Coherence, Propogation and Fluctuations of Light: A Generalized Ray Approach*

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Radiative transfer theory, wave optics and paraxial optics including Fourier optics can be treated uniformly and without any approximation in terms of generalized pencils of rays. This development is reviewed and the role of the groups SL(2,R), Sp(4,R) for paraxial propogation elaborated. The problem of polarization is studied with special care and correctly treated where explicit use of the Poincaré group is made. Higher order correlations are taken up within the generalized pencil formalism; the differences between quantum and classical wave optics are highlighted. The effects of Bose quantization on ray-ray correlations is calculated and displayed. Many of these developments build on the work by Leonard Mandel and Emil Wolf and were inspired by their work.

1. Introduction

The rectilinear propagation of light with constant speed is the immediate recognition of "a body continuing in its state of uniform motion". To the extent one had a corpusculur theory of light, this would be viewed as a special case of the First Law of Motion It is true that careful examination of the shadow cast by an opaque object showed a diffraction pattern that was most naturally explained in terms of the wave theory of light; but then the diffraction pattern itself in the far (Fraunhofer) zone exhibited "scaling" and had linear dimensions grow with the distance in proportion characteristic of rectilinear propagation. The same scaling obtains for interference patterns as well. We are therefore led to look for light rays that travel in straight lines, but generalized in such a fashion that they could be used to generate interference and diffraction.

Some further necessary property of such generalized light ray realization of a field of illumination may be deduced from the crucial experiment done to discriminate between corpusculur and wave theories. According to the wave picture the speed of propagation in our optically denser medium is less than that in an optically rarer medium For a point particle the situation would be the opposite Experiments done by Fizeau and Foucault[1] showed that light travelled more slowly in water than in air, and this was taken as evidence ruling out the corpuscular theory. But a more careful analysis shows that an extended object would behave just like the (extended) wave, so as to bend towards the normal in a medium in which it travels more slowly. Thus, the generalized light rays must constitute a pencil with finite extension: single light rays are mere idealizations, and any realization is of a pencil.

When we consider a collection of freely moving particles propagating in space, we would see that each particle is characterized by a position and a momentum i.e by a point in phase space For a large collection of particles we would use with advantage a phase space density. On the other hand if we consider

a wave, the phase is constant on the wavefront which may be locally approximated by a plane but the momentum is normal to this plane. This would correspond to an ordered streamline motion of particles. But if we have particles moving in many directions at each point, we should make many wavefronts at each point adding incoherently. Therefore an assembly of multidirectional pencils is to be viewed in correspondence with an ensemble of waves.

Whether we deal with an assembly of randomly moving free particles or an ensemble of waves propagating freely, the propagation over distances large compared to the original assembly tends to select those which move in the particular direction of propagation. Given a ball of gas of molecules in random free motion, after a large distance of propagation there is a definite correlation between the direction of propagation and the position of individual particles, the subset selected in this manner develops streamline motion. This must obtain for collections of light rays also: after propagation over distances large compared with the primary source dimensions the light rays appear to be practically parallel at least over small areas. In relation to ensembles of issues this corresponds to a small group of waves with very similar orientations being selected, so that over small areas of illumination one finds coherence. This is the essential content of the van Cittert-Zernike theorem on the development of coherence by propagation.

The problem of radiative transfer has had a long history. For the sun as well as for other stars it is clear that the energy that eventually emerges as radiation is generated by nuclear and gravitational processes in the interiors and is transmitted through material media. In the process there is a spatial redistribution of energy, momentum and angular momentum. Once the radiation leaves the star this redistribution is continuous. At very large distances from the sources simple radiometric and photometric laws will be expected to hold for the radiant intensity. But the propagation involves more than the intensity

alone, and the additional correlations involved also propagate by laws that could be derived and displayed While many celebrated authors including Kirchhoff, Planck, Schwarzschild, Milne and Chandrasekhar have contributed to this field[2], much of the modern work was initiated and inspired by Emil Wolf[3] whom we celebrate on this occasion

In a wave theory the primary quantity is the wave amplitude which as a relatively simple equation of propagation. However, in the case of radiative transfer (at optical frequencies) the amplitude itself is not a measurable quantity but belongs to a statistical ensemble. The measurable quantities are the expectation (mean) values of product of the amplitudes. For most cases of interest the mean of the amplitude vanishes, and the relevant quantity is the mean valued of the bilinear quantity made up of the product of the field amplitude and its hermitian adjoint This quantity is variously called the two-point correlation function or the mutual intensity $\Gamma(x,y)$:

$$\Gamma(x,y) = \langle \phi^{\dagger}(x)\ \phi(y) \rangle \qquad 1.1$$

It obeys the wave equation

$$\left\{\frac{\partial^2}{\partial x_0^2} - \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\right\} \Gamma(x,y)$$

$$= \left(\frac{\partial^2}{\partial y_0^2} - \left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2}\right)\right) \Gamma(x,y) = 0, \qquad 1.2$$

in regions free of sources. On the other hand by identifying x with y, $\Gamma(x,Y)$ is the specific intensity of illumination at the point $\underline{x}$ at time x_0.

Even though we have written a scalar wave equation for simplicity, we could equally well write Maxwell equations for the tensor two-point correlation function

$$\Gamma_{\alpha\beta,\gamma\delta}(x,y) = \langle F^{\dagger}_{\alpha\beta}(x)\, F_{\gamma\delta}(y) \rangle, \qquad 1.3$$

which will satisfy Maxwell's equations with respect to either set of variables. We will then have a complete description of the correlation behaviour including the polarization aspects, at the two-point level.

In the study of the coherence, propagation and fluctuations of light considerable simplification obtains when we restrict our attention to "analytic signals" which are positive frequency parts of $\Phi(x)$ or $F_{\alpha\beta}(x)$ and their complex (hermitian) conjugates. In quantum field theory this choice is obvious since it corresponds to restriction to the annihilation part of the field; and equally well if the field quantity is thought of as the Schrodinger wave function (apart from normalization!) the time dependence should be restricted to positive frequency. In classical theory the arguments are much more subtle: one could appeal to the elimination of the redundancy of both signs of frequency, or the need for first order equations so that the two-point correlation at one time defines it for all times, or to the positivity of the classical "photon density". It is remarkable that the idea of analytic signals introduced by D. Grabor[4] and developed by E. Wolf[5] in the classical context was precisely the natural framework for quantum optics as developed by R. J. Glauber[6] and by E.C.G. Sudarshan[7] to create the quantum theory of partial coherence in particular and quantum optics in general. This theory has the immediate result [called the Second Fundamental Theorem of Quantum Optics][8] that the equations of motion and propagation laws of quantum optics in free space or homogeneous media is the same

as in classical optics but the initial specification of the ensemble density exhibits a more subtle positivity than simple classical pointwise positivity.

According to the First Fundamental Theorem of quantum optics[9], both classical and quantum descriptions lead to identical predictions at the two-point function level. However, the situation becomes different when one considers phenomena involving higher order correlations. A well known early application of the four-point functions occurs in the intensity interferometry of Hanbury-Brown and Twiss[10]. Several inequalities which will be expected to be valid for classical higher order correlation functions are violated by quantum correlation functions[11]. Much of modern interest in quantum optics has centered around nonclassical behaviours like antibunching, squeezing and sub-poissonian statistics and the nonlinear phenomena which give rise to these[12]. Pioneering contributions, both theoretical and experimental, have been made in these areas by Leonard Mandel whom we celebrate today[13].

The above considerations regarding the coherence, propagation and fluctuations of light raise the following important questions:

1. Can the phenomenological radiative transfer theory built on the notion of pencils of light rays be traced to the fundamental electromagnetic theory? Can the notion of light pencils be generalized to obtain an exact ray picture of light fields including the associated interference and diffraction behaviour?

2. It has been known for long that the symplectic groups Sp(2,R), Sp(4,R) play an important role in first order geometrical optics. It has also been known that the metaplectic groups play an identical role in the context of first order Fourier optics. Can these symplectic metaplectic structures be traced to something more fundamental like the Poincaré invariance of the basic wave equation (Maxwell equations)? What is the framework in which a direct action of the symplectic group, exact within wave optics, obtains?

3. Since electromagnetic excitations are vector waves, any complete description of the correlations must handle both coherence and polarization. It is clear that the usual descriptions of (partial) polarization apply only to plane waves in free space, and not applicable in the presence of even a simple optical system like a thin lens. Can one develop a procedure by which description of the action of lens systems and other optical systems on vector electromagnetic waves be obtained in a manner consistent with Maxwell system of equations? Does the symplectic structure survive such a vectorization procedure?

4. Can the description in terms of pencils of generalized rays be extended to cover higher order correlation functions, both classical and quantum? How can the Bose symmetry be implemented in such a ray picture? How will the nonclassical aspects manifest themselves in this new description?

These are but a few questions: one can raise many more. Much progress has been made in recent years towards obtaining answers to many of these questions. Today as we felicitate Leonard Mandel and Emil Wolf, we are convinced that the most fitting manner in which to honour them on the occasion is to present an outline of these developments since, as will be seen, much of these developments have been inspired by their own works. And we do hope this review will please them.

2. The Connection Between Radiative Transfer Theory and Electrodynamics: Generalized Light Rays.

Radiative transfer theory[2] is built on the phenomenological notion of pencils of rays, and shares several formal similarities with Boltzmann's transport theory. The basic object in this theory is the specific intensity $I_\omega(\underline{r},\hat{s})$ which gives the distribution of the pencils of frequence ω in position and direction, very much like the phase space distribution of particles in the transport theory; all other quantities of interest are derived from this object. The specific intensity can be operationally defined through the expression

$$dE_\omega = I_\omega\ (\underline{r},\hat{s})\ \cos\varphi\ d\sigma\ d\Omega\ d\omega\ dt, \qquad 2.1$$

and represents the radiant energy transported per unit time per unit frequency range per unit solid angle in the direction represented by the unit vector $\hat{s}$ per unit area normal to $\hat{s}$ in the neighbourhood of the point $\underline{r}$. In a medium with a distribution of sources, scatterers and absorbers the specific intensity is governed by the so-called equation of transfer which is an integro-differential equation similar in form to the Boltzmann transport-equation, for free space this equation of transfer reduces to the fact that the light pencils travel in straight lines:

$$\hat{s} \cdot \nabla_{\underline{r}} I_\omega(\underline{r},\hat{s}) = 0 \qquad 2.2$$

Though this theory of light radiation has proved to be successful and extremely useful, it should be noted that wave electrodynamics, quantum or classical, was not taken into account in its formulation. Clearly, since electrodynamics is the fundamental theory of light, to the extent that this

phenomenological theory has been successful it should be derivable from electrodynamics.

The existence of radiative transfer theory on its own, seemingly independent of electrodynamics, had no doubt bothered several researchers in the earlier past. But it is only through the landmark work of Wolf in the mid--seventies that a systematic attempt to clarify its relationship with electrodynamics was initiated.

Implicit in the radiative transfer theory is the assumption that energy associated with different frequencies is independently propagated. In the context of the statistical electromagnetic wave theory, this will imply a time-stationary ensemble[15]. Hence, we must look for a correspondence between time-stationary ensembles in wave theory and radiative transfer theory. Because of stationarity, the analysis for each frequency can be done entirely independently. The analysis in the following should be understood as for one such arbitrary fixed frequency ω, but ω and the associated wavenumber $k = \omega/c$ will often be suppressed.

The specific intensity has the dimension of energy density (except for a factor of c) on the one hand and obeys a propagation equation (the equation of transfer) on the other. The correlation functions and correlation tensors of electrodynamics are precisely such objects. Let $E_j(\underline{r},t)$ and $B_j(\underline{r},t)$, $j = 1,2,3$ be the analytic signals representing the components of the electric and magnetic vectors. Define the two-point function

$$\Gamma(\underline{r}_1,\underline{r}_2;\tau) = \frac{1}{4\pi}\,[\,<\underline{E}^*(\underline{r}_1,t+\tau)\cdot\underline{E}(\underline{r}_2,t)> + \underline{B}^*(\underline{r}_2,t+\tau)\cdot\underline{B}(\underline{r}_2,t)>]\,, \qquad 2.3$$

and its Fourier transform $\Gamma(\underline{r}_1,\underline{r}_2;\omega)$ through

$$\Gamma(\underline{r}_1,\underline{r}_2,\tau) = \int_0^\infty d\omega\, \Gamma(\underline{r}_1,\underline{r}_2;\omega)e^{-i\omega\tau}. \qquad 2.4$$

By virtue of the fact that $\underline{E}(\underline{r},t)$ and $\underline{B}(\underline{r},t)$ obey the wave equation, the cross-spectral density $\Gamma(\underline{r}_1,\underline{r}_2,\omega)$ obeys the Helmholtz equation in both $\underline{r}_1$ and $\underline{r}_2$:

$$(\nabla_1^2 + k^2)\,\Gamma(\underline{r}_1,\underline{r}_2;\omega) = 0\ ,$$

$$(\nabla_2^2 + k^2)\,\Gamma(\underline{r}_1,\underline{r}_2;\omega) = 0\ . \qquad 2.5$$

By construction $\Gamma(\underline{r}_1,\underline{r}_2;\omega)$ is hermitian, i.e. $\Gamma(\underline{r}_2,\ \underline{r}_1;\omega) = \Gamma^*(\underline{r}_1,\ \underline{r}_2;\omega)$, and positive semidefinite. Its diagonal elements $\Gamma(\underline{r},\underline{r};\omega)$ give the energy density.

