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WOLF'S NEW THEORY OF PARTIAL COHERENCE IN THE SPACE-FREQUENCY DOMAIN: APPLICATION TO BEAM PROPAGATION PROBLEMS

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Second order coherence theory of stationary planar sources and the fields generated by them are analysed within the framework of Wolf's new theory of partial coherence in the space-frequency domain, in a form suitable for beam propagation problems. In the cases where the map from the source plane to the field plane is lossless it is shown that the natural modes of the field are in simple one-to-one relation to those of the source. As a simple application we use this result to clarify the behaviour of Gaussian Schell-model beams as they pass through first order systems; a generalization of Kogelnik's abcd-law of coherent Gaussian beams to the partially coherent Gaussian Schell-model beams is shown to emerge in a natural way. Two methods for dealing with lossy maps are presented and illustrated with example. Possible extensions and applications of the analysis are pointed out.

1. INTRODUCTION

The theory of partial coherence is a well established subject[1,2]. Both the classical and the quantum formulations of this theory are usually carried out through space-time correlation functions such as the mutual coherence function $\Gamma(\underline{r}_1, \underline{r}_2; \tau)$. It is only recently that a new formulation of this theory in the space-frequency domain has been given by Wolf, first for the second order coherence functions[3] and subsequently for the higher order coherence functions[4]. Mention should be made in this connection also of the important work of Gori[5]. This new theory has been formulated in such a way as to find useful applications in various problems involving the generation, propagation and detection of partially coherent light. Reports of applications of this theory have already started mounting. We have applications in coherence properties of laser modes[6], production of partially coherent light fields with special shape-invariant property[7], the analysis of radiation field generated by Gaussian Schell-model sources[8], estimation of number of degrees of freedom of partially coherent sources[9], studies relating to radiometry[10], inverse problems involving partially coherent sources[11], propagation of band limited wave fields[12], and the synthesis of fields of prescribed partial coherence[13].

One of the basic features of this theory is that the cross-spectral density across a source at a frequency ω is written as the incoherent superposition of certain fully coherent orthonormal normal modes of oscillation of the source at that frequency[3]. Clearly, the cross-spectral density of the field generated can also be given such a natural mode decomposition. Hence it is of interest to know what is the relationship between the modal-decomposition of the source and that of the field. We present in this paper a systematic analysis of this problem and apply the results to some specific situations. The process which maps the source distribution in the plane D to the field distribution in the plane D' is taken to be a general linear process, not restricted to free propagation. The analysis of processes where the map is nonunitary (lossy processes) turns out to be qualitatively different from that for lossless processes corresponding to unitary maps.

In Section 2, we summarize the more important results of Wolf's new theory as applicable to planar sources. This helps also to introduce the notation; we follow the notations and definitions of Wolf[3,4] as closely as possible. In Section 3 the problems to be analysed in this paper are laid out and classified. Unitary maps are analysed in Section 4 and it is shown that in this case the natural modes of the field are in simple one-to-one relation to those of the source, and that the expansion coefficients are left unaffected by the map. The propagation of Gaussian Schell-model beams through first order systems is analysed as an illustrative example. It is shown that the first order system produces a one-to-one map on the Gaussian Schell-model fields, the degree of global coherence being an invariant of the map. Further, for every such field there exists a one-parameter group of first order systems which leaves it invariant. And finally, it is shown that our analysis leads to a natural generalization of Kogelnik's abcd-law[14] of coherent Gaussian beams to partially coherent Gaussian Schell-model beams.

Section 5 presents the analysis of maps which are linear but nonunitary. In this case there is no immediate one-to-one relation between the natural modes of the source and those of the field. Two different approaches to handle such situations are presented. In the first approach which applies to cases where the nonunitary map has an inverse a new inner product, which we call the $\star$-inner product, is defined in the field space. It is shown that the component fields produced by the orthonormal natural modes of the source form $\star$-orthonormal modes of the field, the expansion coefficients themselves remaining invariant under the nonunitary map. In the second approach which does not require the map to have an inverse a way to construct the natural modes of the field from those of the source is presented. The results are illustrated by considering the problem of propagation of Gaussian Schell-model beams through Gaussian transparencies. The final Section 6 contains several concluding remarks.

2. SOURCE CROSS-SPECTRAL DENSITY AND ITS COHERENT MODEL DECOMPOSITION

Let $Q(\underline{r};t)$ represent the scalar random analytic signal of a real planar source distribution with $\underline{r}$, t representing the two-dimensional position and time variables. The

source which may be either primary or secondary is assumed to occupy a region in the source plane D. It is further assumed that we have a steady source in the sense that the statistical ensemble which characterizes the source is (time-) stationary, and is of zero mean. Let

$$\Gamma_Q(\underline{r}_1, \underline{r}_2; \tau) = \langle Q(\underline{r}_1; t - \tfrac{\tau}{2})\, Q^*(\underline{r}_2; t + \tfrac{\tau}{2}) \rangle_t \qquad 2.1$$

represent the cross-correlation function of the source distribution. Here the sharp brackets denote the ensemble average and the subscript t indicates that the averaging is over an ensemble of time-dependent realizations[3]. (Note that if the source is a secondary one, then the analytic signal Γ_Q will be the mutual coherence function of the field distribution in the plane D). If we assume that

$$\int_{-\infty}^{\infty} |\Gamma_Q(\underline{r}_1, \underline{r}_2; \tau)|\, d\tau < \infty, \quad \forall\, \underline{r}_1, \underline{r}_2 \in D, \qquad 2.2$$

then we can define the cross-spectral density (the cross-power spectrum) through the invertible Fourier transform relation

$$W_Q(\underline{r}_1, \underline{r}_2; \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma_Q(\underline{r}_1, \underline{r}_2; \tau)\, e^{i\omega\tau}\, d\tau. \qquad 2.3$$

Its diagonal element

$$S_Q(\underline{r}; \omega) \equiv W_Q(\underline{r}, \underline{r}; \omega) \qquad 2.4$$

is the spectral density (or the spectrum) of $Q(\underline{r}; t)$ at $\underline{r}$.

