximal commuting set of dynamical variables to have values. The values for the dynamical variables themselves are not necessarily the classical spectrum of values but a specific eigenvalue spectrum.

The standard method of description is to have a configuration space $\{q\}$ and the corresponding momenta $\{p\}$. The Schrödinger realization is to represent $\{q\}$ by a real variable $-\infty < x < +\infty$ and define momenta by the differential operator $\frac{\hbar}{i}\frac{\partial}{\partial x}$. This choice is sufficiently general to give all cases with one degree of freedom with unbounded motion (J. von Neumann's theorem: show that $|0>$ state exists and $\sum |n><n| = I$).

For N degrees of freedom we will take the configuration space to be R^N with $\frac{\hbar}{i}\nabla$ as the momenta. These act on a wavefunction $\psi(x)$. The space of wavefunctions is given a norm by associating a dual vector $\psi^\dagger(x)$ to every $\psi(x)$ in an antilinear manner and taking the integral

$$\int \psi^\dagger(x)\psi(x)dx \equiv |\psi|^2$$

as the square of the norm of $\{\psi\}$; this provides us with a vector space over complex numbers and is considered separable (countable basis) and complete (containing limit points of all Cauchy Sequences). The set of all such vectors of finite length with an antilinear correspondence between vectors and their

duals (modulo all vectors of zero norm) is a Hilbert space; often the space of wavefunctions (state vector) is taken to be a Hilbert space.

Given two vectors $\psi^\dagger$ and ϕ we can calculate the matrix element of an operator A in the form

$$\psi^\dagger A\phi \equiv (\psi, A\phi)$$

$$\psi^\dagger A\psi \equiv (\psi, A\psi) =< A >_\psi$$

The matrix element of the unit operator is the scalar product.
Strictly speaking $A^\dagger$ is defined as the dual space and A on the primary space. For a Hilbert space since the dual space is the same we can talk of $A = A^\dagger$ otherwise we must restrict the domain of application before comparing A and $A^\dagger$

$$(\psi^\dagger\phi) = (\psi, \phi)$$

with

$$\psi^\dagger\psi \equiv (\psi, \psi) = 1 \ .$$

The hermitian conjugate $A^\dagger$ of an operator A is given by

$$(\psi, A\phi) = (A^\dagger\psi, \phi) \ .$$

For $A^\dagger = A$ are have hermitian operators. Hermitian operators have real expectation values in any state.

$$(\psi, A\psi)^* = (A\psi, \psi) = (\psi, A\psi) \ .$$

The correspondence $A \to A^\dagger$ is one-to-one, reflexive and antilinear.

If for our operator A and a vector ϕ

$$A\phi = a\phi$$

we say ϕ is an eigenvector and a is an eigenvalue. There may be more than one eigenvector for one eigenvalue.

If the eigenvectors provide a basis we have a spectrum of eigenvalues which are a discrete set. The resolvent operator (Green's function)

$$R(z) = (A - zI)^{-1}$$

is non-singular, analytic at all points except the discrete set of points constituting the spectrum.

These are hermitian operators which do not have a complete spectrum of eigenvectors or even no spectrum at all in terms of eigenvectors. Examples are the hydrogen atom hamiltonian and the free particle energy or momentum. But in these cases we can still define a spectrum as the set of singularities of the resolvent.

When we use the Schrödinger realization (for $N = 1$) the Hilbert space is obtained by considering all square integrable functions (of one variable). With the usual (Lebesgue) measure integration this space is complete and separable (with the L^2 norm).

A subset of such functions are provided by the functions analytic in a domain including the real axis. They form (an overcomplete) basis and they are dense in the space of square integrable functions (Any such function can be approximated arbitrarily closely by analytic functions: any open neighbourhood contains analytic functions). The conjugates of the vector $\psi(z)$ is the vector $\psi^*(z^*)$ which is also an analytic function. They form dual pairs. The choice of the domain of analyticity is at our disposal.