A moment's reflection based on these considerations convinces one that the specific intensity, if it is compatible with electrodynamics, should be a linear functional of $\Gamma(\underline{r}_1,\underline{r}_2;\omega)$. Wolf's original choice[3] corresponds to the linear functional

$$Q_\omega(\underline{r},\hat{s}) = \int \Gamma(\underline{r},\underline{r}+\underline{r}';\omega)e^{-ik\hat{s}\cdot\underline{r}'}\, d^3r'. \qquad 2.6$$

It is clear that $Q_\omega(\underline{r},\hat{s})$ is complex in general. One attractive feature of this linear functional is that it gives a distribution over a five dimensional manifold $(\underline{r},\hat{s})$ as in the phenomenological theory. Yet, it did not meet all the requirements on the phenomenological specific intensity. Wolf made his analysis within both classical and quantum electrodynamics and demonstrated that in both cases one obtains identical results. This, of course, was to be expected in view of the fundamental theorem of quantum optics referred to earlier: this too was pointed out by Wolf (Recall that one is dealing here with phenomena at the level of the two-point functions and no higher order!)

A linear functional which respects the reclilinear propagation condition was subsequently introduced by Sudarshan[16]. This corresponds to the Weyl-Wigner ordering of operators and is given by the Wigner-Moyal transform of the cross-spectral density

$$W_\omega(\underline{r},\underline{p}) = \left(\frac{k}{2\pi}\right)^3 \int d^3\sigma\ e^{-ik\underline{p}\cdot\underline{\sigma}}\ \Gamma(\underline{r} + \frac{1}{2}\ \underline{\sigma}, \underline{r} - \frac{1}{2}\ \sigma,\ \omega). \qquad 2.7$$

Sudarshan[16] who introduced this function $W_\omega(\underline{r},\underline{p})$ named it the Wolf function in honour of Emil Wolf.

The Wolf function is real for all values of $\underline{r}$, $\underline{p}$; this follows from the hermiticity of the cross-spectral density. The Helmholtz equations (2.5) imply the following equations for the Wolf function:

$$\underline{p} \cdot \nabla_{\underline{r}}\ W(\underline{r},\underline{p}) = 0, \qquad 2.8$$

and

$$\frac{1}{4}\ \nabla_r^2\ W(\underline{r},\underline{p}) + k^2(1-|\underline{p}|)^2)\ W(\underline{r},\underline{p}) = 0 \qquad 2.9$$

Rectilinear propagation is ensured by (2.8), but there are two new features now. Unlike the phenomenological specific intensity, $W(\underline{r}, \underline{p})$ takes negative values for some values of $(\underline{r},\underline{p})$. That is, now we have both bright and dark pencils. Further, from (2.9) it is seen that $\underline{p}$ is not a unit vector. Thus, whereas the phenomenological pencils were distributed over a five dimensional manifold $(\underline{r}, \hat{s})$ the Wolf function is a distribution over a six-dimensional region. This phenomenon is called ray-dispersion. It turns out[18] that absence of

ray-dispersion and absence of dark rays are mutually equivalent, and equivalent to the radiation field under consideration being statistically homogeneous.

Thus one is led to the following question: Is there a linear functional of the cross-spectral density which will respect all the requirements on the phenomenological specific intensity? It can be shown[19] that there does not exist one, and consequently the phenomenological theory is not compatible with electrodynamics in a strict sense. Yet, the concept of rectilinear pencil gives such a useful and appealing graphical picture of the propagation process that one would certainly like to have an exact description of wave fields in terms of such pencils, generalized suitably but without sacrificing the rectilinear property.

Since interference phenomena involves addition of light to light producing darkness, it is clear that we cannot retain the pointwise positivity of $I_\omega(\underline{r}, \hat{s})$ if we are to have exact correspondence between wavefields and assemblies of rectilinear pencils. Given this, all the other requirements can be satisfied by taking the Wolf function to represent the density of generalized rays in the six-dimensional phase space $(\underline{r}, \underline{p})$. It should be noted that, even though the Wolf function can assume negative values, any measured intensity corresponds to a suitable phase space integral of $W(\underline{r}, \underline{p})$ and this is always guaranteed to be nonnegative. This is so because $W(\underline{r}, \underline{p})$ meets certain integral positivity condition inherited from and equivalent to the positive semidefiniteness of the cross-spectral density through the Wigner-Moyal transform. That the description in terms of the generalized rays represented by the density $W(\underline{r}, \underline{p})$ is exact follows from the fact that the Wigner-Moyal transform is invertible.

$$\Gamma(\underline{r}_1, \underline{r}_2) = \int d^3p \; W\left(\frac{\underline{r}_1 + \underline{r}_2}{2}, \underline{p}\right) e^{ik\underline{p}\cdot(\underline{r}_1 - \underline{r}_2)} , \qquad 2.10$$

Explicit calculations in the context of Young's double-slit arrangement and

diffraction by a single slit shows that the Wolf function indeed gives an exact generalized ray description of interference and diffraction phenomena[20].

With application to radiometry in mind, the generalized pencils, due to partially coherent planar sources have been computed[21]. It turns out that, as a consequence of the straight line propagation property, the Wolf function exhibits a scaling behaviour[19,21] sufficiently away from the source. This leads to a generalization of the far-zone (Fraunhofer) range criterion to partially coherent sources: naturally, this criterion now involves not only the linear dimension of the source but also the coherence length across the source. Further, a generalization of the van Cittert-Zernike theorem[22] [originally derived for incoherent sources] to partially coherent sources results as a direct consequence of this scaling behaviour[21]. Finally, most of the results on modern radiometry with partially coherent sources can be shown to be simple manifestations of this behaviour[19,21].

Turning now to the Wigner-Moyal transform (2.7) one may be tempted to say that the Wolf function is nothing but a Wigner distribution [except for a real positive multiplicative factor corresponding to the trace of $\Gamma(\underline{r},\underline{r}')$ But there exists a subtle and important difference]. It is true that both the cross-spectoral density and the quantum mechanical density matrix are herm.tian positive semidefinite. In the quantum mechanical case we require unit trace, but any real positive trace is allowed for the cross-spectral density. The Wigner distribution which is by definition the Wigner-Moyal transform of the density matrix is real by virtue of the hermiticity of the density matrix. Since any hermitian positive semidefinite operator of unit trace is a legitimate density operator, it follows that any phase space distribution whose integral over the entire phase space is unity is a Wigner distribution provided it satisfies the integral positive semidefiniteness condition mentioned above in connection with the Wolf function. A Wolf function on the other hand has to satisfy the

additional condition (2.8), (2.9). Since the cross-spectral density has to respect the Helmholtz equations (2.5) [The density matrix on the other hand is purely kinematical, and there is nothing like a Helmholtz equation it has to satisfy!]. Thus, every Wolf function [normalized to unit phase space integral] is a Wigner distribution, but the converse is not true[23].

Motivated by the popular use of smoothed Wigner distributions[24] in quantum mechanics, one may be tempted to "smooth" the Wolf function by convolving it with a standard Wolf function with the aim of getting rid of the dark rays. This, however, is not an attractive proposal. Even in the case of the Wigner distribution, while the convolution of two Wigner distributions is a pointwise nonnegative phase space distribution the result is not a Wigner distribution in general as has been shown recently[25]. In the context of the Wolf function the situation becomes even more restricted. In view of the already noted theorem that absence of dark rays is equivalent to the field being statistically homogeneous and in view of the fact that the Wolf function is necessarily a Wigner distribution [apart from a multiplicative constant, irrelevant for the present context], we find that the convolution of two Wolf functions is a Wolf function only if the result is independent of the position variable r?. Consequently, the only standard Wolf functions which can be used as kernel in the convolution process are the ones which depend only on $\underline{p}$. Such Wolf functions correspond to homogeneous fields, and the result of the convolution will correspond to a homogeneous field for every input Wolf function.

Finally, any description of electromagnetic wave fields should necessarily pay attention to their polarization properties as well. This had indeed been done even in the phenomenological theory[26]: every pencil was assumed to be transversely polarized, and then its state of (partial) polarization was described by four Stokes parameters.

But the generalized rays we have talked so far were defined ignoring the polarization aspect. It is possible to define generalized pencils endowed with polarization[16]. To this end one must use in place of the two-point function (2.3), the two-point electric tensor

$$\Gamma_{ij}(\underline{r}_1,\underline{r}_2;\tau) = \frac{1}{2\pi} < E_i^* (\underline{r}_1, t+\tau) E_j(\underline{r}_2,t)>, \qquad 2.11$$

and then form the associated cross-spectral density tensor $T_{ij}(\underline{r}_1, \underline{r}_2;\omega)$ by Fourier transformation as in (2.4). Thus, one will be led to the Wolf tensor $W_{ij}(\underline{r}, \underline{p})$ in place of the Wolf function $W(\underline{r},\underline{p})$:

$$W_{ij}(\underline{r},\underline{p}) = (\frac{k}{2\pi})^3 \int d^3\sigma\ e^{-ik\underline{p}\cdot\sigma}\ \Gamma_{ij}(\underline{r} + \frac{1}{2}\underline{\sigma},\underline{r} - \frac{1}{2}\underline{\sigma}). \qquad 2.12$$

The state of polarization of the pencil with position $\underline{r}$ and propagating in the direction of $\underline{p}$ is now given by the 3x3 hermitian matrix $W_{ij}(\underline{r},\underline{p})$ [the strength of this pencil is given by the trace of the Wolf tensor evaluated at $\underline{r},\underline{p}$]. It will be seen that whereas the nondispersive rays with $|\underline{p}| = 1$ are transversely polarized, the dispersive rays are not. Combining this result with equivalence of dispension and statistical inhomogeneity, one concludes that the transverse polarization of the phenomenological theory obtains for the generalized pencils only for the statistically homogeneous fields.

By expanding the 3x3 polarization matrix in a complete set of nine hermitian matrices [the SU(3) generators plus identity, for example], it can be seen that a set of nine generalized stokes parameters are needed to specify the state of polarization of the generalized pencils[19]. Further, the Wolf tensor itself satisfies the rectilinear propagation condition (2.8), hence as a generalized

pencil undergoes reclilinear propagation it carries with it, unaltered, the nine generalized Stokes parameters describing its state of polarization[19].

To summarize, it is possible to give an exact ray picture of the electromagnetic light field in terms of generalized light rays. Whether or not one takes the polarization aspect into account, the phenomenological theory obtains only for the statistically homogeneous fields. This should be interpreted as delimiting the validity of the phenomenological theory within the framework of electrodynamics.

3. Generalized Rays and First Order Systems

At optical frequencies it turns out that many problems are paraxial [All beam propagation problems are paraxial by definition]. An important class of paraxial problems goes under the name of first order optics[27]. The method of generalized rays provide an extremely powerful and elegant framework for handling these problems.

In the system approach to optics, the optical system effects a map from the input plane to the output plane. The quantity being mapped is the ray [or more specifically the phase space coordinates of the ray] in geometrical optics, and the field amplitude or the cross-spectral density in wave optics; in the generalized ray method it is the Wolf function. Even free propagation will be conveniently viewed as a map of this kind. We will arrange the coordinate system in such a way that the z-axis is along the axis of the system. Then $\underline{x} = (x,y) = (x_1,x_2)$ forms a coordinate system in any transverse plane.

A first order system is represented by a numerical symplectic matrix s acting on the transverse phase space coordinates of the (geometrical optic) ray arranged into a column

$$\xi = \begin{pmatrix} \underline{x} \\ \underline{s} \end{pmatrix} . \qquad 3.1$$

Here $\underline{s}$ is the transverse part of the momentum vector of the ray: $\underline{x}$, $\underline{s}$ are 2-vectors. The symplectic group is the three-parameter group Sp(2,R) = SL(2,R) in the axisymmetric case, and it is the ten-parameter group Sp(4,R) for systems not invariant under rotation about the system axis. The group Sp(2,R) consists of all 2x2 real matrices S such that detS = +1. But Sp(4,R) consists of all 4x4 real matrices S which obey

$$S^T \eta S = \eta ,$$

$$\eta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} . \qquad 3.2$$

In the case of Sp(2,R) systems it is important to remember that $\underline{x},\underline{s}$ in ξ should be interpreted as single entries so that ξ becomes a 2-element vector.

