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GENERALIZATIONS OF THE POMERANCHUK-OKUN' THEOREM*

by

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Pomeranchuk and Okun'[1] conjectured several years ago that in the scattering of one isospin multiplet by another, all the charge exchange scattering amplitudes must become very small at high energies when compared with the elastic amplitudes. Foldy and Peierls[2] subsequently formulated and proved a weaker form of this conjecture which states that the contribution from the exchange of the isospin zero system[3] to the unpolarized or non-spin flip forward scattering amplitude cannot be negligible when compared with the contributions from the exchange of systems with non-zero isospin. In this note, we shall generalize the Foldy-Peierls formulation of the Pomeranchuk-Okun' theorem to the scattering of multiplets belonging to the finite dimensional unitary irreducible representations of any internal symmetry group. Corresponding theorems for non-forward and spin flip amplitudes will also be proved. These results seem to be of particular significance in view of the current speculations about the approximate symmetries which may underlie strong interactions. If it is true that these symmetries become asymptotically exact at high energies, then, together with an assumption to be stated below, a variety of equalities can be derived among the scattering amplitudes of one multiplet by another at these energies. Another consequence of this investigation is that approximations to partial-wave dispersion relations for the scattering of spinless particles with $\ell \geqslant 1$ which retain only the exchange of a system with definite internal symmetry quantum numbers in one of the crossed channels have a chance of being consistent only if the exchanged system belongs to the trivial representation of the group. Thus, in K-π scattering, for example, the retention of the exchange of a ρ-meson alone cannot lead to satisfactory equations. This inconsistency is not due to the bad asymptotic or threshold behavior of this contribution. It will persist, for example, even if ρ is reggeised.

We shall first summarize our results. Let A and B denote two finite dimensional unitary irreducible multiplets of a suitable internal symmetry group and let r and α denote the members of these multiplets. Further~~more~~, let λ_i and λ_f denote the spin states of the initial and final systems in the scattering $r + \alpha \longrightarrow s + \beta$. If the energy dependence is suppressed, the T-matrix for this process can be written as $T_{s\beta,\, r\alpha}(\cos\theta, \lambda_f, \lambda_i)$ or as $T_{s\beta,\, r\alpha}(\hat{e}_f \cdot \hat{e}_i, \lambda_f, \lambda_i)$ where θ is the scattering angle and $\hat{e}_f$ and $\hat{e}_i$ are unit vectors along the incident and outgoing momentum directions in the centre-of-mass system. We will also use the symbol $T_{s\beta,\, r\alpha}$ for either the non-spin flip or the spin averaged scattering amplitude in the forward direction. With this notation, it will be proved that:

i) The contribution to the amplitude $T_{r\alpha,\, r\alpha}$ arising from the exchange of a system[3] which belongs to the trivial representation of the group cannot be negligible compared with the contribution from the exchange of a system which belongs to any other representation.

ii) The same is true if $T_{r\alpha,\, r\alpha}$ is replaced by

$$\sum_{\lambda_f, \lambda_i} \int d\Omega_f \, d\Omega_i \, g^*(\hat{e}_f, \lambda_f)\, g(\hat{e}_i, \lambda_i)\, T_{r\alpha,\, r\alpha}(\hat{e}_f \cdot \hat{e}_i, \lambda_f, \lambda_i) \tag{1}$$

where g is any arbitrary function of its arguments such that the expression (1) exists.