By definition $W_Q(\underline{r}_1, \underline{r}_2; \omega)$ is hermitian and nonnegative definite. That is

$$W_Q(\underline{r}_2, \underline{r}_1; \omega) = W_Q^*(\underline{r}_1, \underline{r}_2; \omega), \qquad 2.5$$

and

$$\iint_{D\,D} W_Q(\underline{r}_1, \underline{r}_2; \omega)\, f^*(\underline{r}_1)\, f(\underline{r}_2)\, d^2r_1\, d^2r_2 \geq 0, \qquad 2.6$$

for every function $f(\underline{r})$ square integrable in the plane D. Thus, the underlying Hilbert space of interest is $\mathcal{H} = L^2(\mathbb{R}^2)$, the space of functions square integrable over a plane. It follows from (2.5) and (2.6) that $W_Q(\underline{r}_1, \underline{r}_2; \omega)$ is the kernel of an operator which is hermitian and positive semidefinite in $\mathcal{H}$. To these properties of $W_Q(\underline{r}_1, \underline{r}_2; \omega)$ we append the assumption that it is continuous and satisfies the Hilbert-Schmidt property

$$\iint_{D\,D} |W_Q(\underline{r}_1, \underline{r}_2; \omega)|^2 d^2r_1\, d^2r_2 < \infty . \qquad 2.7$$

If, $W_Q = 0$ outside a finite region in D, then (2.7) is satisfied by every bounded cross-spectral density.

In view of (2.5) - (2.7) it follows from Mercer's theorem[15] that $W_Q(\underline{r}_1, \underline{r}_2; \omega)$ can be written in the canonical form

$$W_Q(\underline{r}_1, \underline{r}_2; \omega) = \sum_n \lambda_n(\omega)\, \varphi_n(\underline{r}_1; \omega)\, \varphi_n^*(\underline{r}_2; \omega), \qquad 2.8$$

the series being absolutely and uniformly convergent[16]. The functions $\varphi_n(\underline{r}; \omega)$ are the eigenfunctions and the coefficients $\lambda_n(\omega)$ the eigenvalues of the homogeneous Fredholm integral equation.

$$\int_D W_Q(\underline{r}_1, \underline{r}_2; \omega)\, \varphi_n(\underline{r}_2; \omega)\, d^2r_2 = \lambda_n(\omega)\, \varphi_n(\underline{r}_1; \omega), \qquad 2.9$$

whose kernel $W_Q(\underline{r}_1, \underline{r}_2; \omega)$ is hermitian, positive semidefinite, and Hilbert-Schmidt. These properties of the kernel imply that the eigenvalues are real nonnegative, and the eigenfunctions are orthonormal:

$$\lambda_n(\omega) \geq 0 , \qquad 2.10$$

$$\int_D \varphi_n^*(\underline{r}; \omega)\, \varphi_m(\underline{r}; \omega)\, d^2r = \delta_{nm} . \qquad 2.11$$

Here n,m are suitable single or multiple indices labelling the eigenfunctions. As one consequence of the orthonormality of the mode functions we deduce that the sum of the coefficients $\lambda_n(\omega)$ equals the trace of the cross-spectral density:

$$\int_D W_Q(\underline{r},\underline{r};\omega)\, d^2r = \sum_n{}' \lambda_n(\omega) . \qquad 2.12$$

Note that in the context of radiometry this trace is, in fact, the total irradiance of the source. In terms of $\lambda_n(\omega)$ the Hilbert-Schmidt property (2.7) reads $\sum_n{}' \lambda_n(\omega)^2 < \infty$.

We may now consider the new random variables

$$U_Q(\underline{r};\omega) = \sum_n a_n(\omega)\, \varphi_n(\underline{r};\omega), \qquad 2.13$$

where the random coefficients $a_n(\omega)$ obey

$$\langle a_m^*(\omega)\, a_n(\omega)\rangle = \lambda_n(\omega)\, \delta_{mn} . \qquad 2.14$$

The ensemble describing the set $a_n(\omega)$ is fixed by the original ensemble in (2.1), the second moment of these random coefficients being fixed by the two-point function. Clearly,

$$W_Q(\underline{r}_1,\underline{r}_2;\omega) = \langle U_Q(\underline{r}_1;\omega)\, U_Q^*(\underline{r}_2;\omega)\rangle_\omega . \qquad 2.15$$

Thus, the ensemble of monochromatic oscillations $\{U_Q(\underline{r};\omega)e^{-i\omega t}\}$ provides a realization of the cross-spectral density $W_Q(\underline{r}_1;\underline{r}_2;\omega)$ of the stationary source as a correlation function in the <u>space-frequency</u> domain. We have again attached a subscript ω to indicate the fact that now the averaging is over an ensemble of frequency-dependent realizations.

Further insight is gained by reexpressing (2.8) in the following way:

$$W_Q(\underline{r}_1,\underline{r}_2;\omega) = \sum_m{}' \lambda_m(\omega)\, W_Q^{(m)}(\underline{r}_1,\underline{r}_2;\omega), \qquad 2.16$$

where

$$W_Q^{(m)}(\underline{r}_1,\underline{r}_2;\omega) = \varphi_m(\underline{r}_1;\omega)\, \varphi_m^*(\underline{r}_2;\omega) . \qquad 2.17$$

From the structure of (2.17) it is clear that $W_Q^{(m)}$, for each m, represents a fully (spatially) coherent source mode. It follows that (2.8) can be viewed as the cross-spectral density of the partially coherent stationary source at frequency ω being given by a

weighted incoherent superposition of certain fully coherent natural modes of oscillation of the source, the mode functions $\varphi_m(\underline{r};\omega)$ and the weight factors $\lambda_m(\omega)$ being determined by the cross-spectral density itself. This modal-decomposition also implies, conversely, that the cross-spectral density at frequency ω will be fully (spatially) coherent if and only if it consists of a single nonzero nondegenerate eigenvalue, i.e.a single natural mode.

3. MODAL DECOMPOSITION AND PROPAGATION PROBLEMS

In this and the following Sections we study the implications of the modal decomposition of a planar stationary source on the field it generates. Let $V(\underline{r};t)$ be the random analytic signal of the field generated in the plane D' by the source $Q(\underline{r};t)$ in the plane D. We assume D' is parallel to D. In a manner similar to the case of the source distribution, we define the cross-correlation function of the field

$$\Gamma_V(\underline{r}_1,\underline{r}_2;\tau) = \langle V(\underline{r}_1;t-\tfrac{\tau}{2})\, V^*(\underline{r}_2;t+\tfrac{\tau}{2})\rangle_t , \qquad 3.1$$

and its cross-spectral density

$$W_V(\underline{r}_1,\underline{r}_2;\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma_V(\underline{r}_1,\underline{r}_2;\tau)\, e^{i\omega t}\, d\tau . \qquad 3.2$$

Here $\underline{r}_1$, $\underline{r}_2$ are position variables in the plane D'. We assume that the map from the source distribution to the field distribution is linear and nondispersive. We will call such a map a <u>propagation process</u> though, most often, this map may <u>not</u> correspond to the <u>free</u> propagation process. For example, we will study in detail situations in which the "source" will be a secondary one (a field distribution) in the input plane D of a first order lens system[17] and the "field" will be the field distribution it generates in the output plane D'. Clearly, in this case the propagation process should be identified with the linear operator which acting on the input field gives the output field; the "source" to "field" map is actually a field to field map. In such situations involving secondary sources there is no sharp distinction between sources and fields - the field in one process

may very well become the (secondary) source for a subsequent process.