Another sense in which one considers dual spaces is when ψ is a non-normalizable state. There would still exist many vectors ϕ such that (ϕ, ψ) is finite. When the set $\{\psi\}$ expands the set $\{\phi\}$ contracts and vice versa. If the dual pairs have each a basis in the sense that $(\phi_m, \psi_n) = \delta_{mn}$ then $\sum_m \psi_m \phi_m^\dagger = I$ and any state can be expanded in the $\{\psi_m\}$ basis.

Analytic continuation of the state space is between dense sets. When the states are analytically continued, for suitable operators the matrix elements also can be continued. If whenever ψ is an analytic vector $A\psi$ is also analytic, then A can be extended to the new space. This extended operator may be called by the same name (by abuse of language!). This new possibility means that in the continued space the spectrum (of the same operator!) is different.

We can thus include resonances and metastable states in the spectrum of a hermitian hamiltonian.

Simplest example is the Fourier spectrum of a time signal $f(t)$:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int e^{-i\omega t} g(\omega) d\omega \ ;$$

usually the integral is taken along the real axis. But if the functions are analytic we can take an integral along a new contour within the domain of analyticity giving a spectrum of complex frequencies: not in addition to, but in place of the real frequencies.

All along we have taken a complete set of dynamical variables so that we can go from any state to any other state. We call this "scalar quantum mechanics" with a single state function ψ. If on the others had we take a vector ψ_1, ψ_2, ψ_3 to represent the state vector with scalar product

$$(\Phi, \Psi) = (\phi_1, \psi_1) + (\phi_2, \psi_2) + \cdots$$

Then we call it "vector quantum mechanics". Before spin was discovered as a dynamical variable a two component electron wavefunction would have been an example of vector quantum mechanics.

In the last lecture we discussed quantum mechanics in the Schrödinger realization with the coordinates being unrestricted. But in many cases we want to consider a bounded range for the coordinate. For one variable we choose

$$a \leq x \leq b \quad ,$$

or, for $\mathcal{R}^N$, rectangles

$$a_j \leq x_j \leq b_j \ ; \qquad 1 \leq j \leq N \ .$$

The problem now is entirely different. Any bounded functions is now square integrable (and some not bounded ones too, say $x^{-1/4}$) but we must, restrict the boundary conditions.

$$\left(f(x), \frac{\hbar}{i}\frac{\partial}{\partial x}g(x)\right) = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}f(x), g(x)\right) + f^*(x)g(x)\Big|_a^b \ .$$

So unless

$$f^*(a)g(a) = f^*(b)g(b)$$

the momentum $\frac{\hbar}{i}\frac{\partial}{\partial x}$ is not hermitian. This is assured if,

$$f(a) = f(b) = g(a) = g(b) = 0$$

but also if

$$f(b) = e^{i\alpha}f(a) \quad ; \quad g(b) = e^{i\alpha}g(a) \ .$$

where α is any arbitrary real quantity. So we have a one-parameter infinity of possibilities. They are inequivalent.

If $\alpha = 0$ we could think of the interval $a \leq x \leq b$ as a circle S' with the points a and b identified: "periodic boundary conditions". But what if $\alpha \neq 0$, the twisted periodic case. In this case the identification of b with a circle S' may be made but the wavefunction is no longer single valued.

With b and a identified we have new manifold, topologically a circle S'. For two dimensions with opposite sides identified we get a torus T^2. On these and other general manifolds we have new features of quantum mechanics.

With quantum mechanics on S' (periodic twisted boundary conditions) we have a new spectrum for the momentum and hence free kinetic energy

$$-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2} = \frac{\hbar^2}{2I}(l+\alpha)^2 \quad , \qquad l = 0, \pm 1, \pm 2, \cdots$$

Note particularly the constant solution for $\alpha = 0$ which does not obtain for the box

$$\begin{aligned} \text{For} \qquad \alpha &= 0 \quad , \quad E = \tfrac{\hbar^2}{2I} l^2 \quad l = 0, \pm 1, \cdots \\ \alpha &= \pi \quad , \quad E = \tfrac{\hbar^2}{2I}(l + \tfrac{1}{2})^2 \end{aligned}$$

Before dealing with general manifolds we consider "point singularities". The Hamiltonian is free everywhere except at $x = 0$ where the wave function or its derivative or both are discontinuous. But we require that the current $\frac{1}{2im}(\psi^*\psi' - \psi'^*\psi)$ is continuous.

