"Nature of the Physical World"

by

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The world of physics is a postulate that enables us to systematize those of our experiences that seem to be shared in such fashion that the efficient cause of these experiences could be taken to be existing independent of us. To this (restricted) aspect of reality we attach the label "physical". The objects of which the physical world is composed are, then, expected to obey laws independent of any particular observer. The degree to which the decomposition of experience can be made into a physical and a personal domain is a matter deserving more searching analysis; the theory of measurement in quantum mechanics has highlighted a variety of points of view that may be considered admissible. We observe, in passing, that these questions are at once more intimate and more profound than submicroscopic physics and are fundamental to natural philosophy as so ably pointed out by Erwin Schrödinger. (1)

Physics is the study of patterns of existence and change. Change is measured against the background of the changeless. To the extent we have eliminated the witnessing intelligence of the observer from "physical" theory we have to fall back on physical objects which are changeless. The price that is being extracted for this performance is a reduction of our theory of change to a
description of change. And the best theory is that in which the description is the simplest and most elegant.

At the more primitive levels of dynamical theory there were the two distinct aspects of existence and change. Existence had its manifestation in a class of physical objects: particles, rigid bodies, fluids and fields. Change was transcribed as interaction between physical objects: forces, potentials, local couplings, action-at-a-distance and so on. These two interdependent components of reality could be merged into one if we could derive interactions from objects. Existence and change would then become two aspects of the same reality.

The need for such a synthesis was recognized even in the last century. Heinrich Hertz\(^{(2)}\) attempted to construct a theory in which there were only physical objects and their interaction by contact. To deal with the obviously action-at-a-distance phenomena he introduced the notion of "concealed particles" and "concealed motions". His work remained incomplete; and it had to be so since in classical physics in addition to particles (and rigid bodies) there are also fields which represents an essentially distinct kind of physical object. The rigid body and the fluid could be thought of as being composed of particles but the field could not be so reduced.
1. **QUANTUM THEORY OF FIELDS: A SYNTHESES**

The possibility of unification of the concepts of a field and of a particle came with the work of Satyendranata Bose and the coming of quantum field theory. Bose showed\(^3\) that the states of the electromagnetic field could be identified with the totality of states of a collection of strictly identical photons. The coupling between a particle with a quantized field then corresponds to emission and absorption of quanta by the particle. Local field theory then becomes the transmission of interactions manifested by emission and absorption of quanta. In some form or other this idea has dominated the theory of fundamental interactions ever since Hideki Yukawa's meson theory\(^4\) of the nuclear interaction.

Local quantum field theory has presented us with its share of troubles and difficulties; and at times people have made efforts to construct non-Lagrangian models of interactions, especially of the scattering matrix. To reintroduce the notion that particles are responsible for interactions certain analyticity properties are postulated for the interactions. Among them is the idea of "macroscopic causality" under which "...interactions are transmitted over macroscopic distances only by physical objects. This idea is a macroscopic version of the primitive idea that the world consists only of physical objects, and that these objects act on each other only by direct contact".\(^5\) From a suitably precise form of this requirement it is
possible to derive certain analytic properties of the amplitude in the physical region.\(^{(6)}\)

If we take the principle that interactions come about only by exchange of quanta, incorporating the strict principle of action-by-contact, particle spectra and particle interactions become two aspects of the same framework. Particles could be observed either directly, by kinematic studies, or indirectly by deducing their existence from a dynamical study of interaction phenomena. It is then imperative that all particles "seen" indirectly must also be seen directly. This is certainly an attractive hypothesis.

Modern particle physics is so well at ease with this notion of particle exchanges that we would be surprised if the dynamical effects of known particles are not found. We even talk of seeing particles like the intermediate vector boson, the pomeron and the tachyon.

II. NEED FOR A NEW THEORETICAL FRAMEWORK

Yet there is a virtue in being alert to the possibility that this is not necessarily so. There are basically two reasons that appeal to me. The first is that there are a number of particles which have been postulated, which serve some useful purpose if they existed, and sometimes they have even been used in dynamical calculations, but which have not been found. I include in this catalogue of missing particles magnetic monopoles, intermediate
vector bosons, tachyons and quarks. Some of them may appear more probable than others but none of them have been found. Is it possible to have the dynamical effects of those particles without the particles themselves being found? Could they be like the serpent-rope in our traditional knowledge: something that appears to be a serpent but not really there?