From the assumed linearity of the map it follows that the ensemble in (3.1) is essentially the same as the one in (2.1), since each realization Q(.) of the ensemble (2.1) uniquely leads to a definite V(.) in the ensemble (3.1).

Since $W_V(\underline{r}_1, \underline{r}_2; \omega)$ is itself hermitian, positive semidefinite and we assume it to be a Hilbert-Schmidt kernel, it does possess a natural orthonormal modal-decomposition:

$$W_V(\underline{r}_1, \underline{r}_2; \omega) = \sum_n{}' \beta_n(\omega)\, \chi_n(\underline{r}_1; \omega)\, \chi_n^*(\underline{r}_2; \omega),$$

$$\beta_n(\omega) \geq 0,$$

$$\int_{D'} \chi_n(\underline{r}; \omega)\, \chi_m^*(\underline{r}; \omega)\, d^2r = \delta_{nm},$$

$$\int_{D'} W_V(\underline{r}_1, \underline{r}_2; \omega)\, \chi_m(\underline{r}_2; \omega)\, d^2r_2 = \beta_m(\omega)\, \chi_m(\underline{r}_1; \omega). \quad 3.3$$

Then the problem is to find the relationship between this modal-decomposition of the field and the modal-decomposition (2.8) of the source, that is to find the relationship between the sets $\{\varphi_n(\underline{r}; \omega), \lambda_n(\omega)\}$ and $\{\chi_m(\underline{r}; \omega), \beta_m(\omega)\}$.

If the propagation process does not involve stochastic elements, it will correspond to a map which is linear in the source amplitude itself. On the other hand, if it involves stochastic elements as in phase conjugation or propagation through a turbulent medium, then the map associated with it will be <u>linear only at the cross-spectral density level.</u> The point is that, whereas a map linear in the amplitude induces a linear map on the cross-spectral density the converse is not true. The set of all possible maps linear in the cross-spectral density includes the maps linear in the amplitude as a <u>proper</u> subset; the maps which are not included in the smaller set are precisely those processes which involve stochastic elements[18,19]. In the former situation corresponding to maps linear in the amplitude, it turns out that the analysis is sensitive to whether the propagation process is a lossless one corresponding to a unitary operator, or a lossy one. Thus

there are three cases to be distinguished: (i) lossless processes linear in the source amplitude, (ii) lossy processes linear in the source amplitude, and (iii) stochastic processes linear in the source cross-spectral density. We analyse the first two cases in the following Sections and comment on the last in the concluding Section.

4. PROPAGATION THROUGH LOSSLESS DETERMINISTIC SYSTEMS

Let the field distribution $V(\underline{r}';t)$ in the plane D' be related to the source distribution $Q(\underline{r};t)$ in the plane D through a linear operator $\mathcal{L}$:

$$V(\underline{r}';t) = \mathcal{L}[Q(\underline{r};t)], \tag{4.1}$$

where $r \in D$ and $r' \in D'$. Corresponding to the linear process $\mathcal{L}$ there exists a Green's function G such that

$$V(\underline{r}';t) = \int_{-\infty}^{\infty} dt' \int_D d^2r\, G(\underline{r}',\underline{r};t-t')\, Q(\underline{r};t'). \tag{4.2}$$

The Green's function depends on t and t' only through the difference t - t', owing to the assumed nondispersive nature of $\mathcal{L}$. Further, $G(\underline{r}',\underline{r};t-t')=0$ for t - t' < 0. Forming the cross-correlation function Γ_V from (4.2) we find that a stationary Γ_Q is mapped to a stationary Γ_V by a linear deterministic nondispersive process. Further, the cross-spectral densities are related in the following simple manner:

$$W_V(\underline{r}_1',\underline{r}_2';\omega) = \iint_{DD} d^2r_1\, d^2r_2\, G_\omega(\underline{r}_1',r_1) \times W_Q(\underline{r}_1,\underline{r}_2;\omega)\, G_\omega^*(\underline{r}_2',\underline{r}_2), \tag{4.3}$$

where $G_\omega(\underline{r}',\underline{r})$ is the Fourier transform of $G(\underline{r}',\underline{r};t-t')$ with respect to t-t'. Thus the field distribution at a frequency ω is fully determined by the source distribution at the same frequency. This is a consequence of the assumed nondispersive nature of the linear process.

Now assume that the propagation process under consideration is lossless in the sense that the traces of W_Q and W_V are equal:

$$\int_{D'} d^2r' \, W_V(\underline{r}', \underline{r}'; \omega) = \int_{D} d^2r \, W_Q(\underline{r}, \underline{r}; \omega) . \qquad 4.4$$

In terms of the Green's function, this condition reads

$$\int_{D'} G_\omega^*(\underline{r}', \underline{r}_2) \, G_\omega(\underline{r}', \underline{r}_1) \, d^2r' = \delta(\underline{r}_1 - \underline{r}_2) , \qquad 4.5$$

with $\underline{r}_1, \underline{r}_2 \in D$. As an important class of such lossless processes we cite the first order systems[17]. In this case D and D' are respectively the input and output planes. Free propagation without paraxial approximation is not lossless in general. The reason for this can be easily traced to the behaviour of the plane wave spectrum under free propagation. Whereas the homogeneous plane waves propogate in a lossless manner, the situation is different with the evanescent waves[20]. Therefore if one restricts oneself to that class of inputs which are band-limited inthe spacial frequency domain so that there are no evanescent wave components then free propagation, paraxial or otherwise, is indeed lossless. Secondary sources in particular belong to this class if the primary source is sufficiently far from the. input plane of the optical system.