Topological properties of manifolds. Manifold is region locally mapped into Euclidean space $\mathcal{R}^N$ with suitable transfer functions. We shall deal only with differentiable maps. Manifolds may or may not have boundaries.

Examples of $\mathcal{R}^N$. With choice of boundary conditions we may convert it into a manifold without boundaries. Differentiability is necessary so that the derivatives can be defined for the action of momentum.

Connected and disconnected; take each connected part by itself. Path connected.

Paths; loops. Can all the loops be continuously contracted; then simply connected.

Loops can be multiplied together by sticking them together, equivalence classes of loops. Product of the equivalence class is an equivalence class. Group under class multiplication. First homotopy group $\pi_1(M)$ of the manfold. π_1 trivial, means simply connected manifolds. The transformation of the wavefunction at a point are a representation of π_1. For sealar quantum mechanics theory only one-dimensional phases, and hence a representation of the abelianized theory. Should π_1 be trivial if there is no nontrivial (scalar) quantum theory? No, if π_1 is perfect then no abelian factor groups, abelianization makes it trivial. Perfect group and perfect manifolds (Icosahedral group). Perfect is one in which the factor group is nonabelian.

Back to the compact interval with end point identified; the topology changes from line segment to circle S'. The loops that can be contracted are those that are closed even with the line open. But if it passes through the endpoint we get a non-contractible loop. Many times looping around this gives other non-ccontractible loops. The loop group is the infinite cyclic

group $\mathcal{Z}$ with elements $\epsilon = L^o, L^{\pm 1}, L^{\pm 2}, L^{\pm 3}, \cdots$. Abelian π_1, irreducible representations are all finite dimensional:

$$\mathcal{Z} : L^N \longrightarrow \epsilon^{in\theta} \qquad \theta \text{ arbitrary real}$$

Infinitely many representations, all twisted periodic condition.

Second example is a disc with opposite points on the bounding circle identified. $\pi_1 = \mathcal{Z}_2$. Two representations

D/Z_2 Moebius Strip

one with periodic and the other with antiperiodic boundary condition.

Of course there are other identifications. For $\mathcal{R}^2$ rectangle identifying all the boundary points gives the sphere S^2 with a trivial π_1. But if opposite sides are identified are get the Torus T^2 with $\pi_1 = \mathcal{Z} \times \mathcal{Z}$.

A physically important case is the manifold of $SO(3)$ the group of proper rotations of rigid body which involves an axis and an angle of rotation. So we can get a representation by taking a solid sphere with the direction as the direction of the axis and the radius equal to the magnifide of the angle of rotation. We need only rotation $0 \leq \theta \leq \pi$. But $\theta = \pi$ rotations

about opposite axes are the same! So we have the solid sphere of radius π with antipodal points identified. Two kinds of closed loops, those which are contractible and those which are not. Homotopy group is $\pi_1 = \mathcal{Z}_2$.

In all these cases the homotopy group is abelian (except for the icosahedral group that was mentioned). A familiar case with nonabelian π_1 is given by the configuration space of a generic ellipsoid (say, the moment of inertia ellipsoid). It is the rotation group modulo the symmetry Q of the ellipsoid. Q consists of 8 elements

$$\{e, f, a, b, c, af, bf, cf\}$$

with $f^2 = 1, a^2 = b^2 = c^2 = f$; $fab = ba = c$ etc., $fa = af$ etc. This is a presentation of the group of order eight, sometimes called the quaternion group. It has 4 one-dimensional representations and 1 two dimensional representation. The group is nonabelian.

$e = 1, f = +1, a = b = c = 1$		1×1
$e = f = 1, a = +1, b = -1, c = -1$	and two similar ones	3×1
$e = 1, f = -1, a = i\sigma_1, b = i\sigma_2, c = i\sigma_3$		1×4

order of the group is $8 = 4 \times 1^2 + 1 \times 2^2$.

The last case corresponds to vector quantum mechanics with 2-component wavefunctions (which may be identified with internal "spin").

Perfect spaces with nontrivial topology have vector quantum theories associated with them which are truly internal symmetries of the manifold not locally describable.