Free paraxial propagation through distance z, action of thin lens of focal length f, linear magnifier with magnification strength m and scaled Fourier transformers with scale factor c are examples of Sp(2,R) first order systems with ray-transfer matrices.

$$S_P(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \quad , \; S_L(f) = \begin{pmatrix} 1 & 0 \\ -f^{-1} & 1 \end{pmatrix},$$

$$S_M(m) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \quad , \; S_F(c) = \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} . \qquad 3.3$$

Whereas f in $S_L(f)$ can take both positive and negative values, z in $S_P(z)$ can only be positive for physical free propagations. Thus one could wonder whether all elements of Sp(2,R) can be realized by thin lenses separated by free propagation sections. A careful analysis shows that this is so; indeed every Sp(2,R) system, including inverse of free propagation, can be synthesized with at most three thin lenses[28]. The inverse free propagation is an example of SL(2,R) = Sp(2.R) system which only a three lens, and no two lens, configuration can realize. For completeness it should be noted that even in the Sp(4,R) case, every group element can be realized using astigmatic thin lenses separated by

free propagation sections, and one needs only a finite, and in fact a small, number of lenses.

The first order system which acts in geometrical optics through the ray-transfer matrix S belonging to Sp(2,R) or Sp(4,R) as the case may be, acts in wave optics through the corresponding unitary metaplectic operator $\bar{U}(S)$ [the generalized Huyghens integral[2]] in the Hilbert space of field amplitudes. It follows that the metaplectic group is relevant to first order Fourier optics in just the same way as the symplectic group is for geometrical optics. This important fact was first pointed out by Bacry and Cadilhac[30], and has been further elaborated by Nazarathy and Shamir[31]. In simple terms, the metaplectic group is a unitary group which is a double cover of the symplectic group since $\bar{U}(S)$ and $-\bar{U}(S)$ correspond to the same S[30].

We are interested in time-stationary ensembles, and hence the field in a transverse plane is best described by the cross-spectral density $\Gamma(\underline{x},\underline{x}')$ where, as usual, we have suppressed ω. For the plane-to-plane-map system problems it is convenient to consider the restricted Wolf function $W(\underline{x},,\underline{s})$ which gives the distribution of generalized rays in the four-dimensional transverse phase space, rather than in the full phase space[32]:

$$W(\underline{x},\underline{s}) = \left(\frac{k}{2\pi}\right)^2 \int d^2x' e^{-ik\underline{s}\cdot\underline{x}'} \Gamma\left(\underline{x} + \frac{1}{2}\underline{x}',\ \underline{x} - \frac{1}{2}\underline{x}'\right) . \qquad 3.4$$

One immediately sees that the total irradiance A in the transverse plane is given by the integral of the Wolf function over the entire transverse phase space:

$$\begin{aligned} A &= \int \Gamma(\underline{x},\underline{x}) d^2x = \int d^2x d^2s W(\underline{x},\underline{s}) \\ &= \int d^4\xi W(\xi), \end{aligned} \qquad 3.5$$

where we have combined $\underline{x}$ and $\underline{s}$ into a column ξ as in (3.1) and, with an abuse of notation, written $W(\underline{x},\underline{s}) = W(\xi)$.

It is important to note that $\Gamma(\underline{x},\underline{x}')$ with $\underline{x}$, $\underline{x}'$ two-vectors in the transverse plane is hermitian positive semidefinite. Any function of $\underline{x},\underline{x}'$ possessing these two properties is a valid cross-spectral density over a plane: it does not have to satisfy any further condition like the Helmholtz equation, which it had to in the full three dimensional case. This $\Gamma(\underline{x},\underline{x}')$ is equivalent to the configuration space density matrix of a system with two degrees of freedom, except for a trivial trace normalization factor. It follows that, but for this trivial real multiplicative constant, the Wolf function in this context is identical to a Wigner distribution. Thus, in the context of $\Gamma(\underline{x},\underline{x}')$ over a plane the terms Wolf function and Wigner distribution can be used interchangeably.

When the first order system S acts on the field amplitude through the unitary metaplectic operator $\bar{U}(S)$ transforming the cross-spectral density

$$\Gamma \rightarrow \Gamma' = \bar{U}(S)\ \Gamma\ \bar{U}(S) \quad , \tag{3.6}$$

the Wolf function transforms in the following simple way[32,33]:

$$W(\xi) \rightarrow W'(\xi) = W(S^{-1}\xi). \tag{3.7}$$

The output Wolf function evaluated at the phase space point ξ is the same as the input Wolf function evaluated at $S^{-1}\xi$. Thus, the generalized rays in the first order system follow the same trajectories as the rays of geometrical optics:

$$\xi \rightarrow \xi' = S\xi. \tag{3.8}$$

Since detS=1, we see from (3.5) and (3.7) that the total irradiance is conserved

under first order action. This is consistent with the fact that first order systems act unitarily on the field amplitude. It is important to realize that, whereas S acts on the cross-spectral density through $\bar{U}(S)$ which is an integral transform, it is on the Wolf function that its action, exact within wave optics, is most direct.

The power of the Wigner distribution approach is best illustrated by applying it to the study of the passage of Gaussian Schell-model (GSM) beams[34] through first order systems. These beams have played an important role in the modern study of radiometry of partially coherent planar sources which was started by Walther[35], and extensively developed by the schools of Wolf[36], Baltes[37] and Gori[32]. The cross-spectral density of a GSM beam in a transverse plane has the form[32]

$$\Gamma(\underline{x},\underline{x}') = A\cdot\frac{2}{\pi}\cdot\frac{1}{\sigma_I^2}\exp\left[-\frac{\underline{x}^2+\underline{x}'^2}{\sigma_I^2}-\frac{1}{2}\frac{(\underline{x}-\underline{x}')^2}{\sigma_g^2}\right]$$

$$\cdot\exp\left[-\frac{ik}{2R}(\underline{x}^2-\underline{x}'^2)\right]. \qquad 3.9$$

Here A, σ_I, σ_g and R are respectively the total irradiance, beam width, transverse coherence length and radius of phase curvature of this rotationally invariant beam. It is clear that these beams are generalizations of the coherent Gaussian beams to accommodate partial coherence. The Wolf function for GSM beam is easily computed to be

$$W(\xi) = \frac{A}{\pi^2} \det G \exp(-\xi^T G \xi) ,$$

$$G = \begin{pmatrix} \frac{2}{\sigma_I^2} + \frac{k^2\gamma^2}{2R^2} & \frac{-k^2\varphi^2}{2R} \\ \frac{-k^2\gamma^2}{2R} & \frac{k^2\gamma^2}{2} \end{pmatrix} , \qquad 3.10$$

where we have found it convenient to define a new parameter γ through

$$\frac{1}{\gamma^2} = \frac{1}{\sigma_I^2} + \frac{1}{\sigma_g^2} . \qquad 3.11$$

It is understood that in the exponent of (3.10), ξ is treated as a two-element column with $\underline{x}$ and $\underline{s}$ treated as single entries. One finds that there is a one-to-one correspondence between the three-parameter family (the invariant A is not counted) of GSM beams and 2x2 real matrices G satisfying

$$G^T = G,$$

$$G > 0,$$

$$0 < \det G \leq k^2 . \qquad 3.12$$

It is further seen that $\det G = k^2\gamma^2/\sigma_I^2$ is related to the degree of global coherence σ_g/σ_I in an invertable way.

A first order system S acting on a GSM beam transforms it as follows, as can be seen from (3.7) and (3.10):

$$W(Q) = \frac{A}{\pi^2} \det G \exp(-\xi^T G\xi) \rightarrow \frac{1}{\pi^2} \det G' \exp(-\xi^T G'\xi),$$

$$G' = (S^{-1})^T G S^{-1}. \qquad 3.13$$

Since G' satisfies all the requirements (3.12) on the GSM parameter matrix, we see that first order action produces a one-to-one map on the GSM family, and that the degree of global coherence is an invariant of this map [since detG is]. These considerations suggest that we can, with advantage, label the GSM fields with the matrix G. Then, under action of S, we have

$$W_G(\xi) \rightarrow W_{G'}(\xi) ,$$

$$G' = (S^{-1})^T G S^{-1} . \qquad 3.14$$

The last equation gives a concise description of the transformation of GSM beams under first order action. There are two beautiful ways of looking at this result: we shall outline both[32].

First, the symmetric matrix G can be expanded in terms of the Pauli matrices σ_1, σ_3 and the unit matrix:

$$G = \alpha_o 1 + \alpha_1\sigma_1 + \alpha_2\sigma_3 . \qquad 3.15$$

The fact that detG is an invariant reads, in terms of the real parameters α_o, α_1, α_2 as

$$\alpha_o^2 - \alpha_1^2 - \alpha_2^2 = \text{invariant}. \qquad 3.16$$

It follows that GSM beams can be represented as vectors in a 2+1 dimensional fictitious Minkowski space, the first order systems acting as Lorentz transformations in this space. This geometric picture immediately shows that

under first order action the three-parameter family of GSM fields divide into a continuous family of hyperboloids, each invariant under all Sp(2,R) first order if and only if their representative points are on the same hyperboloid. It is also seen that for every GSM field there is a one-parameter group of first order systems which leaves it invariant. Clearly, these are Lorentz transformations about the Lorentz vector representing the GSM field under consideration.

Secondly, the transformation law (3.14) for the GSM parameter matrix G implies the following transformation for the symmetric variance matrix $V \equiv G^{-1}$:

$$V \to V' = SVS^T. \tag{3.17}$$

In terms of the GSM field parameters, V reads

$$V = \begin{pmatrix} \frac{\sigma_I^2}{2} & \frac{\sigma_I^2}{2R} \\ \\ \frac{\sigma_I^2}{2R} & \frac{2}{k^2\gamma^2} + \frac{\sigma_I^2}{2R^2} \end{pmatrix}. \tag{3.18}$$

Let

$$\det V = \sigma_I^2/(k^2\gamma^2) \equiv \Omega^2, \tag{3.19}$$

with $\Omega > 0$. Since any $S \in Sp(2,R)$ satisfies

$$S^T\eta S = \eta, \quad \eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.20}$$

we can rewrite (3.17) in the following form:

$$V' + i\Omega\eta = S(V + i\Omega\eta)\, S^T \qquad 3.21$$

By the definition of Ω, $V' + i\Omega\eta$ and $V + i\Omega\eta$ are both singular and hence each of these hermitian matrices can be written as the outer product of a complex vector. We have

$$\begin{aligned} V + i\Omega\eta &= YY^\dagger \\ Y &= e^{i\theta}\sqrt{V_{22}}\begin{pmatrix} q \\ 1 \end{pmatrix} \\ q &= Y_1/Y_2 = (V_{12} + i\Omega)/V_{22} \\ q^{-1} &= (V_{12} - i\Omega)/V_{11} \end{aligned} \qquad 3.22$$

where Θ is real arbitrary, and similarly for $V' + i\Omega\eta$. Thus, the equation of transformation (3.17) for the variance matrix becomes

$$Y'Y'^\dagger = SYY^\dagger S^T. \qquad 3.23$$

Since S is real, this equation is seen to imply

$$Y' = SYe^{i\varphi}, \qquad 3.24$$

where φ is an arbitrary phase. Writing out this two-vector equation as two scalar equations, and dividing one by the other we get

$$q' = (aq + b)/(cq + d). \qquad 3.25$$

One immediately sees that this is Kogelnik's abcd-law[39], but now generalized to the partially coherent GSM fields. From the last of the equations in (3.22) we have

$$q^{-1} = \frac{1}{R} - i \frac{2}{k\gamma\sigma_I} \quad . \tag{3.26}$$

The complex radius of curvature q, which transforms according to the generalized abcd-law (3.25), indeed goes over to the conventional complex radius of curvature of coherent Gaussian beams in the coherent limit $\sigma_g \to \infty$ and hence $\gamma \to \sigma_I$: our abcd-law (3.26) is a generalization of Kogelnik's law.

It turns out that a grand generalization of the abcd-law in a form which will apply to <u>every</u> partially coherent beam can be derived[40]. To this end let us define the variance matrix V in the general case as follows: $V_{11} \equiv < \underline{x}^2 >$, a measure of the square of the beam width; $V_{22} \equiv < \underline{s}^2 >$, a measure of the square of the angular spread; and $V_{12} = V_{21} \equiv < \underline{x} \cdot \underline{s} > = < \underline{s} . \underline{x} >$, the position direction correlation. Here $< \ldots >$ implies phase space average of the quantity inside. It is clear that for the variance matrix V thus defined, we have

$$V = \int d^4\xi (\xi\xi^T)\, W(\xi) \quad . \tag{3.27}$$

For the GSM beams, this new definition (3.27) reproduces the matrix V as (3.28), but (3.27) itself applies to every beam.