Since we are dealing with stationary sources and nondispersive processes, different frequency components do not get mixed. Hence it is convenient to suppress the label ω , and treat the $\underline{r}_1, \underline{r}_2, \ldots$ variables as matrix indices so that φ_n, χ_m, U_Q etc. become column vectors and W_Q, W_V, G etc. become matrices. For example, we could in place of (2.5), (2.7), (2.15) and (4.3) write

$$W_Q^\dagger = W_Q , \qquad \mathrm{Tr}(W_Q^2) < \infty , \qquad 4.6$$

$$W_Q = \langle U_Q U_Q^\dagger \rangle_\omega , \qquad 4.7$$

$$W_V = G W_Q G^\dagger . \qquad 4.8$$

The statement that the process is lossless is then equivalent to requiring that the matrix G is unitary:

$$G^{\dagger} G = \mathbb{1} . \tag{4.9}$$

Using the modal decomposition (2.8), (2.16) for W_Q in (4.8) we have

$$\begin{aligned} W_V &= \sum_n{}' \lambda_n G W_Q^{(n)} G^{\dagger} = \sum_n{}' \lambda_n W_V^{(n)} , \\ W_V^{(n)} &= \chi_n \chi_n^{\dagger} , \\ \chi_n &= G \varphi_n . \end{aligned} \tag{4.10}$$

That is

$$\begin{aligned} W_V(\underline{r}_1, \underline{r}_2 ; \omega) &= \sum_n{}' \lambda_n(\omega) W_V^{(n)}(\underline{r}_1, \underline{r}_2 ; \omega) , \\ W_V^{(n)}(\underline{r}_1, \underline{r}_2 ; \omega) &= \chi_n(\underline{r}_1 ; \omega) \chi_n^*(\underline{r}_2 ; \omega) , \\ \chi_n(\underline{r} ; \omega) &= \int_D G_\omega(\underline{r}, \underline{r}') \varphi_n(\underline{r}' ; \omega) d^2 r' , \\ \int_{D'} W_V(\underline{r}_1, \underline{r}_2 ; \omega) \chi_n(\underline{r}_2 ; \omega) d^2 r_2 &= \lambda_n(\omega) \chi_n(\underline{r}_1 ; \omega) . \end{aligned} \tag{4.11}$$

Since χ_n's are unitarily related to the φ_n's, orthonormality of the latter implies orthonormality of the former. Hence comparing (4.11) with (3.3) we find that (4.10), (4.11) indeed constitute the modal-decomposition for the field.

We conclude that for lossless processes the modal-expansion of the field is related to that of the source in this simple way: The mode functions $\chi_n(\underline{r}; \omega)$ (eigenfunctions) of the field are unitarily related to the mode functions of the source $\varphi_n(\underline{r}; \omega)$ in a one to one manner as in (4.11) and the expansion coefficients (eigenvalues) $\beta_n(\omega)$ of the field are identical with the expansion coefficients $\lambda_n(\omega)$ of the source.

4.1 Gaussian Schell-model fields and first order systems

As an application of the results developed so far in this Section, we study the

transformation properties of Gaussian Schell-model (GSM) fields under the action of first order systems from the point of view of Wolf's new theory[21]. These fields have played an important role in recent studies in radiometry of partially coherent sources, and several aspects of these fields have been analysed by a number of authors[22]. The field distribution in a transverse plane z = constant is GSM if the intensity distribution and the modulus of the normalized degree of coherence are both Gaussian so that the cross-spectral density in the plane has the form[21]

$$W(\underline{r}_1,\underline{r}_2;\omega) = [I(\underline{r}_1;\omega)]^{1/2}[I(\underline{r}_2;\omega)]^{1/2}\mu(\underline{r}_1-\underline{r}_2;\omega) \times \exp\left[i\frac{k}{2R(\omega)}(\underline{r}_1^2-\underline{r}_2^2)\right], \qquad 4.12$$

where $\underline{r}_1, \underline{r}_2$ are now two dimensional variables in the plane, and the intensity distribution $I(\underline{r};\omega)$ and the modulus of the degree of coherence $\mu(\underline{r}_1-\underline{r}_2;\omega)$ are Gaussian functions:

$$I(\underline{r};\omega) = \frac{2A(\omega)}{\pi\,\sigma_I(\omega)^2}\exp\left[-2\underline{r}^2/\sigma_I(\omega)^2\right],$$

$$\mu(\underline{r}_1-\underline{r}_2;\omega) = \exp\left[-(\underline{r}_1-\underline{r}_2)^2/2\,\sigma_\mu(\omega)^2\right],$$

$$\sigma_I(\omega)^2,\ \sigma_\mu(\omega)^2 > 0,$$

$$-\infty < R(\omega)^{-1} < \infty. \qquad 4.13$$

The quantities $\sigma_I(\omega)$, $\sigma_\mu(\omega)$ and $R(\omega)^{-1}$ are respectively the intensity width, the (transverse) coherence length and the phase curvature of the field. In most discussions the GSM field is defined without this phase curvature. But as we have demonstrated elsewhere[21] it is necessary to include such a phase curvature in any complete description of GSM fields, since even simple transformations like the action of a thin lens or free propagation on a GSM field without phase curvature impacts to it a phase curvature. Further, we have included a normalization factor $[\pi\sigma_I(\omega)^2/2]^{-1}$ in the defining expression for $I(\underline{r};\omega)$ so that $A(\omega)$ becomes the integral of $I(\underline{r};\omega)$ over the plane:

$$A(\omega) = \int_D I(\underline{r};\omega)\, d^2r. \qquad 4.14$$

It is the total irradiance of the field at frequency ω. Since we are interested in analysing the transformation of GSM fields under the action by FOS's this is a natural normalization to do, for with this normalization $A(\omega)$ becomes a physical invariant as will be shown later.

Gori[5], and Starikov and Wolf[8] have obtained the coherent-mode decomposition of GSM fields by solving the integral equation (2.9). We will directly deduce such a decomposition by a slightly different procedure. Let us first consider the one dimensional case wherein we denote the position variable by x. Define a new parameter $\gamma(\omega)$ through

$$\frac{1}{\gamma(\omega)^2} = \frac{1}{\sigma_I(\omega)^2} + \frac{1}{\sigma_\mu(\omega)^2}, \qquad 4.15$$

so that the cross-spectral density (4.12, 13) can be rewritten as (after dropping ω)

$$W(x_1, x_2) = A\left(\frac{2}{\pi\sigma_I^2}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma_I\gamma}\left[\frac{\gamma}{\sigma_I}(x_1+x_2)^2 + \frac{\sigma_I}{\gamma}(x_1-x_2)^2\right]\right\}$$
$$\times \exp\left[i\frac{k}{2R}(x_1^2 - x_2^2)\right]. \qquad 4.16$$