The way out of this difficulty is not to invent an entirely new kind of interaction but to see if we can alter our perspective so that a given interaction structure gives rise to new physical predictions: in this case, to see if we can get the advantages of the dynamical effects of these missing particles without their physical appearance kinematically.

The second reason to question the requirement that all virtual quanta should become real is that local quantum field theory failed in its original programme. Local interactions lead to meaningless, formally divergent answers when the effects of the interaction are computed to higher order. The local coupling inevitably leads to these divergences: from a rigorous mathematical viewpoint even the interaction densities themselves are meaningless.

About twenty-five years ago it was realized that meaningful results to a variety of questions in a large class of field theories could be extracted which were explicitly free of infinities as long as the physical transition amplitudes could be expressed in terms of the finite observed masses and coupling constants. While these higher order
corrections so computed for quantum electrodynamics agree with experiment to a remarkably high degree there are several questions which are not answerable in these terms.

It would be essential to avoid all infinities in the theories. The only method of doing this consistently is to make use of a quantum field theory in which (suitable weighted integrals of) the fields are linear operators in a vector space with an indefinite scalar product. But in such an indefinite metric space the usual probability interpretation could give rise to difficulties. The only way is to define a new probability interpretation in which the probability amplitude is not simply the scalar product in the vector space but a related (but different) bilinear expression.

III. A NEW FRAMEWORK: SHADOW STATES

Based on these considerations we ask: Is the world of dynamics of interacting elementary entities composed of directly observed particles only? If we think of the dynamical effects as the "shadows" cast by the particles which are the "substance" we may rephrase this question: Are all shadows cast by entities with substance?

This nonordinary world view in which there are shadows without substance is directly in conflict with the more orthodox view in which all interactions are obtained by exchange of physical particles. But I believe that this departure from orthodoxy is essential and I expect that
eventually one will find inevitable whether the theory takes the outline that I propose or not.

I find a number of compelling aesthetic reasons to consider the world to include such non-ordinary objects as particles which affect dynamics but as yet are not seen; shadows without substance: a world full of surprises and yet orderly. Whether you share this adventurous spirit or not the quest itself is worthwhile.

These various possibilities emerge from the discovery of what I call shadow state quantum theory. It has its birth in the recognition that scattering (and transition) amplitudes depend not only on the total Hamiltonian and the comparison ("free") Hamiltonian but also on the choice of boundary conditions and that there is much freedom in this choice. Shadow state theory is the culmination of two decades of study on the structure of quantum field theory and elementary particle physics. It is only a theoretical framework and not a theoretical model: so it is not something that you can directly compare with experiment. The possibility of accommodating the missing particle is as yet only an attractive possibility.

The search for the fundamental theory underlying our computational schemes in particle theory, the search for a genuine quantum field theory seems to force on us such a substantial change in the nature of the world. This kaleidoscopic change is inevitable once one starts searching for the real theory. I am reminded of the Kathakali drama called "Karnasapathani" depicting an
incident in the life of Karna the child of Sūrya (the Sun god) and Kunti (the mother of the pândava princes). Karna, abandoned as a child by Kunti (who was then an unwed princess) grows up as the son of a charioteer, but his prowess as an archer is so great that Duryodhana (the Kaurava king) raises him to the status of a noble and a general in his army. But on the eve of the great battle Karna asks himself, "Who am I?" The dance drama depicts the agony of this man who, at the height of his powers would give anything to know who he is; his very being is being consumed by the question. At this point Kunti comes up to ask him for a favour. Ever gallant, Karna offers the favour to the dowager queen but she wants Karna to join the pândavas and tells him that he is her first-born that had been abandoned. Karna cannot do this but he tells her that he will promise not to kill four of her sons but that he will fight to death with Arjuna her third son who is an equally great archer: either way she will have five sons left at the end of the war. This is the Karnasapathani. Karna now knows his origin from the Sun and he knows that royal blood flows in his veins but he is now led to fight for his very life. This conflict between Arjuna and Karna on the Kuruksetra battlefield brings forth their great abilities. Eventually Karna is mortally wounded. But his essential being leaves its material sheath and his tejas, the glory that was Karna, returns to Sūrya the effulgent One.