Under first order action $W(\xi) \to W(S^{-1}\xi)$, and since det S = 1 it follows from (3.27)

$$V \to V' = SVS^T. \tag{3.28}$$

Since (3.28) has the same form as (3.17) the arguments following (3.19) can be used to write it in the form of a abc-law. We can define using (3.22) a complex quantity

$$Q = (V_{12} + i\Delta)/V_{22} = \langle \underline{x} \cdot \underline{s} \rangle / \langle \underline{s}^2 \rangle + i\,\Delta / \langle \underline{s}^2 \rangle \,. \qquad 3.29$$

Then as the (arbitrary) beam passes through a first order system S, it can be seen that Q transforms as follows

$$Q \to Q' = (aQ + b)/(cQ + d), \qquad 3.30$$

while $\Delta = (\det V)^{1/2}$ remains invariant. Equation (3.30) represents our grand generalization of the abcd-law and applies to every partially coherent beam. For the GSM beams, V goes over to V, Q goes over to q and (3.30) goes over to (3.25). Needless to say, for the special case of coherent Gaussian beams Kogelnik's original abcd-law obtains.

So far we have considered (isotropic) GSM beams and their transformation by Sp(2,R) systems. It is possible to generalize the analysis to general anisotropic Gaussian Schell-model (AGSM) fields and their behaviour under action of Sp(4,R) first order systems, leading to richer results and mathematical structures.

The most general AGSM form consistent with hermiticity is[41]

$$\Gamma(\underline{x}, \underline{x}') = \exp\Big[-\underline{x}^T L\, \underline{x} - \underline{x}' L\, \underline{x}' - \tfrac{1}{2}(\underline{x}-\underline{x}')^T M (\underline{x}-\underline{x}') + \tfrac{i}{2}(\underline{x}-\underline{x}')^T K (\underline{x}+\underline{x}')\Big], \qquad 3.31$$

where the 2x2 matrices L,M,K are real with L,M symmetric. What additional conditions need to be imposed on AGSM parameter matrices L,M,K so that Γ will be positive semidefinite and hence a bonafide cross-spectral density?

Finite irradiance (trace) condition will require L to be positive definite, and Schwarz inequality will require M to be positive semidefinite. But that is not all. While the symmetric part of the phase matrix K has no role to play in

determining the positive semidefiniteness of Γ, the antisymmetric part has a tricky one to play through subtle combination with L, M. A complete solution to this problem is accomplished by going over to the Wolf function[33,41]. We have

$$W(\xi) \doteq \exp(-\xi^T G \xi)$$

$$G = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$$

$$A = 2L + \tfrac{1}{2} K^T (L+M)^{-1} K$$

$$B = \tfrac{1}{2} (L+M)^{-1} \quad ; \quad C = -\tfrac{1}{2} K^T (L+M)^{-1} . \qquad 3.32$$

The conditions $L > 0$, $M \geq 0$ are equivalent to the condition $G > 0$. What additional conditions should be imposed on G to make Γ positive semidefinite or, equivalently, for a Gaussian distribution in a four-dimensional phase space to be a Wigner distribution?

The crucial fact to use is that first order systems act unitarily on Γ. Under action of S we have

$$\Gamma \to \Gamma' = \bar{U}(S)\, \Gamma \bar{U}(S)^\dagger ,$$

$$G \to G' = (S^{-1})^2\, G S^{-1},$$

$$W_G(\xi) \to W_G'(\xi) = W_{G'}(\xi) . \qquad 3.33$$

Thus, Γ is positive semidefinite $\Longleftrightarrow$ Γ' is positive semidefinite. Equivalently, W_G is a Wolf function $\Longleftrightarrow$ $W_{G'}$ is a Wolf function. Now since G is positive definite, a classic theorem due to Williamson[42] guarantees the existence of an $S(G) \in Sp(4,R)$ which will take it to the standard diagonal form G_o through

$$G \longrightarrow G_o = (S(G)^{-1})^T\, G\, S(G)^{-1} , \qquad 3.34$$

$$G_o = \begin{pmatrix} \beta_1 & 0 & & \\ & \beta_2 & & \\ & & \beta_1 & \\ 0 & & & \beta_2 \end{pmatrix} .$$

To compute the values of β_1, β_2 we need not construct S(G) explicitly; they can be determined in terms of the Sp(4,R) invariants of G.

In the diagonal form the positive semidefiniteness requirement is trivial to implement: both β_1, β_2 should be less than or equal to k. Since positive semidefiniteness is unitary invariant, we see that

$$\Gamma \text{ positive semidefinite} \iff \beta_1,\ \beta_2 \leqslant k. \qquad 3.35$$

With η as in (3.2), let the Sp(4,R) invariant traces be $S_1 = -\frac{1}{2}k^{-2}\ \mathrm{tr}(\eta G)^2$ and $S_2 = \frac{1}{2}k^{-4}\ \mathrm{tr}(\eta G)^4$. Then the condition for positive semidefiniteness becomes

$$S_1 \leqslant 2\ ,\quad S_1 - \frac{1}{2}S_1^2 + \frac{1}{2}S_2 \leqslant 1. \qquad 3.36$$

Writing (3.36) explicitly in terms of L,M,K it will be seen that the symmetric part of K does not enter these conditions whereas the antisymmetric part does in an important way[33,43]. In particular, it turns out that if M=0, then (3.36) requires K to be symmetric. That is, an antisymmetric phase factor can not be imparted to an otherwise fully coherent anisotropic Gaussian beam. Finally, (3.36) and the positive semidefiniteness of G are inequalities on the real symmetric 4x4 matrix G. Hence the AGSM fields form a ten-parameter family, three parameters each coming from L and M and four from K. The AGSM beams studied by Li and Wolf[44], and others[45] are subsets of this family.

Just as in the GSM case, it turns out that the action of the ten-parameter group of Sp(4,R) first order systems on the AGSM fields can be described with advantage through a geometric picture[41]. The idea is to use the fact that Sp(4,R) is a double cover of the pseudo-orthogonal group SO(3,2), in just the same way as Sp(2,R) is for SO(2,1). Thus, the AGSM fields can be represented by antisymmetric second rank tensors in a space-time of 3+2 dimensions. Then

Sp(4,R) systems act as de Sitter transformations in this space. A detailed analysis of this picture will be found in Ref. [41]. The structure of the orbits in the Lie algebra of SO(3,2) plays an important role in this analysis[46]. There are two invariants, and these can be related to the degrees of global coherence in two orthogonal transverse directions. Under Sp(4,R) action AGSM fields separate into two families. In Family I the two invariant degrees of global coherence are equal, and fields of this family are related to the GSM fields through Sp(4,R) action. In Family II these are unequal, and as a consequence it is not possible to convert a AGSM field of this family into GSM beam by Sp(4,R) actions.

When AGSM beams are acted on by Sp(4,R) systems the relevant geometric picture is the 2+1 dimensional space[41]. Each AGSM field is represented by three Lorentz vectors and one Lorentz scalar. The optical system again acts as Lorentz transformations. It is clear that now there are seven invariants: the Lorentz scalar, the norm of the three Lorentz vectors, and the three Lorentz inner-products between pairs of these vectors. It turns out that the transformation on the two Lorentz vectors corresponding to the diagonal blocks of the 4x4 symmetric G can be reduced to abcd-laws, but the one corresponding to the off-diagonal block cannot be reduced to an abcd-law in general, since it need not be positive semidefinite.

As a final comment on AGSM fields we note the following: From (3.7) it may be tempting to see if this action on the Wolf function can be extended to the larger group $SL(4,R) \supset Sp(4,R)$. It turns out that for every $S \in SL(4,R)$ but outside of Sp(4,R) there exists an AGSM cross-spectral density which when mapped using (3.7) will lose its positive semidefiniteness[41]. Thus, Sp(4,R) and not SL(4,R) is the group relevant to optics: this, of course, we know from geometrical optics and dynamics.

All the results presented here in connection with AGSM fields are relevant to quantum mechanics as well: in fact in the context of quantum mechanics these results undergo a useful generalization[33]. First order action corresponds in quantum mechanics to evolution under Hamiltonians quadratic in the positions and momenta. AGSM cross-spectral densities correspond to configuration space density matrices of thermal states and their transforms under these Hamiltonians[47]. Consider a quantum mechanical system with n-degrees of freedom evolving under a quadratic Hamiltonian. Then the thermal state density matrix and its transforms under quadratic Hamiltonians are described by density matrices whose configuration space kernel has the form (3.31) with $\underline{x},\underline{x}'$ now n-dimensional vectors and L, M, K known matrices. In the place of Gaussian Wolf functions we will now have Gaussian Wigner distributions in a 2n-dimensional phase space. It is clear that the relevant symplectic group is Sp(2n,R). We may ask: Given a Gaussian phase space distribution in a 2n-dimensional phase space, how to test if it is a bonafide Wigner distribution? The solution to this problem (Littlejohn's problem[48]) is a generalization of the corresponding solution in the n=2 AGSM cases. The Gaussian Wigner distribution is again characterized by a symmetric positive definite 2nx2n matrix G. Williamson's theorem and the unitary action of Sp(2n,R) enable us to take G to its standard diagonal form G = diag. $(\beta_1,\beta_2\cdots\beta_n,\beta_1\beta_2\cdots\beta_n)$, and the solution to Littlejohn's problem reads as follows: the Gaussian phase space distribution is a Wigner distribution if and only if $\beta_j \geq \hbar^{-1}$, j =1,2,...n. These conditions can again be written in terms of the n independent Sp(2n,R) invariant traces of G. The details can be found in Refs. [33,43] and will not be discussed here any further.

The Wolf function (Wigner distribution) method can be used to advantage for a description of Gaussian pure states, of quantum mechanical systems with n degrees of freedom, and their evolution under quadratic Hamiltonians[49]. In the context of classical system optics with n=1,2 this analysis will apply

respectively to isotropic[50] and anisotropic[51] coherent Gaussian laser beams and their transformation on passage through first order optical systems.

In the context of quantum optics it will become clear that it applies to coherent states and squeezed coherent states of n-mode systems and their evolution under quadratic Hamiltonians[52]. A special orbit in the Lie algebra of Sp(2n,R) plays an important role in such an analysis. We state here without proof only the important results. Details of the analysis can be found in Ref. [49].

The normalized Gaussian pure state of a system with n degrees of freedom can be labelled by two nxn real symmetric matrices U, V with U positive definite:

$$\psi_{U,V}(\underline{x}) = \pi^{-n/4}(\det U)^{1/4} \exp\left[-\tfrac{1}{2}\,\underline{x}^{T}(U+iV)\,\underline{x}\right]. \qquad 3.37$$

It is convenient to combine U and V and construct a complex symmetric nxn matrix q such that

$$q^{-1} = -V + iU\,. \qquad 3.38$$

Since the imaginary part of q^{-1} is positive definite, it follows that the imaginary part of q is negative definite.

Clearly, such Gaussian pure states form a n(n+1)-parameter family, n(n+1)/2 parameters each coming from U and V. Under first order symplectic action corresponding to $S\varepsilon$ Sp(2n,R), the Gaussian states are mapped into Gaussian states in a one-to-one manner, and it turns out that this map can be concisely described through

$$q \to q' = (Aq + B)\,(Cq + D)^{-1}, \qquad 3.39$$

where the nxn real matrices A,B,C,D are blocks constituting S:

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} . \qquad 3.40$$

This is recognized as a matrix generalization of the Möbius transformation. Thus, the family of symmetric complex matrices with negative definite imaginary part is invariant under Sp(2n,R) action. The same is true for the family with positive definite imaginary part as can be seen by taking the complex conjugate of (3.39) (remember S is real!) This can be viewed as a generalized matrix form of the invariance of the lower/upper half of the complex plane under Sp(2,R) = SL(2,R) action. Further, (3.39) can be viewed as a generalized Kogelnik matrix ABCD-law on the one hand and as a generalization of a recent result of Bialynicki-Birula[53] on the other.

The special orbit in the Sp(2n,R) Lie algebra relevant to this symplectic dynamics of Gaussian pure states has dimension n(n+1), numerically equal to the number of independent parameters in the Gaussian family. It turns out that V, U^{-1} indeed form a coordinate system for this orbit which is both global and canonical. As a consequence of this important fact, it turns out that the quantum symplectic evolution of Gaussian pure states can be given an elegant classical Poisson bracket description on this special orbit. The details of such a derivation is, however, beyond the scope of this review and will be found in Ref. [49].

To conclude this section we note that even though we have been principally concerned with coherent and partially coherent Gaussian beams and their transformation by first order systems in the context of classical optics, it is clear that the technology presented and the results derived are equally applicable to quantum mechanical pure and mixed states with Gaussian wave functions and density matrices, and their transformation under action of

quadratic Hamiltonians. In particular they are applicable to multimode squeezed states and to the squeezing operation. At the level of single mode system (n=1) this has been shown in Ref. [54], but the validity is general and applies to systems with arbitrary number of modes.

4. Polarization and Front Form Analysis

There have been several methods in use for the description of the state of polarization of the radiation field and its change under the action of optical systems. Fully polarized (pure) states can be described by the Jones vector[55] or by points on the Poincaré sphere[55] whereas the coherency matrix[56] and the Stokes vector[57] can describe both fully polarized and partially polarized (mixed) states. Deterministic systems are described by the complex 2x2 Jones matrices acting on the Jones vector or the coherency matrix, but the real 4x4 Mueller matrix which acts on Stokes vector can handle both deterministic and nondeterministic systems. An interesting question in this connection which has drawn considerable interest recently[58] is the following: Given the Mueller matrix of a system (experimentally measured one, say) how to test if the system was deterministic or otherwise?