Now consider the identity[23]

$$A(1-t)\sum_{n=0}^{\infty} \frac{\alpha}{\sqrt{\pi}} \frac{t^n}{2^n n!} H_n(\alpha x_1) H_n(\alpha x_2) \exp[-\alpha^2(x_1^2+x_2^2)/2]$$
$$\times \exp\left[i\frac{k}{2R}(x_1^2 - x_2^2)\right]$$
$$= A\left(\frac{1-t}{1+t}\right)^{1/2} \frac{\alpha}{\sqrt{\pi}} \exp\left\{-\frac{\alpha^2}{4}\left[\frac{1-t}{1+t}(x_1+x_2)^2 + \frac{1+t}{1-t}(x_1-x_2)^2\right]\right\}$$
$$\times \exp\left[\frac{k}{2R}(x_1^2 - x_2^2)\right]. \qquad 4.17$$

Here H_n are the Hermite polynomials, and $0 \leq t < 1$. The right hand sides of the last two equations become identical if we identify

$$\alpha = \left(\frac{2}{\sigma_I\gamma}\right)^{1/2},$$

$$\frac{1-t}{1+t} = \frac{\gamma}{\sigma_I} \Longleftrightarrow t = \frac{1-\gamma/\sigma_I}{1+\gamma/\sigma_I}. \qquad 4.18$$

Thus we deduce the coherent-mode decomposition of the one dimensional GSM field:

$$W(x_1, x_2) = \sum_{n=0}^{\infty} \lambda_n \varphi_n(x_1) \varphi_n^*(x_2) ,$$

$$\varphi_n(x) = \left(\frac{\alpha}{\sqrt{\pi}\, n!\, 2^n}\right)^{1/2} H_n(\alpha x) \exp[-\alpha^2 x^2/2] \exp\left[i \frac{k}{2R} x^2\right] ,$$

$$\lambda_n = A(1-t)t^n . \qquad 4.19$$

This result for one degree of freedom can now be easily generalized to our original GSM field (4.12, 13) with two degrees of freedom. Denoting the position vector $\underline{r}$ (in the plane) explicitly in terms of its cartesian components $\underline{r}$ = (x,y) we have

$$W(\underline{r}_1, \underline{r}_2) = W(x_1, y_1, x_2, y_2)$$

$$= \sum_{n_1, n_2 = 0}^{\infty} \lambda_{n_1, n_2} \varphi_{n_1, n_2}(x_1, y_1) \varphi_{n_1, n_2}^*(x_2, y_2) ,$$

$$\varphi_{n_1, n_2}(x, y) = \left(\frac{\alpha^2}{\pi\, n_1!\, n_2!\, 2^{n_1+n_2}}\right)^{1/2} H_{n_1}(\alpha x) H_{n_2}(\alpha y)$$

$$\times \exp[-\alpha^2(x^2+y^2)/2] \exp\left[i \frac{k}{2R}(x^2+y^2)\right] ,$$

$$\lambda_{n_1, n_2} = A(1-t)^2 t^{n_1+n_2} . \qquad 4.20$$

Again α, t are related to the GSM field parameters σ_I, γ (and in turn to σ_I, σ_μ) as in (4.18). The mode decomposition (4.20) is consistent with the results of Gori[5], and Starikov and Wolf[8]. The mode functions $\varphi_{n_1, n_2}(x, y)$ are formally the same as the energy eigenfunctions of a two-dimensional isotropic quantum harmonic oscillator (except for the phase curvature) and they are manifestly orthonormal.

A striking feature of the GSM field is that the modal expansion coefficients (eigenvalues) λ_{n_1, n_2} form a geometric progression. We have

$$\sum_{n_1, n_2} \lambda_{n_1, n_2} = A \qquad 4.21$$

as expected. Another feature to which we will make reference later on is that the eigenvalues are independent of R^{-1} and depend on σ_I and σ_μ only through the ratio

$$\eta = \sigma_\mu / \sigma_I , \qquad 4.22$$

sometimes known as the degree of global coherence[8]. To see this note from (4.15) that

$$\gamma / \sigma_I = \left(\frac{\eta^2}{1+\eta^2} \right)^{1/2}. \qquad 4.23$$

The Hermite-Gaussian mode functions $\varphi_{n_1,n_2}(x, y)$ transform in a simple way under the action of first order systems. A first order system is described by a numerical ray transfer matrix $S \in Sp(2,\mathbb{R}) = SL(2,\mathbb{R})$ acting on the phase space coordinates of the rays[17]:

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1. \qquad 4.24$$

In wave optics the same first order system is described by the Green's function (impulse response function, or point spread function)[17]

$$G_S(\underline{r}, \underline{r}') = \left(\frac{i\,2\pi b}{k}\right)^{-1} \exp\left[i \frac{k}{2b} \left(d\underline{r}^2 - 2\underline{r}\cdot\underline{r}' + a\underline{r}'^2 \right) \right]. \qquad 4.25$$

where b, a and d are the elements of S and $\frac{2\pi}{k}$, k are respectively the wave length and wave number corresponding to the frequency ω under consideration. It is instructive to label the Hermite-Gaussian wave fields $\varphi_{n_1,n_2}(x, y)$ with the parameters α, R occuring on the right hand side of the defining expression (4.20). Equivalently, we can combine α, R into a complex parameter q (the complex radius of curvature) through

$$\frac{1}{q} = \frac{1}{R} + \frac{i\alpha^2}{k}, \qquad 4.26$$

so that $\varphi_{n_1,n_2}(x,y)$ can be labelled by q as $\varphi_{n_1,n_2}(x,y;q)$. One could thus write the mode decomposition of the GSM field as

$$W(x_1, y_1, x_2, y_2) = \sum_{n_1,n_2=0}^{\infty} A(1-t^2)t^{n_1+n_2} \varphi_{n_1,n_2}(x_1,y_1;q)\, \varphi^*_{n_1,n_2}(x_2,y_2;q). \qquad 4.27$$

Under the action of a first order system S the functions $\varphi_{n_1,n_2}(x,y;q)$ transform in the following simple manner:

$$G_S \varphi_{n_1,n_2}(x,y;q) = \varphi_{n_1,n_2}(x,y;q'), \qquad 4.28$$

where

$$q' = \frac{aq+b}{cq+d}. \qquad 4.29$$

This is Kogelnik's abcd-law for coherent (Hermite) Gaussian beams[14].

Let the GSM field (4.27) be in the input plane of a first order system S. The cross-spectral density in the output plane is

$$\begin{aligned} W_{out}(x_1,y_1,x_2,y_2) &= G_S W_{in}(x_1,y_1,x_2,y_2) G_S^{\dagger} \\ &= \sum_{n_1,n_2=0}^{\infty} A(1-t)^2 t^{n_1+n_2} G\varphi_{n_1,n_2}(x_1,y_1;q)\left(G\varphi_{n_1,n_2}(x_2,y_2;q)\right)^* \\ &= \sum_{n_1,n_2=0}^{\infty} A(1-t)^2 t^{n_1+n_2} \varphi_{n_1,n_2}(x_1,y_1;q')\varphi^*_{n_1,n_2}(x_2,y_2;q'), \end{aligned} \qquad 4.30$$

where q' is related to q through the abcd matrix S as in (4.29). This is the modal decomposition for the output field. The eigenvalues of the output field are the same as those of the input field, and its mode functions are unitarily related to the mode functions of the input field.

