COHERENCE INDICES IN QUANTUM OPTICS*

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During the past few years the theory of partial coherence has been transcribed into a suitable quantum mechancial framework. In classical optics the (normalized) degree of partial coherence between two points P_1 and P_2 had the immediate physical interpretation[1] as the maximum visibility of interference fringes obtained by using ~~narrow~~ pinholes (of suitable size) at the points P_1 and P_2. The transcription to quantum theory preserves this physical interpretation of the (normalized) degree of partial coherence. With more sophisticated optical experiments involving intensity interferometry and photoelectric counting it becomes necessary to consider higher order coherence functions.[2] Without any essential difficulties, by virtue of the optical equivalence theorem,[3] we could consider both the classical and quantum theory at the same time. It would be desirable to introduce the generalization of the (normalized) degree of coherence for these higher order coherence functions. Some work on this problem has been done,[4] but we feel the "degree of higher order optical coherence" discussed in these papers to be not the natural generalization of the visibility index for ~~this~~ the second order coherence function. In the present paper we introduce "coherence indices" for the higher coherence functions which have a direct physical interpretation as "visibility" indices.

The higher coherence functions of interest in intensity interferometry and photoelectric counting are all of even order defined as the expectation value of the normal-ordered product of ν annihilation operators and ν creation operators in the form

$$\Gamma(x_1, \ldots, x_\nu ; y_1, \ldots, y_\nu)$$
$$= \langle A^\dagger(x_1) \ldots A^\dagger(x_\nu) A(y_1) \ldots A(y_\nu) \rangle$$
$$= \mathrm{tr}\left\{ \rho A^\dagger(x_1) \ldots A^\dagger(x_\nu) A(y_1) \ldots A(y_\nu) \right\}.$$

We may denote these functions, for short, as $\Gamma(x;y)$. With x , y considered as matrix indices it is easy to verify that $\Gamma(x\,;\,y)$ is a nonnegative matrix by virtue of the nonnegativity of the density matrix ρ . For convenience we have suppressed the polarization indices for the field amplitudes; otherwise we should include the polarization indices ξ , η along with the space-time indices x , y so that we would denote the higher order coherence functions as $\Gamma(x\xi\,;y\eta)$. In intensity correlations what we are interested in are functions of the kind

$$\iint \Gamma(x\xi\,;\,y\eta)\, D(\underline{x}\,\underline{\xi}\,;\,\underline{y}\,\eta)\, d^3x\, d^3y$$

where $D(\underline{x}\,\underline{\xi}\,,\,\underline{y}\,\eta)$ is a nonnegative function that is characteristic of the detector. In the case of an idealized detector, the efficiency function D may be considered as being a delta function associated with a set of points $\underline{x}$ (and polarization indices ξ). We could then refer to the nonnegative coherence function $\Gamma(x;x) = \Gamma(x_1,\ldots,x_\nu\,;\,x_1,\ldots,x_\nu)$ as the intensity correlation with regard to the points $x = \{x_1,\ldots,x_\nu\}$. We now define the coherence index $S(x\,;\,y)$ by the relation

$$S(x_1,\ldots,x_\nu\,;\,y_1,\ldots,y_\nu) = S(x;y) = \frac{\Gamma(x;y)}{\sqrt{\Gamma(x;x)\;\Gamma(y;y)}}\,.$$

It can be demonstrated (see below) that these coherence indices always satisfy the inequalities

$$0 \leq |S(x;y)| \leq 1$$

and that the upper limit is obtained only for a very special class of fields. It is important to note that $\varsigma(x\,;\,y)$ so defined has the physical interpretation as a visibility index in the following sense: the highest value of the ratio between the maximum and minimum intensity correlations in an optical field illuminated by 2ν slits at the points $x_1, \cdots, x_\nu\,; y_1, \cdots, y_\nu$ is given by

$$\frac{1 + |\varsigma(x\,;\,y)|}{1 - |\varsigma(x\,;\,y)|} \quad .$$

The optical arrangement for achieving this maximum "visibility"[5] in the general case would involve beam mixing, but in the important case of $|\varsigma(x\,;\,y)| = 1$ this can be achieved without any beam mixing. It is also worthwhile to note that for the special case of $\nu = 1$, the coherence index coincides with the standard definition of the normalized degree of partial coherence.

To prove the inequalities satisfied by the coherence function it is sufficient to note that

$$\langle \{A^\dagger(x) - \lambda K(x;y) A^\dagger(y)\} \{A(x) - \lambda K^*(x;y) A(y)\} \rangle \geq 0$$

where we have written

$$A(x_1) \cdots A(x_\nu) = \mathcal{A}(x)$$

so that

$$A^\dagger(x_1) \cdots A^\dagger(x_\nu) = \mathcal{A}^\dagger(x)$$

and $K(x\,;\,y)$ is any function of its 2ν variables and λ is a real parameter. This can be rewritten in the form

$$\Gamma(x;x) - \lambda \{K(x;y)\Gamma(y;x) + K^*(x;y)\Gamma(x;y)\} + \lambda^2 |K(x;y)|^2 \Gamma(y;y) \geq 0 \quad .$$