Implicit in all these methods is the assumption that the field is polarized transverse to the axis of the system (z = axis) at all points in the input (transverse) plane and that the action of the optical system under consideration is such as to preserve this property. Thus, they are applicable only to plane wave fields and to systems like polarizers and phase plates which map plane waves into plane waves with no change in the wave vector. But one is often interested in finite beam fields (a Gaussian beam, for example) which cannot always be treated as a plane wave, and their transformation by lens systems.

Study of the dependence of the field quantity (over transverse planes) on the transverse coordinates and the mapping of this dependence encompasses virtually the entire systems optics. In these situations one invariably settles down for a scalar field description of the vector (electromagnetic) light field. Thus, while describing polarization the spatial variation is completely ignored and while describing the spatial radiation as in the case of imaging systems the polarization aspect (vector nature) is completely ignored. This is a highly

unsatisfactory situation, and a criticism of this position with particular reference to Gaussian beams can be found in an early paper by Lax, Louisell and McKnight[59].

It cannot be argued that the scalar function used to describe the radiation field in the so-called "scalar optics" represents the amplitude of the field vector polarized in the same direction at all points. The field vectors in the free radiation field have to respect the Maxwell system of equations:

$$\frac{1}{c}\frac{\partial}{\partial t}\underline{E} - \underline{\nabla}\wedge\underline{B} = 0, \qquad 4.1a$$

$$\frac{1}{c}\frac{\partial}{\partial t}\underline{B} + \underline{\nabla}\wedge\underline{E} = 0, \qquad 4.1b$$

$$\underline{\nabla}\cdot\underline{E} = 0, \qquad 4.1c$$

$$\underline{\nabla}\cdot\underline{B} = 0. \qquad 4.1d$$

The first two are true equations of motion for $\underline{E}$, $\underline{B}$ but the last two are equations of constraint to be obeyed at each time. The former preserve the latter in time: if the constraints are obeyed at the initial time, the equations of motion ensure that they are obeyed at all times. Now, $\underline{\nabla}\cdot\underline{E} = 0$ implies that the electric vector cannot have any spatial variation in its direction of polarization. As a consequence, even in a simple case like the Gaussian beam, the electric field cannot be polarized in the same sense at all points. That is, the equations of constraint couple the polarization and the spatial dependence of the field vectors in an intimate way and, therefore, any attempt to analyze or affect one without due regard for the other aspect will at best be inconsistent with the Maxwell system of equations. Thus it is desirable to look for a procedure which will handle polarization and spatial modulation in a unified and systematic manner. It turns out that such a procedure can be derived through an analysis of the paraxial propagation problem in the front form[60,61] of

relativistic description. Such an analysis traces, as a bonus, the symplectic structure of first order optics to the basic Poincare invariance of the Maxwell equations/wave equation.

The most familiar description of the generators of the Poincare (inhomogeneous Lorentz) group $\mathcal{P}$ is in the so-called instant form[60], wherein one uses a complete set of dynamical variables describing physical conditions at all points of the three-dimensional space at a common-time. The generators of $\mathcal{P}$ are then listed as follows[62]: four generators P^{μ} corresponding to space-time translations; three generators $\underline{J} = (J_1, J_2, J_3)$ corresponding to spatial rotations about x_1, x_2, x_3; and three generators $\underline{K} = (K_1, K_2, K_3)$ corresponding to pure Lorentz transformations along the three orthogonal directions.

We have shown recently[63] that for problems of paraxial wave optics a more convenient way to display the action and generators of $\mathcal{P}$ is in the front form.

In place of the usual time and space coordinates $x^o = ct, \underline{x}$ one introduces the combinations

$$\tau = \frac{1}{2}(x^o + x^3) \quad , \quad \sigma = x^o - x^3. \quad x \quad = (x_1, x_2). \tag{4.2}$$

A front is a hyperplane consisting of all space-time points for which τ is a constant-we may call it "the front τ". Corresponding to the choice (4.2) of space-time coordinates, one rearranges the Poincare generators as follows:

$$\begin{aligned} &J_3; \quad G_{\perp} = (K_1 - J_2)/2, \quad (K_2 + J_1)/2; \quad P_{\perp} \; ; \\ &M = (P^o + P^3)/2; \quad K_3 - \tau(P^o - P^3); \end{aligned} \tag{4.3a}$$

$$\mathcal{H} = P^o - P^3; \quad H_{\perp} = K_1 + J_2, \; K_2 - J_1 \; . \tag{4.3b}$$

The seven generators of $\mathcal{P}$ listed in (4.3a) give rise to transformations that map

the front τ onto itself, the remaining ones in (4.3b) generate transformations that move the front. In particular the combination $\mathcal{H}$ shifts a front parallel to itself.

A particular feature of the rearrangement (4.3) of the Poincaré generators is that the six generators, J_3, G_1, P_1, M drawn from (4.3a), along with $\mathcal{H}$ taken from (4.3b), obey commutation relations corresponding to a two-space one-time Galilei algebra[62]. In this algebra $\mathcal{H}$ plays the role of "energy" – it causes shift in τ but not in σ or $x_\perp$ – while M is like the "mass". Now the Galilei algebra with nonzero "mass" always contains with it, as is well known from nonrelativistic dynamics, quantities obeying the canonical or Heisenberg type commutation relations. These quantities are the (transverse) spatial translation generators $P_\perp$ and the boost generators $G_\perp$ divided by the "mass" M. Thus, if we define

$$Q_\perp = \frac{1}{M} G_\perp \ , \tag{4.4}$$

and recall that M commutes with both $G_\perp$ and $P_\perp$, we find as a consequence of the Poincaré commutation relations,

$$[Q_a, P_b] = i\,\delta_{ab}, \quad [Q_a, Q_b] = [P_a, P_b] = 0 \tag{4.5}$$

It should be emphasized that the Galilei structure and hence the canonical commutation relations (4.5) are embedded in the overall Poincaré structure; the front form has only exposed them. From the commutation relations (4.5) it follows that hermitian and at most quadratic expressions in the four hermitian generators $Q_\perp$, $P_\perp$ form generators of the metaplectic group[30]; exponentials of i times these quadratic expressions give elements of the metaplectic group.

Thus, we have traced the symplectic structure of first order optics to the basic Poincaré invariance. The fact that the wave equation of scalar optics and the Maxwell system of equations of vector wave optics both possess Poincaré symmetry implies that this symplectic structure has to vector optics the same relevance it has for scalar optics.

Consider the paraxial propagation of scalar quasi-monochromatic waves about the x_3 axis (z - axis). All Poincaré generators (4.3) in the front form can be easily computed[63]. We have, in particular,

$$Q_{\perp} = x_{\perp} ,$$

$$P_{\perp} = i\nabla_{\perp} = (-i\frac{\partial}{\partial x_1} , -i \frac{\partial}{\partial x_2}) . \qquad 4.6$$

Further, as explained in Ref. [63] when the so-called "quasi-henochromaticity" condition is satisfied (most paraxial situations do satisfy this condition) the "mass" $M \simeq k = \omega/c$, and an optical system located in space in the plane $x^3 = a$ appears, to a good approximation, effectively located at $\tau = a$ in the front form; and propagation in τ appears as propagation in z. A paraxial scalar field is specified by its amplitude $\psi(x_{\perp} ; z_o)$ over a transverse plane $z = z_o$; the propagation equation then gives $\psi(x_{\perp} ; z)$ for any value of $z > z_o$. The paraxial assumption on $\psi(x_{\perp} ; z_o)$ demands, of course, that its Fourier transform $\tilde{\varphi}(k_{\perp} ; z_o)$ with respect to x have appreciable values only for those $k_{\perp}$ such that $|k_{\perp}| << k$. And there is no other condition which $\psi(x_{\perp} ; z_o)$ in the initial plane has to satisfy. But the situation is different with vector waves. Let us arrange the components of $\underline{E}$, $\underline{B}$ vectors into a six element column

$$F = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} \qquad 4.7$$

When the front form Poincaré generators acting on F are computed[64] one finds that $P_\perp$ has the same expression as in (4.6) but $Q_\perp$ picks up an extra 'polarization' operator. We have

$$P_\perp = -i \mathbb{1} \nabla_\perp \,, \qquad Q_\perp = \mathbb{1} x_\perp + \frac{1}{k} \tilde{G}_\perp \,, \tag{4.8}$$

where

$$G_1 = \tfrac{1}{2}\begin{pmatrix} -S_2 & S_1 \\ -S_1 & -S_2 \end{pmatrix}, \quad G_2 = \tfrac{1}{2}\begin{pmatrix} S_1 & S_2 \\ -S_2 & S_1 \end{pmatrix};$$

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}. \tag{4.9}$$

Here, $\mathbb{1}$ represents the 6x6 identity matrix but this will be suppressed henceforth. Clearly, for a paraxial beam each component of F in the initial plane must be paraxial – but that is not enough. These components must be so chosen that the constraint part of the Maxwell equation (4.1c,d) is satisfied, so that F will indeed represent a Maxwell beam. Then the propagation equation in z gives F at a later plane, and this evolution preserves the constraint as noted earlier. Now the important point is that the Poincaré invariance expressed in the front form enables us to disentangle these aspects properly, and leads to a compact expression for the most general quasi-monochromatic Maxwell beam. We state here only the result, the proof can be found in Ref. [65]: F in the initial plane must necessarily have the form

$$F(x_\perp) = \exp\left[\frac{i}{k}(\tilde{G}_a P_a)\right] F_T(x_\perp) \,,$$

$$F_T(x_\perp) = \begin{pmatrix} E_1(x_\perp) \\ E_2(x_\perp) \\ 0 \\ -E_2(x_\perp) \\ E_1(x_\perp) \\ 0 \end{pmatrix} \tag{4.10}$$

where $E_1(x_\perp)$ and $E_2(x_\perp)$ are two arbitrary paraxial scalar functions.

That is, if $F_T(x_\perp)$ has the first and the fifth entries identical, second and fourth equal but opposite in sign, and the third and sixth entries zero then the matrix differential operator $\exp[\frac{i}{k}(\tilde{G}_a \tilde{P}_a)]$ acting on $F_T(x_\perp)$ produces $F(x_\perp)$ which is guaranteed to respect the constraints (4.1c,d).

These results enable one to find description of the action of paraxial optical elements on Maxwell beams in a manner consistent with Maxwell equation. In the scalar case, an optical system acts through a characteristic operator $\omega(x_\perp, P_\perp)$ on the field in the input plane to produce the field in the output plane. In the vector case, even if the system under consideration is not "sensitive" to polarization in the usual sense, we cannot have $\omega(x_\perp, P_\perp)$ act on each component of F to represent the action of the optical system. The reason for this is easy to see; whereas $P_\perp$ continues to be a generator of the Poincare group for the vector field, $x_\perp$ is not, and hence the F resulting from such an action of $\omega(x_\perp, P_\perp)$ will not be a Maxwell field.

A clue to the generalization of the action of optical systems from scalar optics to vector optics can be drawn from (4.8): replace $x_\perp$ of the scalar theory in $\omega(x_\perp, P_\perp)$ by $Q_\perp$ in (4.8) appropriate to the Maxwell system to get the system operator $\Omega(x_\perp, P_\perp)$ for the vector case:

$$\omega(x_\perp', P_\perp) \longrightarrow \Omega(x_\perp, P_\perp) = \omega\left(x_\perp + \frac{1}{k}\tilde{G}_\perp, P_\perp\right). \qquad 4.11$$

Since $Q_\perp$ and $P_\perp$ obey the same algebra (4.5) as do $x_\perp$ and $P_\perp$, it follows that

$$\omega'(x_\perp, P_\perp)\,\omega(x_\perp, P_\perp) = \omega''(x_\perp, P_\perp) \qquad 4.12$$

implies

$$\Omega'(x_\perp, P_\perp)\,\Omega(x_\perp, P) = \Omega''(x_\perp, P_\perp)\,. \qquad 4.13$$

Thus, the prescription (4.11) automatically guarantees that the association of first order systems with elements of the group Sp(2,R) or Sp(4,R), depending on their symmetry, is maintained in vector optics also.