Several comments are in order now. It is easy to see that there is a one-to-one correspondence between GSM fields and expressions of the form of the right hand side of (4.27) with $0 \le t < 1$ and q complex with $R_e(q) < 0$. Since the transformation (4.29), in view of det S = +1, maps the lower (upper) half of the complex q plane onto itself we deduce that W_{out} in (4.30) is GSM; first order systems produce one-to-one maps on the family of GSM fields. Further, since the modal expansion coefficients are invariant under this transformation and since they are functions only of the degree of global coherence η, we deduce that the degree of global coherence itself is an invariant of the map[24]. Since these coefficients are individually invariant their sum which is the irradiance A is invariant also.

The three-parameter SL(2, $\mathbb{R}$) group of first order systems produce only a two-parameter change on the three-parameter GSM family. This can be traced to the following

property of the transformation (4.29): for every q there exists a corresponding one-parameter sub-group of SL(2, $\mathbb{R}$) transformations which leaves q invariant. In fact an explicit form for this one-parameter group can be written as

$$S(q;\theta) = S^{-1}(q)\, S(\theta)\, S(q), \qquad 4.31$$

where

$$S(q) = \begin{pmatrix} \frac{1}{\sqrt{y}} & -\frac{x}{\sqrt{y}} \\ 0 & \sqrt{y} \end{pmatrix}, \quad x + iy = q, \qquad 4.32$$

and $S(\theta)$ is the rotation matrix. Clearly, we can replace S(q) in (4.31) by $S(\theta')\,S(q)$ for any real value θ' and still get the same one-parameter group. In this manner we could write, for example,

$$S(q;\theta) = [S'(q)]^{-1}\, S(\theta)\, S'(q),$$

$$S'(q) = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}, \quad \alpha - i\beta = q^{-1}. \qquad 4.33$$

To conclude this Section we can summarize the transformation properties of GSM fields on passage through first order systems in the following manner. The GSM field is specified by three real parameters $(\sigma_I, \sigma_\mu, R)$, and under the action of first order system

$$(\sigma_I, \sigma_\mu, R) \longrightarrow (\sigma_I', \sigma_\mu', R). \qquad 4.34$$

Now rearrange the three real parameters into a real parameter $\eta = \sigma_\mu/\sigma_I$ and a complex parameter q defined through

$$\frac{1}{q} = \frac{1}{R} + \frac{2i}{k\sigma_I}\left(\frac{1}{\sigma_I^2} + \frac{1}{\sigma_\mu^2}\right)^{1/2}. \qquad 4.35$$

Then under the action of a first order system S

$$(\eta, q) \longrightarrow \left(\eta, q' = \frac{aq+b}{cq+d}\right). \qquad 4.36$$

Eq.(4.36) completely describes the transformation properties of GSM fields on passing through first order systems. Further it forms a generalization of the usual abcd-law for coherent Gaussian beams[14] to the partially coherent GSM fields[21]. Recall that for the coherent Gaussian beams the complex radius of curvature

$$q = \left(\frac{1}{R} + \frac{2i}{k\sigma_I^2}\right)^{-1} \qquad 4.37$$

transforms according to the abcd-law. We have shown that the GSM beam transformation is also governed by the same law provided we replace in the imaginary part of q^{-1} the factor σ_I^2 by $\sigma_I^2\left[1 + \frac{\sigma_I^2}{\sigma_\mu^2}\right]^{-1/2}$.

5. PROPAGATION THROUGH DETERMINISTIC SYSTEM WITH LOSSES

In the last Section we analysed the situation where the map connecting the source and the field was unitary corresponding to a lossless system. In such a situation the orthonormal natural modes of the field were shown to be related to those of the source in a simple one-to-one manner. In this Section we extend that analysis to dissipative systems.

Again assuming that the map is linear in the source amplitude we have

$$W_V = G\, W_Q\, G^\dagger, \qquad G^\dagger G \neq \mathbb{1}. \qquad 5.1$$

Using the model decomposition (2.8) for W_Q, we can rewrite W_V as

$$W_V = \sum_n \lambda_n \psi_n \psi_n^\dagger, \qquad \psi_n = G\varphi_n, \qquad 5.2$$

where φ_n are the natural modes of the source W_Q. Even though (5.2) has the same form as (4.10), there is an essential difference: φ_n's in (4.10) were orthonormal, but by virtue of the fact that G in the present case is not unitary the ψ_n's in (5.2) are neither orthogonal nor normalized. Further, these ψ_n's are not eigenvectors of W_V:

$$W_V \psi_n \neq (\text{constant}) \cdot \psi_n . \tag{5.3}$$

Thus (5.2) is not the modal decomposition we are looking for. We note immediately that since W_V is a cross-spectral density it has an orthonormal natural mode decomposition in its own right. We could write the model decomposition of W_V as

$$W_V = \sum_m \chi_m \chi_m^{\dagger} \beta_n , \tag{5.4}$$

where χ_m are the orthonormal eigenfunctions of W_V with eigenvalues β_n. But these modes χ_m have no simple and immediate relation to the source modes φ_n, or to the functions ψ_n in (5.2).

To find the relationship between these sets of functions, let $\mathcal{H} = L^2(\mathbb{R}^2)$ be the Hilbert space of functions square integrable over a plane and let $\mathcal{H}'$ be the null space of G in $\mathcal{H}$. Denote by $\mathcal{H}_0$ the complement of $\mathcal{H}'$ in $\mathcal{H}$. Clearly, $\mathcal{H}_0$ is the image of $\mathcal{H}$ under G. Note that because of the nonunitarity of G, it can have nontrivial null eigenvectors and $\mathcal{H}'$ is spanned by such vectors. As a result the field Hilbert space $\mathcal{H}_0$ is in general isomorphic to a proper subspace of the source Hilbert space $\mathcal{H}$.