Statistics of Strictly Identical Particles.

The problem of strictly identical particles may be viewed as quantum mechanics in nontrivial manifolds. Since we already know how to characterize π_1 we know the type of statistics we can obtain. This we shall proceed to do.

But as a preleminary let us recall the usual scheme of Bose and Fermi statistics. For Bose/Fermi systems

$$\psi(x_1 x_2) = \begin{cases} \psi(x_2 x_1) & B \\ -\psi(x_2 x_1) & F \end{cases}$$

So the functions are defined only on $\mathcal{R}^m \times \mathcal{R}^m / S_2$ but this is not a manifold, so we remove the diagonal points $x_1 = x_2$ and take as configuration space

$$M = (\mathcal{R} \times \mathcal{R} - \Delta)/S_2$$

Now since $x_1 \neq x_2$ is one point a path from (x_1, x_2) to (x_2, x_1) is closed but not contractible, while a path from (x_1, x_2) to (x_1, x_2) is closed and contractible. The homotopy group is $\pi_1(\mathcal{R}^m \times \mathcal{R}^m - \Delta/S_2) = S_2 \equiv \mathcal{Z}_2$ and its two representations are Bose and Fermi.

What happens when there are more particles? The configuration space is $(((\mathcal{R}^m)^n - \Delta)/S_n)$ and $\pi_1 = S_n$. There are two one dimensional representations of S_n which are Bose and Fermi. All the other representations are multidimensional; and realize parastatistics.

The comment applies when the motion is in 3 or more dimensions; we get $\pi_1 = S_n$ and hence parastatistics.

Digression on Parabose and Parafermi statistics

The Hamiltonian (and other generator) equations of motion for a Fermi system is

$$H = \frac{\omega}{2}(a^\dagger a - aa^\dagger) \quad [a, H] = \omega a$$

Question: are there things other than Fermi oscillator which satisfy this relation? The answer is yes! Write

$$a = J_- \quad , \quad a^\dagger = J_+ \quad \text{Then} \quad [J_-,\ (J_+ - J_- - J_- J_+)] = 2J_-$$

which is the same as $J_+ J_- - J_- J_+ = +2J_3$ and the $J_\pm, J_3$ satisfy angular momentum commutation relations. Spin $\frac{1}{2}$ realization corresponds to Fermi operators. The general spin $s \geq 1$ correspond to "parafermions". A way of building higher spins from spin $\frac{1}{2}$ is to add commuting spins.

$$\left.\begin{array}{ll} a = \sigma_- + \tau_- \cdots, & a^\dagger = \sigma_+ + \tau_+ \cdots \\ \frac{1}{2}(a^\dagger a - aa^\dagger) = \sigma_3 + \tau_3 + \cdots & \end{array}\right\} \text{ but this is not irreducible.}$$

For many degress of freedom we can extend this:

$$\begin{aligned} [a_j, \tfrac{1}{2}(a_k^\dagger a_l - a_l a_k^\dagger)] &= \delta_{jk} a_l \\ [a_j, \tfrac{1}{2}(a_k a_l - a_l a_k)] &= 0 \\ [a_j^\dagger, \tfrac{1}{2}(a_k a_l - a_l a_k)] &= -\delta_{jk} a_l + \delta_{jl} a_k \end{aligned}$$

We can realize this in terms of the angular momentum for $2N+1$ dimensions (Lie algebra B_N). A reducible realization may be contructed by the Green ansatz

$$\begin{aligned} a_j &= \sum a_j^{(4)} & a_j^{(r)} a_j^{(s)} &= -a_j^{(s)} a_j^{(r)} \quad r \neq s \\ [a_j^{(r)}, a_j^{\dagger(s)}] &= \delta_{jk}, r = s; & a_j^{(r)} a_j^{(s)} &= -a_j^\dagger a_j^{(r)} \quad r \neq s \end{aligned}$$

This is not irreducible. Just as parafermi could be seen as representation of B_N, these could be realized as representations of $Sp(2N;R)$ (Lie algebra C_N).