Karna's question "Who am I?" is now answered.

It appears to me that the theoretical framework of
\[ H = H_0 + gV = -\frac{v^2}{2m} + gV(r) \]

\[ H_0 = -\frac{v^2}{2m} \]

In the case of a nonlocal potential we may write in place of \( V(r) \) a nonlocal potential \( V(r,r') \) which may take the simple separable form

\[ V(r,r') = gF(r) F^*(r') \]

For the free motion we have the wave-function plane waves or their superpositions; for the actual case we have more complicated wave functions. But if the potential is expected to behave "properly" both at the origin and at infinity we know that the wave function must behave at infinity as a superposition of converging and diverging spherical waves. We could therefore write:

\[ u_{\alpha}(r) = e^{ikr} + f(\cos \theta) \frac{e^{ikr}}{r} \]

\[ + f'(\cos \theta) \frac{e^{-ikr}}{r} \]

for the asymptotic form for the actual wave function. If we now choose the boundary condition that there are to be no converging spherical waves we are led to choose \( f' = 0 \) so that we can get the familiar asymptotic form:

\[ e^{ikr} + f(\cos \theta) \frac{e^{ikr}}{r} \]
particle physics is very much like Karna. Its origins are noble but it does not know its parentage. At the height of its powers it asks "What am I?" It does not want to be identified with quantum field theory, but it does make the promise to fight Lagrangian field theory to a finish. When the battle is over would it not be revealed in its real form, restored to its glorious lineage?

The Concept of Scattering

"Scattering" is the departure from the "straight and narrow path", the deviation of the behaviour of a system from a suitable preassigned natural behaviour of a corresponding idealized system. "Transition" implies the change from one mode of motion or existence to another such mode. The natural motions themselves are not considered as scattering or transition: a particle is scattered when it departs from its natural uniform rectilinear motion, not when it changes its position at a uniform rate. Scattering and transition are, therefore, properties of a system in relation to a comparison system.

In elementary problems of quantum theory of scattering we consider the actual system to contain an interaction which serves to distinguish it from the free system which serves as the comparison system. For example, in considering scattering of a quantum particle by a potential the interacting and free Hamiltonians are respectively given by:
in which the second and third terms together yield no net flux.

IV. QUANTUM THEORY OF SCATTERING

Recognizing these possibilities let us now consider scattering in a more general format. Consider a quantum system described by a total Hamiltonian

\[ H = H_0 + gV \]

For the time being we need not distinguish between quantum field theories and quantum mechanical systems with finite number of degrees of freedom except to note that in the former case the various terms would be integrals over all space of corresponding densities. \( H_0 \) is chosen so that it has the same spectrum as \( H \) and with the same multiplicity. [In the field theoretic case this implies that all mass renormalization terms are included in \( V \).] Let us now choose corresponding (improper) eigenvectors \( \chi, \phi \) of \( H \) and \( H_0 \):

\[ H\chi = E\chi \]
\[ H_0\phi = E\phi \]

A formal choice of \( \chi \) is as a solution of the equation:

\[ \chi = \phi + gGV\chi \]
We now take \( f(\cos \theta) \) to be the scattering amplitude. By standard manipulations we can demonstrate that this scattering amplitude satisfies the nonlinear relation

\[
I_m f_\lambda = k |f_\lambda|^2
\]

where \( f_\lambda \) is the partial wave scattering amplitude.

But we must note here that \( f_\lambda \) is not automatically given by specifying the potential; it was necessary to specify the boundary conditions also. As the boundary condition changes the scattering amplitude also will change. Of course, for this particular problem we do not wish to make any changes since we have a physical picture of an incident plane wave with spherical waves diverging from the scattering centre; correspondingly there is a diverging flux from the second term.

But there are other solutions. The simplest is obtained by taking the complex conjugate of the above wave function and changing the sign of \( k \). We thus obtain

\[
v_k(r) = e^{ik \cdot r} + f^*(-\cos \theta) \frac{e^{-ikr}}{r}
\]

In this case there is a converging flux towards the centre for the second term. We could also take half of these two to obtain

\[
w_k(r) = e^{ik \cdot r} + f(\cos \theta) \frac{e^{ikr}}{r} + f^*(-\cos \theta) \frac{e^{-ikr}}{r}
\]
\[ \chi = \phi + gGV\phi + g^2GVGV\phi + \cdots \]
\[ = (1 - gGV)^{-1}\phi \] .