First consider the restricted case where the map G is such that the null space $\mathcal{H}'$ is empty (We analyse an example of such a situation towards the end of this Section). Then the hermitian matrix $GG^{\dagger}$ is positive definite, and hence

$$g = (GG^{\dagger})^{-1} \tag{5.5}$$

is a valid metric matrix in $\mathcal{H}_0$. We have already noted that while φ_n is an eigenvector of W_Q, ψ_n is not an eigenvector of W_V. Instead of that we have the surprising result

$$W_V g \psi_n = \lambda_n \psi_n . \tag{5.6}$$

To interpret this result introduce a new inner product in $\mathcal{H}_0$ by

$$(\psi_1, \psi_2) \equiv \psi_1^\dagger g \psi_2 , \quad \psi_1, \psi_2 \in \mathcal{H}_0 . \tag{5.7}$$

It is easy to show that this definition satisfies the triangle inequality, positivity etc. necessary to make it a valid inner product. The definition of hermiticity (symmetry) corresponding to this new inner product is no longer the familiar $A^\dagger = A$, but instead

$$A^\star = g^{-1} A^\dagger g = A . \tag{5.8}$$

By this criterian W_V is not hermitian but $W_V g$ is:

$$(W_V g)^\star = g^{-1} (g W_V) g = W_V g . \tag{5.9}$$

From (5.6) we see that ψ_n are the eigenvectors of the $\star$ -hermitian operator $W_V g$; that is ψ_n are the $\star$ -eigenvectors of W_V. By virtue of the orthonormality of φ_n, we see that ψ_n are $\star$ -orthonormal:

$$\begin{aligned} (\psi_n, \psi_m) &= \psi_n^\dagger g \psi_m = \varphi_n^\dagger G^\dagger (G G^\dagger)^{-1} G \varphi_m \\ &= \varphi_n^\dagger \varphi_m = \delta_{nm} . \end{aligned} \tag{5.10}$$

Note that in the absence of losses G is unitary and hence the metric matrix g becomes the identity matrix; our $\star$ -inner product goes over to the conventional inner product, $\star$ -orthonormality, hermiticity becomes orthonormality, hermiticity in the usual sense, and we recover the results of Section 4 for lossless systems. Thus, this point of view can be summarized by saying that the effect of dissipation is to make the metric nontrivial.

There exists another way of relating the orthonormal modes of W_V to those of W_Q which is more general and does not require the map to be invertible. To see this first rewrite (5.2) explicitly in terms of the position variables:

$$W_V(\underline{r}_1, \underline{r}_2) = \sum_n \lambda_n \psi_n(\underline{r}_1) \psi_n^*(\underline{r}_2) . \tag{5.11}$$

Recalling that λ_n are nonnegative, define a matrix ρ in this way:

$$\rho_{mn} = \lambda_m^{1/2} \lambda_n^{1/2} \int_{D'} \psi_m^*(\underline{r}) \psi_n(\underline{r}) d^2r. \qquad 5.12$$

Clearly,

$$\rho^\dagger = \rho,$$
$$\rho \geqslant 0. \qquad 5.13$$

Hence there exists a unitary matrix v diagonalizing ρ so that

$$v \rho v^\dagger = \sigma; \quad \sigma_{\ell\ell'} = \sigma(\ell)\delta_{\ell\ell'}. \qquad 5.14$$

or

$$\sum_{m,n} v_{\ell m} v^*_{\ell' n} \rho_{mn} = \sigma(\ell)\delta_{\ell\ell'},$$
$$\sigma(\ell) \geqslant 0. \qquad 5.15$$

For those values of m for which $\sigma(m) > 0$, define the new functions

$$\chi_m(\underline{r}) = \sigma(m)^{-1/2} \sum_n \lambda_n^{1/2} v^*_{mn} \psi_n(\underline{r}). \qquad 5.16$$

These are orthonormal as a consequence of (5.15):

$$\int_{D'} \chi_m^*(\underline{r}) \chi_n(\underline{r}) d^2r = \delta_{mn}. \qquad 5.17$$

Analogous to ~~(2.13)~~ we can now construct an ensemble

$$U_v(\underline{r}) = \sum_m b_m \chi_m(\underline{r}), \qquad 5.18$$

where we choose the random variables b_m to be such that

$$\langle b_m b_n^* \rangle = \sigma(m)\delta_{mn}. \qquad 5.19$$

Then

$$W_v(\underline{r}_1, \underline{r}_2) = \sum_m \sigma(m) \chi_m(\underline{r}_1) \chi_m^*(\underline{r}_2),$$

$$\int_{D'} W_V(\underline{r}_1, \underline{r}_2)\, \mathcal{X}_m(\underline{r}_2)\, d^2 r_2 = \sigma_m \mathcal{X}_m(\underline{r}_1) , \qquad 5.20$$

which can be verified by substituting for $\mathcal{X}_m(\underline{r})$ from (5.16) and comparing with (5.11). We could also write W_V as an ensemble average in the space-frequency domain by reinstating the frequency variable ω :

$$W_V(\underline{r}_1, \underline{r}_2; \omega) = \langle U_V(\underline{r}_1; \omega)\, U_V^*(\underline{r}_2; \omega) \rangle_\omega . \qquad 5.21$$

Since $\mathcal{X}_m$ are orthonormal and $\sigma(m)$ nonnegative, we conclude that (5.20) is the natural mode decomposition for the cross-spectral density of the field.

As an example of dissipative process let us consider the passage of the GSM field in (4.12, 13) through a Gaussian transparency. In this case D is the input plane immediately before the transparency and D' is the output plane immediately after the transparency. The Gaussian transparency has the amplitude transmittance (Green's function)

$$G(\underline{r}, \underline{r}') = B_0 \exp[-\underline{r}^2/\Delta^2]\, \delta(\underline{r} - \underline{r}') . \qquad 5.22$$

Here $\underline{r}', \underline{r}$ are two-dimensional vectors in the input, output planes respectively.

With an input GSM field with parameters $A, \sigma_I, \sigma_\mu, R$ it follows from (5.1) and (5.22) that the output field is also GSM with parameters

$$\sigma_\mu' = \sigma_\mu , \quad R' = R ;$$
$$\sigma_I' = \sigma_I/(1+F^2)^{1/2} , \quad A' = A B_0^2 (1+F^2)^{-1} ; \qquad 5.23$$

where

$$F = \sigma_I / \Delta . \qquad 5.24$$

Since the transverse coherence length remains unaffected while the intensity width decreases, the degree of global coherence η changes (increases) unlike the case of first order systems. From (5.2) we see that

$$\psi_n = G \varphi_n = B_0 \exp[-\underline{r}^2/\Delta^2]\, \varphi_n(\underline{r}) , \qquad 5.25$$

showing that the ψ_n are not orthonormal owing to the extra Gaussian factor $\exp[-\underline{r}^2/\Delta^2]$. The metric g is easily seen to be

$$g = B_0^{-2} \exp\left[2\underline{r}^2/\Delta^2\right]. \qquad 5.26$$

Clearly, ψ_n are orthonormal with respect to this metric, for g kills the extra Gaussian factor in (5.25) demonstrating that orthonormality of φ_n implies $\star$ -orthonormality of ψ_n.