Parastatistics furnish representations of the permitation groups S_n. Except the two one-dimensional representations B, F all the others represent multicomponent (vector) quantum theories. The corresponding density matrices can be made symmetric by taking $\Sigma\psi_n\psi_n^\dagger$ over all members of an irreducible representation.

Reducible parastatics have been introduced quite elegantly by HS Green. The anasatz for parafermions is to take the sum of R commuting fermion operators

$$b = \sum_{n=1}^{R} a_j^{(r)}$$

$$\{a_j^{(r)}, a_k^{\dagger(s)}\} = \delta_{\kappa j} \quad r = s \quad ; \quad \{a_j^{(r)}, a_k^{(s)}\} = 0, \quad r = s.$$

$$[a_j^{(r)}, a_k^{(s)}] = [a_j^{(r)}, a_k^{\dagger(s)}\} = 0, \quad r \neq s.$$

Then

$$b_j b_k^\dagger - b_k^\dagger b_j \quad = \textstyle\sum_r (a_j^{(r)} a_k^{\dagger(r)} - a_k^{\dagger(r)} a_j^{(r)})$$

$$b_j b_k - b_k b_j \quad = \textstyle\sum (a_j^{(r)} a_k^{(r)} - a_k^{(r)} a_j^{(r)})$$

and the parafermi commutation relations are satisfied. For parabose we take mutually anti-commuting Bose operators.

It is clear that parafermi oscillators are representation of $S0(3)$, parabose oscillators give two representations of $Sp(2,R) \sim S0(2,1)$. This is valid in

general for N degress of freedom $1 \leq j, k \leq N$.

$$\text{parafermi} \quad S0(2N+1) \equiv B_N$$

$$\text{parabose} \quad Sp(2N) \equiv C_N$$

reducible parafermi (parabose) for $N+1$ degrees is not irreducible for N degrees; but Green ansatz is unaffected. The Green index may be considered as a new dynamical label called color. Green ansatz contains all parafermions, but *not all* parabose realizations.

For a base space $\mathcal{R}^d, d \geq 3$ the configuration space of n identical particles is

$$M = ((\mathcal{R}^d)^n - \Delta^{dn})/S_n$$

where Δ^{dn} is the diagonal where any two particles have identical coordinates; and S_n is the permutation group in n variables with $n!$ elements. In this case it is a standard result to state

$$\pi_1((\mathcal{R}^d)^n - \Delta^{dn})/S_n) = S_n$$

which (for $n \geq 3, \geq 3$) is noncommutative. The most general statistics is parastatistics.

For $d = 1$ the space is not path connected and the connected pieces have trivial no homotopy.

For $d = 2$ we get a very new situation. The group is not S_n but a group called the n-string Artin braid group $B_n(\mathcal{R}^2)$

$$\pi_1((\mathcal{R}^2)^n - \Delta)/S_n) = B_n(\mathcal{R}^2)$$

Braids can be visualized as ropes in a plane being twisted in the third dimension (parameter dimension). For two strings keep one fixed and get the other spiral around it through an integer number of turns. These are closed loops which cannot be disentanged and contracted. The groups so obtained is $\mathcal{Z}$ with infinite number of distinct one-dimensional representations $e^{i\theta}$. This is the case of anyon statistics.

The generic n-string group has the presentation

$$B_n(\mathcal{R}^2) = \left\langle \tau_1, \tau_2, \cdots, \tau_{n-1} | \tau_{i+1}\tau_i\tau_{i+1} = \tau_i\tau_{i+1}\tau_i; \quad \tau_j\tau_j = \tau_j\tau_j | i-j| > 2 \right\rangle .$$

For $n \geq B_n(\mathcal{R}^2)$ is nonabelian and infinite.

For $n = 3$ we have

$$B_e(\mathcal{R}^2) = \left\langle \tau_1, \tau_2 | \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2 \right\rangle$$

or by writing $\tau_1\tau_2 = a, \tau_1\tau_2\tau_1 = b$,

$$B_3(\mathcal{R}^2) = \left\langle a, b | a^3 = b^2 \right\rangle$$

There are an infinite number of nonscalar quantum theories on $((\mathcal{R}^2)^3 - \Delta)/S_3$, and the dimensions of the representations are unbounded.