Then

\[ \xi = gV(1 - gGV)^{-1}\phi = (1 - gVG)^{-1}gV\phi \]

and

\[ T_{fi} = (\phi_f, gV(1 - gGV)^{-1}\phi_i) = \langle f | gV(1 - gGV)^{-1} | i \rangle \]

so that

\[ T = gV(1 - gGV)^{-1} \]

is the matrix of transition amplitudes. All these calculations are standard and they are the generalization of the usual method of taking the coefficients of the asymptotic diverging spherical waves in the usual elementary wave mechanical description of scattering.\(^{(8)}\) The scattering is dependent on the interaction \(gV\) as well as the free Hamiltonian \(H_0\).

What is important to recognize is that there is still some freedom in defining the correspondence \(\chi + \phi\) and hence of the "source of the scattered wavelets" \(\xi\). This freedom stems from the freedom in the choice of the Green's function \(G\). We may choose any Green's function; all the relationships would continue to hold as long as the defining relation

\[ (E - H_0)G = 1 \]
where $G$ is Green's function obeying

$$(E - H_0)G = 1 .$$

We verify that this is so by showing that

$$(H - E)\chi = (H - E - gV)\chi + g(E - H_0)GV\chi$$

$$= (H_0 - E)(\chi - gGV\chi) = (H_0 - E)\phi = 0 .$$

The "free state" $\phi$ and the "fully interacting state" $\chi$ differ and this difference $\chi - \phi$ may be thought of as a scattered wave:

$$\psi = \chi - \phi = gGV\chi .$$

It may be thought of as a free wave generated by a source:

$$\xi = (E - H_0)\psi = gV\chi .$$

The "scattering" (transition) amplitude from an initial state $i$ to a final state $f$ is ten given by

$$T_{fi} = <f|T|i> = \langle \phi_f, gV\chi_i \rangle .$$

This expression can be rewritten in a slightly different form by writing down the formal solution:
is maintained. As $G$ changes so do $\xi$ and $T$. The scattering is defined not only by $H_0$ and $H$ but also the specific choice of $G$. Since $G$ may be identified with the propagation function, we can interpret this freedom by remarking that scattering depends not only on the interaction but also the manner in which the wavelets propagate;\(^{9}\) and that is eminently reasonable.

It is natural to ask at this stage as to why this freedom does not seem to show up in the familiar elementary derivation of the scattering of a plane wave by a potential. The solution is unique since we take it for granted that the propagation of the scattered wavelets is forward-in-time and so it is mandatory to choose the so-called retarded Green's function. We choose as the solution $\chi$ of the exact Hamiltonian a state tending to a plane wave $\phi$ in the far past, and to the plane wave $\phi$ plus diverging spherical waves $\psi$ in the far future. Here the choice is already made and no freedom is left.

The theory of shadow states\(^{10,11}\) has its origin in the recognition of this freedom; and in the willingness to keep an open mind about whether all particles have to obey this boundary condition.

The transition amplitude matrix $T$ satisfies certain non-linear conditions which incorporate the law of conservation of probability in the elementary theory of scattering. They are therefore called the generalized "unitarity" condition. These relations relate to the difference between $T$ and $T^\dagger$. We have,
\[ T^+ - T = gV \left( (1 - gG^+V)^{-1} - (1 - gGV)^{-1} \right) \]

\[ = gV(1 - gG^+V)^{-1}(G^+ - G)gV(1 - gGV)^{-1} \]

In writing down this relation we have made use of the hermicity of the interaction \( gV \). The relation could be simplified to read:

\[ T^+ - T = T^+(G^+ - G)T \]

This non-linear relation depends on the antihermitian part of the Green's function \( G \). When we make a new choice for \( G \) we get a new relationship for \( T \). This equation is the generalization of the "optical theorem" which relates the imaginary part of the elastic forward scattering amplitude to the cross section.\(^{\text{(8)}}\) The optical theorem is directly related to probability conservation. Hence any freedom in the choice of the Green's function and the consequent change in the generalized unitarity relation entails a new probability interpretation. The possibility of discounting certain states from contributing to the probability arises out of this circumstance.