The prescription (4.11) can indeed be derived directly from the representation (4.10) for paraxial Maxwell beams. We allow $\omega(x_\perp, P_\perp)$ to act on each component of the input F_T. Clearly, the result is a valid F_T [recall the comment following (4.10)]:

$$F_{T,\,out}(x_\perp) = \omega(x_\perp, P_\perp)\, F_{T,\,in}(x_\perp)\,. \qquad 4.14$$

Now acting on either side of (4.14) from the left with $\exp[\frac{i}{k}(\tilde{G}_a P_a)]$ and making use of the identity

$$\exp\left[\frac{i}{k}\tilde{G}_a P_a\right]\omega(x_\perp, P_\perp)\exp\left[-\frac{i}{k}\tilde{G}_a P_a\right] = \omega\left(x_\perp + \frac{1}{k}\tilde{G}_\perp, P_\perp\right). \qquad 4.15$$

which incidentally is a direct consequence of the commutation relations (4.5), one obtains

$$F_{out}(x_\perp) = \Omega(x_\perp, P_\perp)\, F_{in}(x_\perp)\,. \qquad 4.16$$

This completes the derivation of our rule for going from scalar to vector optics[66]. It is useful to illustrate this rule with a simple example. For an ideal thin lens of focal length f we have (the lens is placed at z=o)

$$\omega(x_\perp, P_\perp) = \exp\left(-\frac{ik}{2f}x_\perp^2\right)\,, \qquad 4.17$$

and hence

$$\Omega(x_\perp, P_\perp) = \exp\left[-\frac{ik}{2f}\left(x_\perp + \frac{1}{k}\tilde{G}_\perp\right)^2\right] . \qquad 4.18$$

This matrix exponential can be simply evaluated in closed form owing to the special properties of the polarization matrices $\tilde{G}_1$, $\tilde{G}_2$

$$\begin{aligned} [\tilde{G}_a, \tilde{G}_b] &= 0 \quad ; \\ \tilde{G}_1^2 + \tilde{G}_2^2 &= 0 \quad , \\ \tilde{G}_a \tilde{G}_b \tilde{G}_c &= 0 \quad . \end{aligned} \qquad 4.19$$

We have for the thin lens

$$\begin{aligned} \Omega(x_\perp, P_\perp) &= \exp\left[-\frac{ik}{2f}x_\perp^2\right] \exp\left[-\frac{i}{f}x_a\tilde{G}_a\right] \\ &= \exp\left[-\frac{ik}{2f}x_\perp^2\right] \cdot \left(1 - \frac{i}{f}x_a\tilde{G}_a - \frac{1}{2f^2}(x_a\tilde{G}_a)^2 \cdots\right) \end{aligned}$$

$$= e^{-\frac{ik}{2f}x_\perp^2} \begin{pmatrix} 1+\frac{y^2-x^2}{8f^2} & -\frac{xy}{4f^2} & -\frac{x}{2f} & \frac{xy}{4f^2} & \frac{y^2-x^2}{8f^2} & \frac{y}{2f} \\ -\frac{xy}{4f^2} & 1+\frac{x^2-y^2}{8f^2} & -\frac{y}{2f} & \frac{y^2-x^2}{8f^2} & -\frac{xy}{4f^2} & -\frac{x}{2f} \\ \frac{x}{2f} & \frac{y}{2f} & 1 & -\frac{y}{2f} & \frac{x}{2f} & 0 \\ -\frac{xy}{4f^2} & \frac{x^2-y^2}{8f^2} & -\frac{y}{2f} & 1+\frac{y^2-x^2}{8f^2} & -\frac{xy}{4f^2} & -\frac{x}{f} \\ \frac{x^2-y^2}{8f^2} & \frac{xy}{4f^2} & \frac{x}{2f} & -\frac{xy}{4f^2} & 1+\frac{x^2-y^2}{8f^2} & -\frac{y}{2f} \\ \frac{y}{2f} & -\frac{x}{2f} & 0 & \frac{x}{2f} & \frac{y}{2f} & 1 \end{pmatrix} . \qquad 4.20$$

This matrix applied to the input F made up of components of the input field vectors E_j, B_j in the plane immediately before the lens produces the output F

consisting of the output components E_j', B_j' in the plane immediately after the lens.

Since we are interested in exhibiting how the prescription (4.11) works, we shall retain only the lowest order term half in the matrix in (4.20). Further, for a paraxial beam, to leading order, $E_1 = B_2$, $E_2 = B_1$ and E_3, B_3 are small quantities of first order relative to E_a, B_a,. Then, we see from (4.20) that the output field vectors are related to the input ones as follows:

$$
\begin{aligned}
E_a'(x_\perp;0) &\simeq \exp\left(-\frac{ik}{2f}x_\perp^2\right) E_a(x_\perp;0) \quad , \\
B_a'(x_\perp;0) &\simeq \exp\left(-\frac{ik}{2f}x_\perp^2\right) B_a(x_\perp;0) \quad , \\
E_3'(x_\perp;0) &\simeq \exp\left(-\frac{ik}{2f}x_\perp^2\right)\left[E_3(x_\perp;0) + (x_a/f)\, E_a(x_\perp;0)\right] \quad , \\
B_3'(x_\perp;0) &\simeq \exp\left(-\frac{ik}{2f}x_\perp^2\right)\left[B_3(x_\perp;0) + (x_a/f)\, B_a(x_\perp;0)\right] .
\end{aligned}
\qquad 4.21
$$

Thus one find that the action of lens contributes a small additional "axial" component[67]. To see a striking effect of this axial component, assume that the input was an axially propagating plane wave, and consider the Poynting vector whose components immediately behind the lens are found to be

$$
\mathrm{Re}\left[\underline{E}'(x_\perp;0) \wedge \underline{B}'(x_\perp;0)^*\right] \simeq \tfrac{1}{2}\left(|\underline{E}|^2 + |\underline{B}|^2\right)\left(-\frac{x_\perp}{f}, 1\right) . \qquad 4.22
$$

For a converging lens with $f > 0$ we see that at each $x_\perp$ immediately after the lens (assuming we consider only paraxial points $|x_\perp| \ll f$) the outgoing Poynting vector points exactly to the focus $(0,0,f)$, which is just what is expected. To appreciate this, note that if the terms $x_a E_a$, x_a, B_a in E_3', B_3' in (4.21) had been absent, then the outgoing Poynting vector, like the incoming one, would have been parallel to the system axis at each point $x_\perp$!

At the level of the field components we see, for example, that the electric vector $\underline{E}'(x_\perp;0)$ and the magnetic vector $\underline{B}'(x_\perp;0)$ are mutually orthogonal and orthogonal to $(-x_\perp, f)$ which is the vector leading from $(x_\perp, 0)$ to the focal

point (o,o,f). Thus, for an incident axial plane wave the lens transformation (4.21) yields an outgoing wave which locally can be described by a set of vector plane waves, all directed to the focal point. Thus we have obtained an extension of the well known Debye integral representation[5] of focused scalar fields to the vector field case.

Due to space limitations, we have presented in detail only the lens action. A complete analysis of all first order systems and their action on vector waves can be found in Ref. [65]. Indeed such an analysis has been extended to "polarization sensitive" devices like phase plates, optical rotators and polarizers[68]. We have presented here our analysis based on the six element column F consisting of the $\underline{E}$ and $\underline{B}$ vectors. It is possible to develop a more economical description based on a three element column consisting of the components of the electric vector[64,65,69]. In that case, as should be expected, the polarization matrices $\tilde{G}_a$ become 3x3 matrices. As a final comment in connection with our formalism, we note that the analysis of polarization presented so far is at the field vector level. It is possible to extend this analysis to the two-point tensor level. Use of Wigner-Moyal technique will then lead to polarized generalized ray[64,65]. Details of these aspects, however, will not be discussed here any further.

We conclude this section with some brief comments on Gaussian beam solutions to the Maxwell system of equations[70]. In the scalar case, the Gaussian beam has the form

$$\psi(x_\perp, z) \doteq \exp\left[\frac{ik}{2q(z)} x_\perp^2\right], \qquad 4.23$$

where q(z) in the complex radius of curvature. We are interested in vector Gaussian beams, polarized in the x_1-direction to the extent permitted by the Maxwell equations. Field vectors in such a beam can be constructed by a slight

reinterpretation of (4.23). we interpret $\psi(x_\perp ;z)$ in (4.23) as the result of passing an axially propagative scalar plane wave through a Gaussian transparency whole amplitude transmittance is $\exp[\frac{iR}{2q(z)}\ x_\perp^2]$. The vector plane wave with the kind of polarization we want is given by

$$F_o = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad 4.24$$

and the Gaussian transparency in the vector case is represented by replacing $x_\perp$ by $x_\perp + \frac{1}{k}\tilde{G}_\perp$ in the transmittance function

$$\Omega(x_\perp) = \exp\ [\frac{ik}{2q}\ (x_\perp + \frac{1}{k}\tilde{G}_\perp)^2]. \qquad 4.25$$

Now, the matrix operator (4.25) has the same form as the lens operator (4.18).

Thus, the Gaussian transparency action (4.25) when evaluated will be represented by (4.20) with f replaced by -q. The matrix operator (4.25) has to act on F_o in (4.24) to produce $F(x_\perp ;z)$ corresponding to the Maxwell Gaussian beam. It follows that in addition to the principal polarization component, there will be present in the beam an axial component weaker by one order in $q(z)^{-1}$. But, unlike the lens case, these two components will not be in phase since q(z) is not real. The second order terms in the matrix in (4.20) has a interesting physical consequence: it results in a cross-polarization component[71], along x_2 for $\underline{E}$ and along x_1 for $\underline{B}$ vector, weaker by one more order in $q(z)^{-1}$. It is precisely this cross-polarization component that was observed in an interesting experiment by Fainman and Shamir[71,72]. Finally, the integrated strength of the

principal, axial, and the cross-polarization components are individually invariant under propagation of the beam[71].

5. Families of Bose Rays and Higher Order Correlations

The generalized ray description of wave optics we presented in Sections 2 and 3 was specifically at the level of the two-point functions. At that level there is no distinction between classical and quantum descriptions [recall the First Fundamental Theorem of Quantum Optics] and hence a partial generalization of the concept of light pencils, allowing for light and dark rays, did suffice. The usefulness of such generalized pencils has been established beyond doubt, and stem principally from their extremely simple behaviour under various conditions.

But a statistical state of the radiation field is decribed by a hierarchy of correlation functions, $\gamma^{(n,m)}(\ldots)$ in classical optics and $G^{(n,m)}(\ldots)$ in quantum optics, rather than just the two-point function. [For simplicity, we deal with scalar waves and ignore polarization; Generalization to take into account the vector nature is straight forwrd, and can be accomplished by replacing the (n,m)-order correlation functions in what follows by the (n,m)-order correlation tensor. For paraxial illumination it is clear that our rule for going from scalar to vector optics, presented in Section 4, is again valid.] Thus one may ask:

1. Is it possible to transcribe all the information of a statistical state contained in the entire collection of classical (quantum) correlation function $\{\Gamma^{(n,m)}\}$ ($\{G^{(n,m)}\}$) into a generalized ray language, in an exact manner?

2. What are the differences between classical generalized rays and quantum generalized rays, which must exist and which could not be seen at the two-point level n=m=1?

3. What are the consequences of "Bose statistics" for generalized rays, classical and quantum?

4. While the collections $\{\Gamma^{(n,m)}\}$, $\{G^{(n,m)}\}$ can be neatly handled by generating functionals, are there similar characteristic functional methods for

handling collections of joint distribution functions for generalized rays by working "up in the exponent"?

5. When the entire hierarchy of classical (quantum) correlation functions are taken into account, what is the most convenient definition of the practically important paraxial situation?

Some of these questions have been resolved recently using Weyl-Wigner-Moyal (WWM) methods. We outline here only the principal results: the details can be found in a forthcoming paper[73] in the Schrodinger Centenary special issue of "Foundations of Physics".

First some notations and definitions. We will work with a scalar field $\Phi(\underline{x},t) \equiv \Phi(x)$, whose analytic signal part we denote by $\psi(x)$, and distinguish operators by a caret symbol over the field symbol. Further, we restrict attention to free space and hence $\psi(x)$ obeys the free wave equation.

Then the classical correlation functions are given by[6,7]

$$\Gamma^{(n,m)}(x_1,x_2,\ldots,x_n;y_1,y_2,\ldots,y_m) = \langle \psi^*(y_1)\ldots\psi^*(y_m)\psi(x_1)\ldots\psi(x_n)\rangle \quad . \tag{5.1}$$

There are several choices available in the quantum theory, but we will use the normal ordered correlation functions:[6,7]

$$G^{(n,m)}(x_1,\ldots,x_n;y_1,\ldots,y_m) = \langle \hat{\psi}^\dagger(y_1)\ldots\hat{\psi}^\dagger(y_m)\hat{\psi}(x_1)\ldots\hat{\psi}(x_n)\rangle$$
$$= \mathrm{Tr}\left[\hat{\psi}(x_1)\ldots\hat{\psi}(x_n)\hat{\rho}\hat{\psi}^\dagger(y_1)\ldots\hat{\psi}(y_m)\right] . \tag{5.2}$$

The average in (5.1) and (5.2) are made over the classical and quantum ensembles, respectively. Clearly, $\Gamma^{(n,m)}$ and $G^{(n.m)}$ have the same dimensions. For the (quasi) monochromatic situation the dependence on the time variables is trivial,

and hence the space-time coordinates x's and y's can effectively be replaced by the 3-vector space parts $\underline{x}$, $\underline{y}$ alone. For propagation problems involving transformation between transverse planes, one simplifies further to $\underline{x}$'s and $\underline{y}$'s which are 2-vectors in the transverse plane.