All 2-dimensional representation are labelled by two angles ϕ and θ

$$0 \leq \theta \leq \pi/2, 0 \leq \phi, \omega = e^{2\pi i/3}$$

(for $\theta = 0$, $0 \leq \phi \leq \pi$)

$$a = e^{2i\phi} \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \quad ; b = e^{3i\phi} \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\cos\theta \end{pmatrix}$$

A three parameter family of 3-dimensional representation is

$$a = e^{2i\phi}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad ; \quad b_{jk} = e^{3i\phi}(-\delta_{jk} + 2n_j n_k)$$

with n_j a unit vector and ω a cuberoot of unity.

In contrast, if we consider particles on a 2-sphere,

$$B_3(S^2) = \left\langle e, d | e^3 = d^3 \quad d = ede \right\rangle$$

of order twelve. Four scalar (one-dimensional) and 2 two-dimensional representations obtain;

$$e = \lambda^2 \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \quad ; \quad d = \lambda^3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \lambda = 1, i$$

$B_n|S^2>$ are infinite and nonabelian for $n \geq 4$.

"Dynamical" Realizations of Anyons

For many degrees of freedom fermions we take first a commuting set of Pauli triplets

$$\{\alpha_j, \alpha_j^\dagger\} = 1 \quad , \quad [\alpha_j, \alpha_k] = [\alpha_j, \alpha_k^\dagger] = 0 \ , \quad j \neq k$$

Then we do a Klein tranformation

$$\alpha_1 \to a_1 \quad ; \quad \alpha_2 e^{i\pi\alpha_1^\dagger\alpha_1} \to a_2 \quad ; \quad e^{i\pi(\alpha_1\alpha_1 + \alpha_2^\dagger\alpha_2)}\alpha_3 \to a_3$$

etc., this makes the different degrees of freedom anticommute, that is anyons with the special phase $e^{i\pi}$. So why not do the following:

$$\alpha_1 \to e_1 \quad \alpha_2 e^{\theta(\alpha_1^\dagger\alpha_1)} \to c_2 \qquad \alpha_3 e^{\theta(\alpha_1^\dagger\alpha_1 + \alpha_2^\dagger\alpha_2)} \to e_3 \quad etc.,$$

Then $\epsilon_1\epsilon_2 = e^{i\theta}\epsilon_2\epsilon_1$ etc.

So any case where there is an operator unitary transformation on the "vector" of annihilation operators with $\exp(i\theta Q)$ where Q is the total number operator will bring about such a transformation. Thus a fractional Klein transformation can transmute the statistics.

This is good to have as an explicit constiruction since the abelianization of the Braid groups give us $< \tau_1, \tau_2, \cdots | \tau_1 = \tau_2 = \cdots >$ so that we get $\mathcal{Z}$ for any $B_n^{ab}(\mathcal{R})^2$ or $B_n^{ab}(S^2)$.

It is also tempting to derive this transmutation from a dynamical theory. Now if ψ is a field $\psi\exp(i\theta F(x))$ will have different derivatives:

$$\begin{aligned}\partial_\mu\{\psi\exp(i\theta F(x))\} &= e^{i\theta F(x)}(\partial_\mu - i\theta\partial_\mu F)\psi(r) \\ &= e^{i\theta F(x)}(\partial_\mu - iA_\mu)\psi(x) \quad , \quad A_\mu = \theta\partial_\mu F \ .\end{aligned}$$

In two dimensions in the charge-current free region we can replace a curlfree field by a pure curl. In fact using the Cauchy-Riemann equation we get

$$\log z = \log r = i\tan^{-1}\frac{y}{x} \quad , \quad z = x + iy = re^{i\theta}$$

as the Green's function in 2 dimensions. We can, to a certain extent, interchange the role of the two (Note that choice $\log r$ is defined everywhere except at 0 and ∞, $\tan^{-1} y/x$ is multiple valued; $d\theta$ is a closed but not exact differential Danger + opportunity!) Using such ideas Semenoff attempted to construct a field theory of free anyons from a gauge coupling in $2+1$ dimensions: Hagen has pointed out some problems in this construction. Polyakov

has realized the necessity to stay away from charges and currents and hence assert the transmutation of statisties only for the low frequencies!