Let us, therefore, pay special attention to the choice of Green's function. The most familiar one is to choose the retarded Green's function:

\[ G_R = (E - H_0 + i\epsilon)^{-1} \]

For this choice
Together with time-reversal invariance this leads to the optical theorem. The probability interpretation is then the usual one with all states contributing to the physical probability. All states are physical states; and the physical probability that is summed over all these states is conserved. We may well say: all virtual states can become real. Or equally well: the world is made up of physical states only.

V. A NEW SCATTERING THEORY

Now let us be adventurous and explore other possibilities. Let us separate the vector space of states $W$ for world into two subspaces $R$ (for real) and $S$ (for shadow) by means of a projection operator $\sigma$:

$$ S = \{ \sigma \psi \mid \psi \in W \} , $$

$$ R = \{ (1 - \sigma) \psi \mid \psi \in W \} . $$

Since

$$ \sigma^2 = \sigma = \sigma^+ , $$

it follows that these two subspaces are orthogonal. We take care to have $\sigma$ commute with the free Hamiltonian $H_0$ so that $R$ and $S$ are invariant under the free Hamiltonian
evolution. In a relativistic theory we may choose $\sigma$ to commute with the generators of a larger invariance group including the "free" inhomogeneous Lorentz group. There is now the possibility of choosing the Green's function

$$G = \sigma \bar{G} + (1 - \sigma)G_R$$

$$= \frac{1}{2} \sigma (G_R + G_A) + (1 - \sigma)G_R,$$

where

$$G_A = (E - H_0 - i\epsilon)^{-1}.$$

Equally well we may write

$$G = (E - H_0 + (1 - \sigma)i\epsilon)^{-1}.$$

The transition amplitude depends on $\sigma$ in a non-linear manner and we must now seek out the proper probability interpretation associated with the framework.

Before doing this we note one point: the "scattered wave" $\psi$ for an "incident wave" $\phi$ is now different from what it would have been with the standard (retarded) choice for $G$. This includes some advanced waves which are expected from the advanced components in the Green's function we have chosen. Hence the "free" state $\phi$ and the state $\chi$ in the far past no longer coincide; but the lack of this coincidence depends on the space $S$ of shadow states.
\[ T^+_R - T^-_R = 2\pi i T^+_R T^-_R . \]

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The correspondence between the "free states" $\phi$ and the "interacting states" $\chi$ is now more subtle: but it is of course well defined when we have settled on the projection $\sigma$.

We now complete the formulation of the new scattering theory by specifying the new probability interpretation. We compute the probability using $(1 - \sigma)$ as the metric operator. The entire subspace of states $S$ now correspond to zero contribution to the probability while the subspace of states $R$ has the usual probability interpretation. There are no probability amplitudes connecting $R$ and $S$. The space $S$ of states, devoid of probability interpretation, is a part of the world that is a mathematical auxiliary.

VI. TRANSITION AMPLITUDES IN SHADOW STATE THEORY

One feature of the theory should be stressed though it should be clear: the Green's function $G$ refers to the entire system and propagates the quantum state of the system. It is not the propagator of a particle, but the propagator corresponding to the system of particles. Misunderstanding on this point seems to prompt many authors including Professor Stapp\(^{(5)}\) to criticize shadow state theory incorrectly. The probability interpretation is for the state not for a particle!

This completes the formulation of shadow state theory.
The discussion so far has been in terms of Hamiltonians and interactions and it looks non-covariant. Is the theory in fact, covariant? We may also wonder whether for the new theory we must develop an entirely new computational calculus similar to the one that we have developed say in perturbation series for a relativistic quantum field theory. The theory is in fact relativistic. To show this and other features of the scattering amplitude in shadow state theory we study the scattering amplitude in perturbation theory carried out in the interaction picture.