By choosing

$$K(x;y) = \sqrt{\frac{\Gamma(x;y)}{\Gamma(y;x)}} = \frac{\Gamma(x;y)}{|\Gamma(x;y)|}$$

we obtain, in particular,

$$\Gamma(x;x) - 2\lambda |\Gamma(x;y)| + \lambda^2 \Gamma(y;y) \geq 0$$

for all real λ. This requires that the discriminant of the quadratic form be nonpositive so that

$$|S(x;y)| = \frac{|\Gamma(x;y)|}{\sqrt{\Gamma(x;x)\,\Gamma(y;y)}} \leq 1.$$

In general the coherence index (of rank ν) so defined will depend explicitly on its arguements and will vary as these vary. The upper limit to the absolute value of the coherence index is unity; it is of particular interest to consider wave fields where the coherence index (of rank ν) is unimodular for all sets of points. The simplest such field is a wavefield with excitation in only one single (arbitrary but fixed) mode with mode function u(x). In this case we have

$$\Gamma(x;y) = u^*(x_1) \cdots u^*(x_\nu)\, u(y_1) \cdots u(y_\nu) \langle (a^\dagger)^\nu a^\nu \rangle$$

where a, a^+ are the annihilation and creation operators for this mode. In this case,

$$S(x;y) = \left\{ \frac{u^*(x_1)}{u(x_1)} \cdots \frac{u^*(x_\nu)}{u(x_\nu)} \frac{u(y_1)}{u^*(y_1)} \cdots \frac{u(y_\nu)}{u^*(y_\nu)} \right\}^{\frac{1}{2}}$$

which is unimodular for all values of x , y. There is however no guarantee that the general unimodular case corresponds only to such a single mode excitation; it will turn out however that except for the admixture of states with less than ν photons, these are the only states.

Let us consider the optical wavefields with $\varsigma(x\,;\,y)$ unimodular for all x , y. We wish to determine the most general density matrix consistent with this condition. Consider the relation

$$\Gamma(x;y) = \varsigma(x;y)\left\{\Gamma(x;x)\,\Gamma(y;y)\right\}^{\frac{1}{2}}$$

which for $|\varsigma(x\,;\,y)| = 1$ implies

$$\mathrm{tr}\left(\left\{A(x) - \frac{\Gamma(y;x)}{\Gamma(y;y)}A(y)\right\}\rho\left\{A^{\dagger}(x) - \frac{\Gamma(x;y)}{\Gamma(y;y)}A^{\dagger}(y)\right\}\right) = 0.$$

If ρ has the eigenvector decomposition

$$\rho = \sum_{\beta} \lambda_{\beta}\,\psi_{\beta}\,\psi_{\beta}^{\dagger}.$$

then it follows that each of the state vectors ψ_{β} must satisfy the relation

$$\left\{A(x) - \frac{\Gamma(y;x)}{\Gamma(y;y)}A(y)\right\}\psi = 0.$$

This together with its adjoint

$$\psi^{\dagger}\left\{A^{\dagger}(x) - \frac{\Gamma(x;y)}{\Gamma(y;y)} A^{\dagger}(y)\right\} = 0$$

implies that

$$\Gamma(x';y) = \frac{\Gamma(x';x)\,\Gamma(x;y)}{\Gamma(x;x)} .$$

In addition the coherence functions must be symmetric under permutation of all the x variables or of all the y variables so that $\Gamma(x, y)$ must factorize[5] in the form

$$\Gamma(x;y) = g_{\nu}\, u^{*}(x_1) \cdots u^{*}(x_{\nu})\, u(y_1) \cdots u(y_{\nu})$$

where g_{ν} is a coefficient[4] so chosen than u(x) can be chosen to be a normalized (mode) function. Thus a unimodular optical field of rank ν implies factorization of the coherence function.

This does not fully specify the density matrix. To complete this specification we consider the special case of the relation satisfied by the state vector ψ in the case when $x_r = y_r$, $2 \leq r \leq \nu$ but $x_1 \neq y_1$. Then we have

$$A(x_2) \cdots A(x_{\nu}) \left\{A(x_1) - \frac{u^{*}(x_1)}{u(y_1)} A(y_1)\right\} \psi = 0.$$

The general solution to this equation consists of a linear combination of all states with the total number of photons ~~(in any mode)~~ less than ν as well as arbitrary states of exictation of the mode with mode function u(x). We thus have (Ω denotes the vacuum state):

$$\psi = N \left\{ f(a^{\dagger}) + P_{\nu-1}(A^{\dagger}) \right\} \Omega$$

all higher ranks, since the "polynomial contribution" to the state does not make any contribution to any of the higher coherence functions but may contribute to the lower coherence functions. If the polynomial is really only of degree $\mu - 1$, $\mu < \nu$ then we could also have unimodular coherence indices for the lower ones up to the rank μ but no lower.

The modifications introduced by the quantum theory as compared with the classical theory can now be easily discerned. In the classical theory the "polynomial contributions" can be ignored and we have then the following result: unimodular coherence index of any rank ν automatically implies, in classical theory, unimodular coherence indices of all ranks. We note also that whenever coherence index of rank ν is unimodular but the index of rank μ is not, then $\mu < \nu$ and the diagonal representation[3] has singularities as high as the $(2\mu)^{th}$ derivative of the delta function or higher.

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