The last result is of importance for high energy phenomenology. If, along with Foldy and Peierls[2], it is assumed that the exchange of only one system dominates at high energies, it must be the exchange of the trivial representation. All the "charge exchange" amplitudes

$$\sum_{\lambda_f, \lambda_i} \int d\Omega_f \, d\Omega_i \, g^*(\hat{e}_f, \lambda_f)\, g(\hat{e}_i, \lambda_i)\, T_{s\beta,\, r\alpha}(\hat{e}_f \cdot \hat{e}_i, \lambda_f, \lambda_i),$$

$$s \neq r \ \text{and/or} \ \beta \neq \alpha \tag{2}$$

then become very small and all the elastic amplitudes (1) become equal at high energies[4]. By a judicious choice of g, one can thus obtain predictions about off-forward and spin flip amplitudes in a form suitable for comparison with experiments. For example, Eq. (1) can be converted into an expression involving only the unpolarized amplitude within the diffraction peak by choosing[5]

$$g(\hat{e}_i, \lambda_i) = 1, \quad \hat{e}_i \cdot \hat{j} \geqslant \cos\theta_o,$$
$$= 0, \quad \hat{e}_i \cdot \hat{j} < \cos\theta_o$$

where $\hat{j}$ is the unit vector along the z-axis and θ_o is an appropriate angle.

To prove (i), we observe that as the system is invariant under the group,

$$T_{s\beta,\, r\alpha} = \sum_{\substack{s',\beta' \\ r',\alpha'}} \mathfrak{D}^{A*}_{s's}(R)\, \mathfrak{D}^{B*}_{\beta'\beta}(R)\, \mathfrak{D}^{A}_{r'r}(R)\, \mathfrak{D}^{B}_{\alpha'\alpha}(R)\, T_{s'\beta',\, r'\alpha'} \tag{3}$$

where $\mathfrak{D}^A(R)$ and $\mathfrak{D}^B(R)$ are the representation matrices acting in the spaces of A and B which correspond to the group element R.

Since $\mathfrak{D}^A(R)$ is unitary, it follows that

$$\sum_r T_{r\beta,\, r\alpha} = \sum_{\beta',\alpha'} \mathfrak{D}^{B\dagger}_{\beta\beta'} \left[\sum_r T_{r\beta',\, r\alpha'}\right] \mathfrak{D}^{B}_{\alpha'\alpha}(R) \tag{4}$$

But $\mathfrak{D}^B(R)$ is also unitary. Therefore, the quantity $\sum_r T_{r\beta,\, r\alpha}$, when regarded as a matrix in the indices β and α, commutes with $\mathfrak{D}^B(R)$. As this result is true for every R and B denotes an irreducible multiplet, $\sum_r T_{r\beta,\, r\alpha}$ must be a multiple of the identity due to Schur's lemma:

$$\sum_r T_{r\beta,\, r\alpha} = T\, \delta_{\beta\alpha} \tag{5}$$

Thus, the averaged amplitude $\sum_r T_{r\beta, r\alpha}$ received contributions only from the exchange of the trivial representation. Let P denote the contribution from this representation to $\mathrm{Im}\, T_{r\alpha,\, r\alpha}$ (this contribution being independent of r) and Q_r from all the other representations. Eq. (5) then gives

$$\mathrm{Im}\, T_{1\alpha,\, 1\alpha} = P + Q_1,$$
$$\sum_{r=2}^{N} \mathrm{Im}\, T_{r\alpha,\, r\alpha} = (N-1)\, P - Q_1 \tag{6}$$

where N is the dimensionality of the representation A. But, because of the optical theorem, the left-hand sides of these equations cannot be negative. Therefore,

$$-P \leq Q_1 \leq (N-1)\, P$$

Similarly,

$$-P \leq Q_r \leq (N-1)\, P \qquad (7)$$

for any r. This proves (i).

As was observed earlier, if the additional assumption is made that the exchange of only one representation contributes significantly to the amplitudes $T_{s\beta, r\alpha}$ at high energies, this theorem tells us that the dominant exchange must belong to the one-dimensional representation. All the charge exchange scattering amplitudes $T_{s\beta, r\alpha}$ ($s \neq r$ and/or $\beta \neq \alpha$) then go to zero at these energies and all the elastic amplitudes $T_{r\alpha, r\alpha}$ become equal. That is, the scattering matrix becomes a multiple of the identity[4].