Let \( gV_i(t) \) be the interaction in the interaction picture. We rewrite the shadow theory Green's function in the form:

\[
iG(E) = \int_{-\infty}^{\infty} \{\theta(t') - 1/\sigma\} e^{-i(H-E_0)t'} dt' .
\]

Following the work of Richard (12) we then get the modified Dyson formula:

\[
iT = \sum_{n=1}^{\infty} \frac{(-ig)^n}{2\pi} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n V_i(t_1) \{\theta(t_1-t_2)-1/\sigma\} V_i(t_2) \cdots \{\theta(t_{n-1}-t_n) - 1/\sigma\} V_i(t_n)
\]

This can be simplified into the form

\[
iT = \sum_{\nu=1}^{\infty} (-\pi i)^\nu (-i\tau)(\sigma \tau)^\nu
\]

where
\[-i\tau = \sum_{n=1}^{\infty} \frac{(-ig)^n}{2\pi i n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n T(V_I(t_1), \ldots, V_I(t_n))\]

is the Dyson expression for the standard transition amplitude computed using the fully retarded Green's function (for the system!). It is familiar to us in quantum field theory in terms of Feynman's diagrammatic computational calculus. (We stress that the interaction \(gV_I(t)\) in the interaction picture does not depend on the choice of the Green's function.) We thus obtain the simple formula for the physical scattering amplitude:

\[\mathcal{T} = (1 - \sigma)T(1 - \sigma) = (1 - \sigma)\tau(1 - \pi i\sigma)\tau^{-1}(1 - \sigma)\]

While we have used perturbation theory and interaction representation to calculate \(\mathcal{T}\) in terms of \(\tau\) we could derive this relationship in other ways not dependent on perturbation theory. We may take this to be an exact relation.

It is clear that \(\mathcal{T}\) so defined is relativistically invariant and possesses invariance under internal symmetry operations that were common to \(\sigma\) and \(V\). Though it is not explicit in the appearance of the projection operator, \(\sigma\) is a piecewise constant operator considered as a function of the four-momenta of the particles. Hence the transition amplitude is piecewise analytic since \(\tau\) itself is an analytic function.
But let us show that \( \tilde{T} \) conserves probability. By direct calculation we obtain:

\[
\tilde{T}^+ \tilde{T} = (1-\sigma)(1+\pi i \tau^+ \sigma)^{-1} \tilde{T}^+ \tau (1-\pi i \sigma) \tau^+ (1-\sigma) = (1-\sigma)(1+\pi i \tau^+ \sigma)(\tau^+ \tau - 2\pi i \tau^+ \sigma \tau)(1-\pi i \sigma)^{-1}(1-\sigma)
\]

\[
= 2\pi i \tilde{T}^+ \tilde{T}
\]

In other words the probability with the metric \( (1-\sigma) \) in the space \( W \) (or equally well with metric 1 in the space \( R \) and 0 in the space \( S \)) is conserved by the transition amplitude. And this is equally true whether we are talking about nonrelativistic multichannel systems, relativistic multichannel systems or relativistic field theory. The only word of caution is that in the last case we must ensure that we have a theory which yields a finite amplitude \( \tilde{T} \).

Thus, given a theory with a unitary scattering amplitude we can construct a shadow state theory with the reduced scattering amplitude

\[
\tilde{T} = (1-\sigma)\tau (1-\pi i \sigma)^{-1}(1-\sigma)
\]

which is restricted to the subspace selected by the projector \( (1-\sigma) \). This new scattering amplitude is unitary in this subspace. With the new probability interpretation we can now be satisfied that probability is conserved in scattering.
We have now a method of constructing a theory with as many shadows as we please. We start out with any theory in which the scattering amplitude is finite and unitary. Divide the space of states into two groups in such a manner as is invariant under the free Hamiltonian (or the free Poincaré or Galilei algebra if we are talking of a relativistic theory). The reduced scattering amplitude now provides the shadow state amplitude. All the states involved in the original theory which have been denied a probability interpretation automatically become shadows. They contribute to dynamics (not quite as before!) but do not enter the unitarity sum.

The relation between the reduced amplitude $\mathcal{T}$ and the amplitude $\tau$ shows that if $\mathcal{T}$ is to be free of infinities we require $\tau$ also should be free of infinities. As long as we are dealing with nonrelativistic models we can invent interactions with suitable cutoffs so that finite unambiguous results are obtained by honest means starting with the total Hamiltonian. This happy situation does not obtain in local relativistic field theories unless we formulate them in a suitable indefinite metric theory.