The Wavefunctions of Anyons

For two anyons with phase $e^{i\theta}$ the wave function is an arbitrary analytic function in $(z_1 + z_2)/2$ and an analytic function $F(\zeta)$ multiplied by $\zeta^{\theta/2\pi}$ where ζ is the relative (Jacobi) coordinate $\zeta = z_1 - z_2$. This is defined at all finite points except the origin.

For three particles z_1, z_2, z_3 we have to consider two Jacobi coordinates $\zeta = z_1 - z_2$ and $\eta = (z_1 + z_2 - 2z_3)/\sqrt{3}$ and have $(\zeta(3\eta^2 - \zeta^2))^{\theta/2\pi}\ F(\zeta, \eta)$; and so on.

If F is finite at the zeroes of the Jacobi coordinates it follows that all the wavefunctions vanish for the particles approaching each other and gets small in the neighbourhood of the diagonal. This is a pseudorepulsion, we can also get a pseudo attract of F has singularities at the zeroes of the Jacobi coordinates. Weak increasing negative powers $\zeta^{\frac{1}{2}-\epsilon}$ are *not* forbidden.

Phases, Exclusion Principle and Wavefunction Symmetry

Electrons obey the exclusion principle, can we conclude that they are fermions? Clearly not, since all (non-Bose) statistics including anyons obey this. So to deduce antisymmetry we need to look at selection rules or the matrix elements of, say, the Coulomb interaction ("exchange integrals"). If we have anyonic behaviour we need the generic many electron wavefunction explicitly to carry out this calculation.

Prof. V. Srinivasan has pointed out another important avenue for search, in the scattering of identical particles. For Bose/Fermi we have $F(\theta) = f(\theta) \pm$

$f(\pi-\theta)$ so only even/odd partial waves will occur for spinless particles or spin symmetric states of spinning particles. What is the modification for anyons? We must use partial waves of fractional order $P_l(\cos\lambda\theta)$ with λ taking care of the phase. We should develop a Faxen-Holtsmark kind of decomposition with such fractional partial waves; needless to say the centrifugal barrier will also change accordingly.

We could do Aharanov-Bohm kind experiments like Tonamura's experiments at Hitachi using whiskers to have two alternative paths of an anyon around (a clump of) anyons. If we can show that such thing do not show a phase then either no anyons exist or anyons of "roots of unity" phases are there but they are confined.

We can also ask: can particles obeying exclusion principle have symmetric wave functions? As long as the exclusion is only for a fixed basis we could do so but if the exclusion is mode independent then the wavefunction must be antisymmetric. Take two modes u_1 and u_2. Then a two-electron state obeying exclusion principle would be $u_1(x_1)u_2(x_2)$ or if we symmetrize $\frac{1}{\sqrt{2}}\{u_1(x_1)u_2(x_2) \pm u_1(x_2)u_2(x_1)\}$. Now change the modes to

$$v_1 = \frac{1}{\sqrt{2}}(u_1 + u_2) \qquad v_2 = \frac{1}{\sqrt{2}}(u_1 - u_2)\ .$$

Then the antisymmetric function is still antisymmetric and obeys exclusion principle but the symmetric wave function is the difference

$$\frac{1}{\sqrt{2}}(v_1(x_1)v_1(x_2) - v_2(x_1)v_2(x_2))$$

which is double occupancy!

Mode independence must be really demanded of any quantization (including q-oscillators!!)

Time Reversal and Nonintegrable Phases

Quantum mechanical time reversal is done for simplest systems by

$$\psi(x,t) \to \psi^*(x,-t) \ .$$

If there is a mechanical spin we should amend this by requiring

$$\psi \to e^{i\pi J_2}\psi^*(x,-t) \ .$$

If there are internal symmetries we must consider the internal symmetry labels (even if it is $SU(2)$) as "real" and leave them alone. (Biedenharn-Sudarshan, Pramana in press).

What about the nonintegrable loop phases? If we imagine the loop being described is the time reversed loop described in the reverse direction and is hence replaced by the conjugate phase. [For non-scalar quantum mechanics the situation is more complicated and requires more delicate handling.]

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