With these definitions we begin our analysis by noting the following: It is impossible to have a classical looking description of quantum mechanics at the level of the wave function ψ, linear in the wave function. While it is true that in the WKB limit classical pictures are useful, any exact classical-like version of quantum mechanics must use bilinears $\psi\psi^*$. Similarly in classical wave optics the eikonal limit gives an approximate ray picture linear in field amplitude, but again it is the bilinear two-point function that leads to an exact ray picture through the Wolf function. An extension of this argument leads us to the following observation, important for our purpose:

A generalized ray description is not possible for all $\Gamma^{(n,m)}$ and $G^{(n,m)}$; it can be achieved, at best, only for $\Gamma^{(n,n)}$ and $G^{(n,n)}$. Thus, we can restrict our attention henceforth to the "diagonal" correlation functions $\Gamma^{(N,N)}$ and $G^{(N,N)}$ $M = 1,2,\ldots$. We will show that for these functions a generalized ray description is indeed possible.

Before venturing into that, however, the following point should be clarified:

For a fixed N, is the set $\{\Gamma^{(N,N)}\}$ of all allowed classical correlation functions $\Gamma^{(N,N)}$ the same as the set $\{G^{(N,N)}$ of allowed quantum correlation functions $G^{(N,N)}$? From the definitions (5.1) and (5.2) it is clear that both $\Gamma^{(N,N)}$ and $G^{(N,N)}$ are (i) hermitian, (ii) positive semidefinite, and (iii) Bose symmetric. One realizes that (apart from a trace condition) these are precisely the defining conditions on the configuration space kernel of the quantum mechanical density matrix $\rho(N)$ of N identical Bose particles (moving in 3 or 2 dimensions as appropriate). Further, it follows from (5.2) that every function

of 2N vector variables possessing these three properties is a valid $G^{(N,N)}$ for some state $\hat{\rho}$. But the situation is different with $\Gamma^{(N,N)}$: it has to satisfy, in addition to these three properties, the additional requirement that its realizations have to have the special form

$$\varphi^*(\underline{y}_1) \ldots \varphi^*(\underline{y}_N)\, \varphi(\underline{x}_1)\ldots\varphi(\underline{x}_N) \ . \tag{5.3}$$

An example of allowed $G^{(2,2)}$ and $\rho^{(2)}$, but not $\Gamma^{(2)}$, is

$\xi^*(\underline{y}_1,\underline{y}_2)\ \xi(\underline{x}_1,\ \underline{x}_2)$ where

$$\xi(\underline{x}_1,\ \underline{x}_2) = u(\underline{x}_1)\ v(\underline{x}_2) + u(\underline{x}_2)\ v(\underline{x}_1) \ , \tag{5.4}$$

or

$$\xi(\underline{x}_1,\ \underline{x}_2) = u(\underline{x}_1 + \underline{x}_2) \ . \tag{5.5}$$

Thus $\{\Gamma^{(N,N)}\}$ is a proper subset of $\{G^{(N,N)}\}$ and we have

$$\{\Gamma^{(N,N)}\} \subset \{G^{(N,N)}\} = \{\rho^{(N)}\} \ . \tag{5.6}$$

For the case N=1, the condition (5.3) becomes void and hence $\{\Gamma^{(1,1)}\} = \{G^{(1,1)}\}$, a fact we already know from the First Fundamental Theorem.

It is useful to view the state space of N identical quantum particles as a subset of the state space of N possibly distinguishable quantum particles. Now recall that for an N particle quantum mechanical system the WWM technique is available and it gives a classical looking exact description. Since both $\Gamma^{(N,N)}$ and $G^{(N,N)}$ are hermitian positive semidefinite, we can compare them with quantum mechanical density matrices for N distinguishable particles and then borrow the WWM technology to define N-fold joint quasi-probability distribution of generalized (classical or quantum) rays. When this is done, $\Gamma^{(N,N)}$ will lead to

an N-fold ray distribution function $\omega_N(\ldots)$; and similarly $G^{(N,N)}$ to a function $W_N(\ldots)$. After doing this we must, of course, impose the "Bose condition". It will turn out that the Bose condition when implemented on $\omega_N(\ldots)$ and $W_N(\ldots)$ leads to certain essentially nonlocal correlations on these multivariate ray distributions.

With these observations we define the N-fold classical generalized ray distribution function for one ray with position $\underline{x}_1$ and ray vector $\underline{k}_1$, another with parameters $\underline{x}_2$, $\underline{k}_2$, ..., and the final one with $\underline{x}_N$, $\underline{k}_N$ through the Wigner-Moyal transform

$$\omega_N(\underline{x}_1,\underline{k}_1;\ \ldots\ ;\underline{x}_N,\underline{k}_N) \equiv (2\pi)^{-Nn}\int\prod_{\alpha=1}^{N} d^n x'_\alpha\ \exp\left(i\sum_{\alpha=1}^{N}\underline{k}_\alpha\cdot\underline{x}'_\alpha\right)\cdot$$

$$\cdot\ \Gamma^{(N,N)}\left(\underline{x}_1-\tfrac{1}{2}\underline{x}'_1,\ \ldots\ ,\underline{x}_N-\tfrac{1}{2}\underline{x}'_N;\ \underline{x}_1+\tfrac{1}{2}\underline{x}'_1,\ \ldots\ ,\underline{x}_N+\tfrac{1}{2}\underline{x}'_N\right). \qquad 5.7$$

Here n = 3 or 2 as appropriate. Similarly, we define the N-fold joint quantum mechanical ray distribution function

$$W_N(\underline{x}_1,\underline{k}_1;\ \ldots\ ;\underline{x}_N,\underline{k}_N) \equiv (2\pi)^{-Nn}\int\prod_{\alpha=1}^{N} d^n x'_\alpha\ \exp\left(i\sum_{\alpha=1}^{N}\underline{k}_\alpha\cdot\underline{x}'_\alpha\right)\cdot$$

$$\cdot\ G^{(N,N)}\left(\underline{x}_1-\tfrac{1}{2}\underline{x}'_1,\ \ldots\ ,\underline{x}_N-\tfrac{1}{2}\underline{x}'_N;\ \underline{x}_1+\tfrac{1}{2}\underline{x}'_1,\ \ldots\ ,\underline{x}_N+\tfrac{1}{2}\underline{x}'_N\right). \qquad 5.8$$

It is a property of the Wigner-Moyal transform that $\omega_N(W_N)$ will in general not be non-negative for all values of its arguments. Hermiticity of $\Gamma^{(N,N)}(G^{(N,N)})$ $\Longleftrightarrow$ reality of $\omega_N(W_N)$. Positive semidefiniteness of $\Gamma^{(N,N)}(G^{(N,N)})$ $\Longleftrightarrow$ (pointwise) product of $\omega_N(W_N)$ with any valid $\omega_N(W_N)$, when integrated over all the arguments yields a non-negative number. Further, the fact that $\omega_N(W_N)$ gives an

exact ray description of all the diagonal $\Gamma^{(N,N)}(G^{(N,N)})$ is obvious from the fact that the Wigner-Moyal transform is invertible.

Now we implement the Bose symmetry condition on $\omega_N(W_N)$. At the $\Gamma^{(N,N)}(G^{(N,N)})$ level the Bose symmetry reads

$$(\Gamma^{(N,N)} \text{ or } G^{(N,N)})\ (\underline{x}_1, \ldots, \underline{x}_\alpha, \ldots, \underline{x}_\beta, \ldots, \underline{x}_N; \underline{y}_1, \underline{y}_2, \ldots \underline{y}_N)$$
$$= (\Gamma^{N,N)} \text{ or } G^{(N,N)})\ (\underline{x}_1, \ldots \underline{x}_\beta, \ldots \underline{x}_\alpha, \ldots \underline{x}_N; \underline{y}_1, \underline{y}_2, \ldots, \underline{y}_N). \qquad 5.9$$

Clearly, there are $N(N-1)/2$ such conditions. A double application of this condition and use of the hermiticity of $\Gamma^{(N,N)}(G^{(N,N)})$ leads to

$$(\Gamma^{(N,N)} \text{ or } G^{(N,N)})\ (\underline{x}_1 \ldots \underline{x}_\alpha \ldots \underline{x}_\beta \ldots \underline{x}_N; \underline{y}_1 \ldots \underline{y}_\alpha \ldots \underline{y}_\beta \ldots \underline{y}_N)$$
$$= (\Gamma^{(N,N)} \text{ or } G^{(N,N)}\ (\underline{x}_1 \ldots \underline{x}_\beta \ldots \underline{x}_\alpha \ldots \underline{x}_N; \underline{y}_1 \ldots \underline{y}_\beta \ldots \underline{y}_\alpha \ldots \underline{y}_N). \qquad 5.10$$

It is important to realize that (5.9) is the primitive and correct Bose condition, and (5.10) is a much weaker one (it applies not only to Bosons but also to Fermions and to even more general situations!)

One may be tempted to expect that on $\omega_N(W_N)$ the Bose condition manifests as follows:

$$(\omega_N \text{ or } W_N)\ (\underline{x}_1, \underline{k}_1; \ldots; \underline{x}_\alpha, \underline{k}_\alpha; \ldots; \underline{x}_\beta, \underline{k}_\beta; \ldots; \underline{x}_N, \underline{k}_N)$$
$$= (\omega_N \text{ or } W_N)\ (\underline{x}_1, \underline{k}_1; \ldots; \underline{x}_\beta, \underline{k}_\beta; \ldots; \underline{x}_\alpha, \underline{k}_\alpha; \ldots; \underline{x}_N, \underline{k}_N). \qquad 5.11$$

That is symmetry under interchange of $(\underline{x}_1, \underline{k}_1) \longleftrightarrow (\underline{x}_2, \underline{k}_2)$, $(\underline{x}_1, \underline{k}_1) \longleftrightarrow (\underline{x}_3, \underline{k}_3)$, $(\underline{x}_2, \underline{k}_2) \longleftrightarrow (\underline{x}_3, \underline{k}_3)$ etc. While the local relation (5.11) is definitely a consequence of the Bose symmetry, it is not an exact rendering of Bose symmetry In fact it is an exact rendering of (5.10)!

It turns out[73] that the primitive Bose symmetry condition (5.9) leads to the following nonlocal correlations on the rays:

$$\omega_N \text{ (or } W_N)(\underline{x}_1, \underline{k}_1; \ \ldots, \underline{x}_N, \underline{k}_N) = \int \frac{d^n x' \, d^n k'}{(2\pi)^n} \exp\left[i \underline{k}'.(\underline{x}_1 - \underline{x}_2) - i \underline{x}'.(\underline{k}_1 - \underline{k}_2) \right] .$$

$$\cdot \ \omega_N \text{ (or } W_N)\left(\tfrac{1}{2}(\underline{x}_1 + \underline{x}_2 - \underline{x}'), \tfrac{1}{2}(\underline{k}_1 + \underline{k}_2 - \underline{k}'); \tfrac{1}{2}(\underline{x}_1 + \underline{x}_2 + \underline{x}'), \tfrac{1}{2}(\underline{k}_1 + \underline{k}_2 + \underline{k}'), \ldots \right). \qquad 5.12$$

and similarly for any other pair of rays. In all there are N(N-1)/2 such relations, and it is these nonlocal ray-ray correlations that form an exact rendering of the Bose symmetry. When applied twice, these nonlocal relations lead to local relations of the form (5.11).