In the following sections we shall deal with a sequence of models of increasing complexity, all of them having the virtue of leading to a finite "unitary" amplitude $\tau$. Given such amplitudes we can compute the finite unitary amplitudes $\mathcal{T}$.

We stress again that if $\tau$ is unitary with respect to an indefinite metric so could $\mathcal{T}$ be unitary with respect
to the restriction of the metric by the projection σ. We should now choose the projected space to have a positive definite metric. Then the amplitude would be truly unitary.

Since the shadow particles are not directly observed they may not only have negative metric but they may also have complex masses. Complex masses for them could have several advantages, but the convergence brought about by means of the indefinite metric with real masses illustrates the general principles. Moreover the results obtained using complex conjugate masses can be obtained by a suitable analytic continuation of the results obtained using real shadow masses. So except when it becomes necessary we shall not distinguish complex shadow mass theories from those with real shadow masses.

VII. TWO-LEVEL SYSTEM COUPLED TO A CONTINUUM OF OSCILLATORS

We consider now a simple but artificial model in which there is a coupling of two fermions (up ↑ and down ↓) to a continuum of oscillators. The essential scheme of this model was first used by Dirac in the context of the problem of line width and is discussed in his book "Principles of Quantum Mechanics". It was rediscovered by T. D. Lee in the context of a model field theory and is usually known as the Lee model. We discuss first the positive norm case. The continuum of oscillators is described by
creation and annihilation operators $a^+(\omega)$ and $a(\omega)$, depending on the real variable $\omega$, and satisfying the relations

$$[a(\omega), a^+(\omega')] = \delta(\omega-\omega'); \quad [a(\omega), a(\omega')] = 0$$

$$[a^+(\omega), a^+(\omega')] = 0$$

A zero excitation (vacuum) state is assumed such that

$$a|\uparrow,0> = a|\downarrow,0> = 0$$

where the arrow up (down) indicates an eigenstate of $\sigma_z$ with eigenvalue plus (minus) one. One easily checks that the appropriate generalization of the number operator $a^+a$ is given by

$$N = \int a^+(\omega)a(\omega)d\omega$$

and that the states

$$\int \phi(\omega)a^+(\omega)d\omega|,0>$$

$$\int \int \phi(\omega_1,\omega_2)a^+(\omega_1)a^+(\omega_2)d\omega_1d\omega_2|,0>$$

correspond respectively to states with one, two, etc., excitations. We now introduce the Hamiltonian of the system

$$H = \frac{1}{2} \sigma_z^3 M + \int \omega a^+(\omega)a(\omega)d\omega + \int g(\omega)[a^+(\omega)\sigma_-a(\omega)\sigma_+]d\omega$$
where we see that \( \omega \) has been given the meaning of the (free) excitation energy; \( g(\omega) \) can be any real function.

Once again, from the fact that the Hamiltonian causes transitions only of the form

\[
|\uparrow, n\rangle \xrightarrow{\tau} |\uparrow, n+1\rangle
\]

we can identify sectors, i.e., subspaces of the Hilbert space, such that upon evolution any vector in the subspace will be transformed into another vector in the same subspace. The structure is the same as in the case of a single oscillator if we replace \( \int a^\dagger(\omega)a(\omega)d\omega \) for \( a^\dagger a \), (Figure 1). The vector with \( \sigma_z = -1, N = 0 \) is unchanged by the interaction. The first non-trivial sector is the one spanned by the vectors \( |\uparrow, 0\rangle; |\uparrow, 1\rangle \). A vector in this sector can be written as

\[
\chi |\uparrow, 0\rangle + \int \phi(\omega)a^\dagger(\omega)d\omega |\uparrow, 0\rangle
\]

or in column vector notation as

\[
\begin{pmatrix}
\chi \\
\phi(\omega)
\end{pmatrix}
\]