Since the output field is GSM, its orthonormal mode decomposition can be easily constructed patterned after (4.26). We have

$$W_{out}(\underline{r}_1, \underline{r}_2) = \frac{B_0^2 A}{(1+F^2)} (1-t')^2 \sum_{n_1, n_2 = 0}^{\infty} t'^{\,n_1+n_2} \chi_{n_1,n_2}(\underline{r}_1) \chi^*_{n_1,n_2}(\underline{r}_2),$$

$$\chi_{n_1,n_2}(\underline{r}) = \chi_{n_1,n_2}(x,y) = \left(\frac{\alpha'^2}{\pi n_1! n_2! \, 2^{n_1+n_2}}\right)^{1/2} H_{n_1}(\alpha' x) H_{n_2}(\alpha' y)$$
$$\times \exp\left[-\alpha'^2 (x^2+y^2)/2\right] \exp\left[\frac{ik}{2R}(x^2+y^2)\right],$$

$$\alpha' = \left(\frac{2}{\sigma_I' \gamma'}\right)^{1/2}, \quad t' = \frac{1-\gamma'/\sigma_I'}{1+\gamma'/\sigma_I'}, \quad \gamma' = \frac{\sigma_I' \sigma_\mu}{[\sigma_I'^2 + \sigma_\mu^2]^{1/2}}. \qquad 5.27$$

Note that $\alpha' > \alpha$, and hence the natural modes of the output field are scaled <u>down</u> versions of the input field modes, the scale factor being α/α'. Comparing with (5.20), we see that the eigenvalues $\sigma(m)$ of the output field are

$$\sigma(m) = \sigma(n_1, n_2) = \frac{B_0^2 A}{1+F^2} (1-t')^2 \cdot t'^{\,n_1+n_2}. \qquad 5.28$$

Further, since the input normal modes form a complete orthonormal set it is clear that the normal modes of the output field are unique linear combinations of the normal modes of the input field.

To conclude this Section we note that since $t' < t$, the eigenvalues of the output GSM field as a sequence in n_1, n_2 decrease much faster than those of the input GSM field. As a consequence the effective number of degrees of freedom of

the output GSM field, using for instance Starikov's[9] definition for the same, is less than that of the input field. This is, of course, consistent with our earlier observation that the degree of global coherence of the output GSM field is larger than that for the input field.

6. CONCLUDING REMARKS

We have analysed time-stationary planar sources and the field they generate, under a very general setting, within the framework of Wolf's new theory of partial coherence. In the lossless case the natural modes of the field were shown to be related to those of the source in a one-to-one and invertible way through a unitary kernel. In lossy processes such a relationship was found to have no simple form; we have presented two alternate ways of analysing this problem.

The analogy between the quantum mechanical density matrix and the cross-spectral density of the radiation field is well known. The defining properties of these two quantities are identical except that the trace of the latter is not required to be unity. The present analysis throws further light on this analogy. In particular, we note from (4.19) that the cross-spectral density of GSM field with A = 1 and $R^{-1} = 0$ is identical to the density matrix of a two-dimensional isotropic quantum harmonic oscillator in thermal equilibrium. In fact, comparing with standard expressions for the energy eigenfunctions of harmonic oscillators, we can formally identify α, t in (4.19) with quantities corresponding to the oscillator in the following way:

$$\alpha = (m\omega/\hbar)^{1/2},$$
$$t = \exp[-\hbar\omega/k_B T], \qquad 6.1$$

so that

$$T^{-1} = -\left(\frac{k_B}{\hbar\omega}\right)\log(t). \qquad 6.2$$

Here m, ω are the mass and classical frequency of the oscillatorand T is the tempera-

ture. Thus, the analogy between GSM field and harmonic oscillator leads to a formal relationship between the degree of global coherence and the ratio ω/T. Action of first order systems then correspond to quantum evolution of the density matrix under Hamiltonians which are quadratic in the position and momentum operators. Such an analogy between first order optics and evolution of quantum states under quadratic Hamiltonians can be used to advantage in the problem of squeezed states and in related problems[25].

There are important situations where the mapping process from the source to the field involves stochastic elements. We cite beam propagation in turbulent atmosphere[26] and scattering of partially coherent light beams by liquid crystals[27] as examples. The analogy between cross-spectral density and density matrix suggests that the machinery of dynamical maps[18] developed for problems involving general stochastic evolution of density matrix can be readily applied to these problems. In particular, the stochastic map can always be decomposed into a sum of deterministic maps. If the former was a "completely positive" map[18], then the coefficients of the decomposition are all positive, implying that such a stochastic map can be physically realized in terms of <u>parallel</u> deterministic maps. On the other hand if the stochastic map was not completely positive (and such maps do exist, phase conjugation being an example), it is not clear as to how it can be realized in terms of linear deterministic maps.

The present analysis has been within the framework of the scalar theory. In situations where the polarization aspects can not be ignored, one should deal with correlation tensors[28] in place of correlation functions. Hermiticity, positivity etc., are again valid and hence modal-decomposition is not expected to pose any formal difficulties. The mode functions in this case will be vector modes, forming solutions to the Maxwell system of equations. The recently developed method[29-31] for passing form scalar to vector optics can be utilized in this case. Each vector mode will have a polarization which can be different at different points in space, and the polari-

zation of two different modes can be completely dissimilar. For instance, in place of the Gaussian scalar beam mode corresponding to $n_1 = n_2 = 0$ in (4.20) we will have Gaussian Maxwell beam[32] which will then have a longitudinal and a cross-polarization component[33], in addition to the principal polarization component. We plan to return to a detailed study of these problems in a subsequent paper.

Finally we note that the modal decomposition of the GSM field in Section 4 helped to clarify the action of $SL(2,\mathbb{R})$ first order systems. Here both the GSM field and the first order system were rotationally invariant about the system axis. We have analysed elsewhere[34] using generalized rays a ten-parameter family of generalized anisotropic Gaussian Schell-model fields and the action of the ten-parameter $Sp(4,\mathbb{R})$ first order systems[17] on them. We believe that a coherent mode decomposition of the general anisotropic Gaussian Schell-model field will further clarify the nature of the $Sp(4,\mathbb{R})$ systems and their action of anisotropic Gaussian Schell-model fields.

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