Assertion (ii) will now be proved. The imaginary part of T- matrix is given by the matrix elements of the positive operator $T^{\dagger}T$. Its diagonal matrix elements in any representation are therefore non-negative. Taking its diagonal matrix element for the state

$$\sum_{\lambda_i} \int d\Omega_i \; g(\hat{e}_i, \lambda_i) \, | \hat{e}_i, \lambda_i \rangle ,$$

we have

$$\sum_{\lambda_f, \lambda_i} \int d\Omega_f \, d\Omega_i \, g^*(\hat{e}_f, \lambda_f) \, g(\hat{e}_i, \lambda_i) \, \mathrm{Im}\, T_{r\alpha, r\alpha}(\hat{e}_f \cdot \hat{e}_i, \lambda_f, \lambda_i) \geqslant 0 \qquad (8)$$

As it is still true that

$$\sum_{r} \sum_{\lambda_f, \lambda_i} \int d\Omega_f \, d\Omega_i \, g^*(\hat{e}_f, \lambda_f) \, g(\hat{e}_i, \lambda_i) \, \mathrm{Im}\, T_{r\beta, r\alpha}(\hat{e}_f \cdot \hat{e}_i, \lambda_f, \lambda_i) = \overline{T}\, \delta_{\beta\alpha} \qquad (9)$$

statement (ii) follows as before. The remarks regarding partial-wave dispersion relations are also readily verified by a simple modification of the proof given elsewhere[6] which shows that the homogeneous parts of these equations have no solutions when $\ell \geqslant 1$.

It may be emphasized that these results apply to any group, continuous or discrete. For example, the exchanged system which is dominant in the amplitude $T_{r\alpha,\, r\alpha}$ must be even under charge conjugation and G-conjugation and must have even parity[7].

An application of the techniques outlined here also yields information on the angular momentum of the exchanged systems. For example, it can be proved that the contribution of the exchanged system with zero spin cannot be negligible for $T_{r\alpha,\, r\alpha}(1, \lambda_i, \lambda_i)$. This result was previously suggested by Foldy and Peierls[2]. A detailed discussion of such matters will be taken up in a later publication.

References

1.) I. Ia. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. 30, 423 (1956) [Translation: Soviet Phys. - JETP 3, 306 (1957)]; L. B. Okun' and I. Ia. Pomeranchuk, ibid. 30, 424 (1956) [Translation: ibid. 3, 307 (1957)].

2.) L. L. Foldy and R. F. Peierls, Phys. Rev. 130, 1585 (1963).

3.) The precise meaning of the term "exchange of a system with definite quantum numbers" is explained in Ref. 2.

4.) Galindo, Morales and Ruegg have proved that if, for any compact continuous internal symmetry group, the "charge exchange" amplitudes are negligible compared to the elastic amplitudes in the forward direction, the forward scattering matrix is a multiple of the identity. See A. Galindo, A. Morales and H. Ruegg, "Theoretical Physics" (International Atomic Energy Agency, Vienna, 1963), p. 265. A similar result is due to C. N. Yang, J. Math. Phys. 4, 52 (1963), who starts from a slightly generalized version of the Pomeranchuk-Okun' theorem in which each of the two colliding particles is supposed to be a coherent mixture of the various components of the corresponding multiplet.

These results can also be quite simply proved for the finite dimensional unitary irreducible representations of any group by the methods used here as will be shown in a later paper.

5.) In using the function g and choosing such forms for it, we follow the work of K. Yamamoto, Phys. Lett. 5, 355 (1963).

6.) A. P. Balachandran and Frank von Hippel, "Consequences of Analyticity and Unitarity for Partial-wave Amplitudes," EFINS-64-18 and Ann. Phys. (in press), Sec. III.

7.) Van Hove has used the result that the dominant exchanged system must be even under charge conjugation in his investigations of the high energy limits of scattering amplitudes. See L. Van Hove, Phys. Lett. 7, 76 (1963).