In order to understand these correlations better let us consider the simplest case of N=2 where the Bose symmetry imposes just one condition on the bivariate ray-ray distribution W_2:

$$W_2(\underline{x}_1, \underline{k}_1; \underline{x}_2, \underline{k}_2) = (2\pi)^{-n} \int d^n x' \, d^n k' \ \exp\left[i \underline{k}'.(\underline{x}_1 - \underline{x}_2) - i \underline{x}'.(\underline{k}_1 - \underline{k}_2) \right] .$$

$$\cdot \ W_2\left(\tfrac{1}{2}(\underline{x}_1 + \underline{x}_2 - \underline{x}'), \tfrac{1}{2}(\underline{k}_1 + \underline{k}_2 - \underline{k}'); \tfrac{1}{2}(\underline{x}_1 + \underline{x}_2 + \underline{x}'), \tfrac{1}{2}(\underline{k}_1 + \underline{k}_2 + \underline{k}') \right). \qquad 5.13$$

We have an identical relation for ω_N. This somewhat odd-looking relationship takes a suggestive form if we use the average and relative coordinates

$$\underline{x} = \frac{1}{2}(\underline{x}_1' + \underline{x}_2), \quad \underline{k} = \frac{1}{2}(\underline{k}_1 + \underline{k}_2),$$

$$\underline{\xi} = \underline{x}_1 - \underline{x}_2, \quad \underline{\eta} = \underline{k}_1 - \underline{k}_2. \qquad 5.14$$

With an abuse of notation we write

$$W_2(\underline{x}, \underline{k}; \underline{\xi}, \underline{\eta}) = W_2(\underline{x}_1, \underline{k}_1; \underline{x}_2, \underline{k}_2). \qquad 5.15$$

Then the Bose property demands

$$W_2(\underline{x},\underline{k};\underline{\xi},\underline{\eta}) = (2\pi)^{-n}\int d^n\xi' d^n\eta' \exp(i\underline{\eta}'\cdot\underline{\xi} - i\underline{\xi}'\cdot\underline{\eta}) W_2(\underline{x},\underline{k};\underline{\xi}',\underline{\eta}'). \quad 5.16$$

For each average $\underline{x}$, $\underline{k}$ the distribution in the relative coordinate and ray vectors $\underline{\xi}$, $\underline{\eta}$ has to be self conjugate in the sense that

$$f(\underline{\xi},\underline{\eta}) = (2\pi)^{-n}\int d^n\xi' d^n\eta' \exp(i\underline{\eta}'\cdot\underline{\xi} - i\underline{\xi}'\cdot\underline{\eta}) f(\underline{\xi}',\underline{\eta}'). \quad 5.17$$

This symplectic-invariant partial Fourier transformation would normally take us from the Wigner-Moyal transform (in the relative coordinates) to the ambiguity function[74] in the relative coordinates; but here the two have to coincide.

The functions obeying (5.17) can be completely characterized[73]. In the general case of W_N there will be $N(N-1)/2$ such conditions of invariance under partial symplectic Fourier transforms. The correlations between light rays that these conditions represent is a manifestation of the inherent Bose symmetry of the rays, and is an alternate manifestation of the positive distance correlation in an ideal Bose gas[75]. Since this ray-ray correlation is a consequence of the symmetry of the (N,N) order correlation functions under permutation of the argument, it is equally valid for both classical wave optics and quantum optics.

So much for the allowed set of (classical or quantum) N-ray distributions for a fixed N, and the nonlocal manifestation of the fundamental Bose symmetry in them. Now we turn our attention to the entire chain of correlation functions of increasing orders associated with a given (classical or quantum) state. In the classical case we have[73]

$$\Gamma^{(n,m)}(\underline{x}_1,\cdots,\underline{x}_n;\underline{y}_1,\cdots,\underline{y}_m) = \langle \psi^*(\underline{y}_1)\cdots\cdots\psi(\underline{x}_n)\rangle$$
$$\equiv \int_S \mathcal{D}\psi^*\mathcal{D}\psi \; P[\psi(\cdot);\psi^*(\cdot)]\,\psi^*(\underline{y}_1)\cdots\psi^*(\underline{y}_m)\,\psi(\underline{x}_1)\cdots\psi(\underline{x}_n). \quad 5.18$$

Here $P[\psi(\cdot);\psi^*(\cdot)]$ is the (non-negative) probability functional completely characterizing the given classical state and S is the set of allowed analytic signal solutions to the free wave equation. In the quantum case too a formally similar expression obtains: all that one has to do is to use Sudarshan's diagonal representation and replace $P[\psi;\psi^*]$ by the quasiprobability functional $\Phi[\psi;\psi^*]$ (recall the Second Fundamental Theorem):

$$G^{(n,m)}(\underline{x}_1,\cdots,\underline{x}_n;\underline{y}_1,\cdots,\underline{y}_m)$$
$$=\int \mathcal{D}\psi\mathcal{D}\psi^*\,\Phi[\psi(\cdot),\psi^*(\cdot)]\,\psi^*(\underline{y}_1)\cdots\psi^*(\underline{y}_m)\psi(\underline{x}_1)\cdots\psi(\underline{x}_n). \quad 5.19$$

All quantum effects are now buried in $\Phi[\psi;\psi^*]$, that is in the manner in which it differs from a classical probability functional.

Let

$$\omega_1^{(\psi)}(\underline{x},\underline{k}) = (2\pi)^{-n}\int d^n x'\, e^{i\underline{k}\cdot\underline{x}'}\,\psi(\underline{x}-\tfrac{1}{2}\underline{x}')\psi^*(\underline{x}+\tfrac{1}{2}\underline{x}'),\quad \psi\in S, \quad 5.20$$

be the random-valued generalized ray distribution (Wolf function) we would have defined at the level of $\Gamma^{(1,1)}$ for a pure state ψ random valued over S. Then from (5.18) we have for the classical N-ray distribution associated with the state $P[\psi;\psi^*]$

$$\omega_N(\underline{x}_1,\underline{k}_1;\cdots,\underline{x}_N,\underline{k}_N)$$
$$=\int_S \mathcal{D}\psi\mathcal{D}\psi^*\,P[\psi(\cdot),\psi^*(\cdot)]\,\omega_1^{(\psi)}(\underline{x}_1,\underline{k}_1)\cdots\omega_1^{(\psi)}(\underline{x}_N,\underline{k}_N)$$
$$\equiv\langle\,\omega_1^{(\psi)}(\underline{x}_1,\underline{k}_1)\cdots\omega_1^{(\psi)}(\underline{x}_N,\underline{k}_N)\,\rangle. \quad 5.21$$

For the quantum state represented by $\Phi[\psi;\psi^*]$ we have from (5.19)

$$W_N(\underline{x}_1,\underline{k}_1;\cdots,\underline{x}_N,\underline{k}_N)$$
$$=\int_S \mathcal{D}\psi\mathcal{D}\psi^*\,\Phi[\psi(\cdot),\psi^*(\cdot)]\,\omega_1^{(\psi)}(\underline{x}_1,\underline{k}_1)\cdots\omega_1^{(\psi)}(\underline{x}_N,\underline{k}_N)$$

$$\equiv \left\langle \omega_1^{(\psi)}(\underline{x}_1,\underline{k}_1) \cdots\cdots\cdots \omega_1^{(\psi)}(\underline{x}_N,\underline{k}_N) \right\rangle_{\Phi} . \qquad 5.22$$

Now, from (5.21) it is clear that in classical statistical optics the generalization in the concept of rays that has been necessitated at the level of dealing with the two-point function $\Gamma^{(1,1)}$, leading to bright and dark rays (recall $\omega^{(\psi)}(\underline{x},\underline{k})$ is not pointwise non-negative), is <u>all that is needed</u> in generalizing light rays to give an exact description of $\Gamma^{(N,N)}$ for all N associated with a state, apart from the Bose correlation of rays. In the quantum case the situation is different and we have additional aspects.

First, we have already noted a particular rendering of the quantum nature in that the set of allowed W_N for a given $N \geq 2$ is larger than the set of permissible ω_N. Secondly, the quantum nature manifests also in the relation between the different W_N's in the chain of W_N's corresponding to a given state. Since $P \geq 0$, it is seen from (5.21) that if we have a nontrivial field of illumination so that the classical ensemble has some nonzero ψ, then

$$\omega_1(\underline{x},\underline{k}) \neq 0 \longrightarrow \omega_N(\underline{x}_1,\underline{k}_1;\ldots;\underline{x}_N,\underline{k}_N) \neq 0 \text{ for all } N \geq 2. \qquad 5.23$$

But in the quantum case since Φ is not a true probability functional this does not obtain: We can have a state with a finite number of photons and then W_N vanish for large N! Thus,

$$W_1(\underline{x},\underline{k}) \neq 0 \not\longrightarrow W_N(\underline{x}_1,\underline{k}_1;\ldots;\underline{x}_N,\underline{k}_N) \neq 0 \text{ for all } N \geq 0. \qquad 5.24$$

The equations (5.21) and (5.22) allow us to set up generating functionals for ω_N's and W_N's quite easily[73]. Since $\omega^{(\psi)}$ as defined in (5.20) is real, it suffices to introduce a real "external source" function $\lambda(\underline{x},\underline{k})$. Then we have

$$\left\langle \exp\left(i\int d^n x\, d^n k\ \lambda(\underline{x},\underline{k})\ \omega_1^{(\psi)}(\underline{x},\underline{k}) \right) \right\rangle$$

$$= 1 + \sum_{N=1}^{\infty} \frac{i^N}{N!} \int d^n x_1 \cdots d^n k_N \, \lambda(\underline{x}_1, \underline{k}_1) \cdots \lambda(\underline{x}_N, \underline{k}_N) \cdot \omega_N(\underline{x}_1 \underline{k}_1, \cdots, \underline{x}_N \underline{k}_N) \cdot \quad 5.25$$

Thus the entire chain of classical generalized ray distributions corresponding to a state $P[\psi;\psi^*]$ can be handled compactly "up in the exponent". Taking the inverse Wigner-Moyal transform of $\lambda(\underline{x},\underline{k})$ and $\omega^{(\psi)}(\underline{x},\underline{k})$ the expression in the exponent of (5.25) could have been written as

$$\int d^n x \, d^n k \, \lambda(\underline{x},\underline{k}) \, \omega_1^{(\psi)}(\underline{x},\underline{k}) = (2\pi)^n \int d^n x \, d^n x' \, \tilde{\lambda}(x,x') \, \psi(\underline{x}') \, \psi^*(\underline{x}) ,$$

$$\tilde{\lambda}(\underline{x},\underline{x}') = \int d^n k \, \lambda\left(\tfrac{1}{2}(\underline{x}+\underline{x}'), \underline{k}\right) \exp\left[i \underline{k} \cdot (\underline{x}-\underline{x}')\right] \cdot \quad 5.26$$

Thus, we could write the generating functional in the alternate form

$$\left\langle \exp\left[i (2\pi)^n \int d^n x \, d^n x' \, \tilde{\lambda}(\underline{x},\underline{x}') \, \psi(\underline{x}') \, \psi^*(\underline{x}) \right] \right\rangle$$

$$= 1 + \sum \frac{i^N}{N!} \int d^n x_1 \, d^n k_1 \cdots d^n x_N \, d^n k_N \, \lambda(\underline{x}_1 \underline{k}_1) \cdots \lambda(\underline{x}_N \underline{k}_N) \cdot \omega_N(\underline{x}_1 \underline{k}_1 \cdots \underline{x}_N \underline{k}_N) \cdot \quad 5.27$$

As for the quantum generating functional, a comparison of (5.21) and (5.22) shows that we will have to do the averaging in (5.27) with respect to ϕ rather than P; but in terms of the density matrix $\hat{\rho}$ this just means normal ordering. Thus we have the quantum generating functional

$$1 + \sum_{N=1}^{\infty} \frac{i^N}{N!} \int d^n x_1 \cdots d^n k_N \, \lambda(x_1 k_1) \cdots \lambda(x_N k_N) \, W_N(x_1 k_1 \cdots x_N k_N)$$

$$= \mathrm{Tr}\left[\hat{\rho} : \exp\left\{ i (2\pi)^n \int d^n x \, d^n x' \, \tilde{\lambda}(\underline{x},\underline{x}') \hat{\psi}^\dagger(\underline{x}) \hat{\psi}(\underline{x}') \right\} : \right] . \quad 5.28$$

We have shown that all diagonal correlations, classical and quantum, can be given an exact ray decription. It is useful to note that the diagonal correlations cover all situations involving photon counting and intensity correlation experiments. In bothe the classical and quantum cases Bose symmetry

manifests itself as some definite nonlocal correlations in the multi-ray distributions. We have also shown how classical and quantum states differ in the ray description. Finally, we have presented in both cases generating functional for the chain or ray distributions associated with a state.

We conclude this section with some comments on paraxial illumination. In the classical case since P is non-negative, it is clear from (5.21) that if a ray appears in $\omega_1(\underline{x},\underline{k})$ it does appear in all ω_N for $N \geq 0$, and no ray which does not appear in ω_1 can appear in higher order ω_N. In the quantum case, while a ray appearing in W_1 need not appear in W_N for $N \geq 2$, it is again clear that no new ray which is not present in W_1 can be present in higher order W_N. Thus, a beam described by a state $P[\psi;\psi^*]$ or $\Phi[\psi;\psi^*]$ is paraxial at all levels of correlation if and only if it is paraxial at the level of the two-point function. This is a simple and satisfying result regarding the definition of paraxial illumination.

6. Concluding Remarks

We have presented an outline of some of the recent developments in relation to coherence, propagation, and fluctuation of light. As it has turned out, this review has centered essentially around our own work, and the unifying theme has been the notion of generalized light rays throughout. Owing to limitations beyond our control, related works of other authors could not be reviewed. In particular it is unpardonable to have made no mention of two recent developments: Emil Wolf has recently introduced a new theory of coherence in the space-frequency domain first at the two-point function level[76] and subsequently for the higher order correlation functions[77]. This has already found several useful applications[78,47]. Secondly, our results presented in Sections 3 and 4 are based on Lie group methods in an essential way. Very important contributions have been made by the groups of Alex Dragt[79] and Kurt Bernardo Wolf[80] in the context of the application of Lie methods in optica[81]. We hope to mend these omissions elsewhere

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