The matrix element \( H_{\omega', \omega} \) between continuum normalized eigenstates of the free Hamiltonian is then given by

\[
H_{\omega', \omega} = \begin{pmatrix}
M & 0 \\
0 & \omega \delta(\omega' - \omega)
\end{pmatrix} + \begin{pmatrix}
0 & g(\omega) \\
g(\omega') & 0
\end{pmatrix}
\]
The eigenvalue equation

\[ H |\lambda \rangle = \lambda |\lambda \rangle , \]

with

\[ |\lambda \rangle = \frac{1}{\sqrt{2}} \left( |\uparrow \rangle + \int \phi(\omega) a^\dagger(\omega) d\omega |\uparrow \rangle \right) \]

can be put in the form

\[
\begin{pmatrix}
M - \lambda & g(\omega) \\
g(\omega') & (\omega - \lambda) \delta(\omega - \omega')
\end{pmatrix}
\begin{pmatrix}
\chi \\
\phi(\omega')
\end{pmatrix} = 0
\]

(7.2)

Since the index \( \omega \) is continuous this is not a problem that can be solved by the method of the secular determinant. To solve it we use an indirect method. From (7.2) and using the \( \delta(\omega - \omega') \) function we get the relations

\[
(\lambda - M) \chi = \int g(\omega) \phi(\omega) d\omega
\]

(7.3)

\[
(\lambda - \omega) \phi(\omega) = g(\omega) \chi
\]

(7.4)

Very naively we could think of (7.4) as an algebraic relation and then get successively

\[
\phi(\omega) = \frac{g(\omega) \chi}{\lambda - \omega}
\]

(7.5)

and replacing in (7.3) we get

\[
\left\{ (\lambda - M) - \int \frac{g^2(\omega) d\omega}{\lambda - \omega} \right\} \chi = 0
\]
We now define the function

\[ a(z) = z - M - \int \frac{g^2(\omega) \delta(\omega)}{z - \omega} \tag{7.6} \]

where \( z \) is a complex variable. We could now say that since \( \chi = 0 \) gives only a trivial solution, the eigenvalues \( \lambda \) will have to be such that

\[ a(\lambda) = 0 \]

There are, however, too many loose ends in this reasoning. The integration in (7.6) extends effectively to the range of values of \( \omega \) such that \( g(\omega) \neq 0 \). We shall assume for definiteness that \( g(\omega) \) is nonzero only for \( \omega > 1 \), and that \( M < 1 \). In that case we see that for \( g^2(\omega) \) "small", we have a solution* near \( M \) and for a value of \( \lambda \) less than \( M \), i.e., we get the expected depression of the ground level. However, if \( g^2(\omega) \) is made zero everywhere, we find from (7.5) that \( \phi(\omega) \) must also be zero, and therefore the continuum states are not reproduced. The situation improves if we notice that instead of (7.5) we could equally well have written

\[ \phi(\omega) = \frac{g(\omega)}{\lambda - \omega} \chi + c\delta(\lambda - \omega) \tag{7.7} \]

where \( c \) is a constant (or a function of \( \lambda \)). The last term

---

*This will be rigorously proved later.
in (7.7) is required to recover the free solutions when \( \gamma^2 \to 0 \). But still, since \( \lambda - \omega \) can be zero, expression (7.7) is not quite precise. To make it meaningful we need a specification of how to handle the denominator in \( \lambda - \omega \) when \( \lambda \to \omega \). At this point there are many different choices, for instance, we could take the principal value, or we could make the replacement

\[
\frac{1}{\lambda - \omega} \to \frac{1}{\lambda - \omega \mp i\epsilon}; \epsilon = \text{real, positive.}
\]

Corresponding to each choice and for each well defined physical situation there is also an appropriate choice for \( c \). One should make immediately clear that any such choice is equally valid and that the physics of the problem is certainly independent of the way in which one specifies the solution. The physical interpretation of the set of states obtained will however depend on the specification, as we shall see. It will be convenient to take the specification

\[
\frac{1}{\lambda - \omega} \to \frac{1}{\lambda - \omega - i\epsilon}
\]

in which the limit \( \epsilon \to 0 \) should be taken at the end of the calculation. But we hasten to say that this is only one of the possible conventions and that we are free to change it if we want to do so. Equation (7.7) is now written as

\[
\phi(\omega) = \frac{g(\omega)\chi}{\lambda - \omega - i\epsilon} + \delta(\lambda - \omega) \tag{7.8}
\]
where we have made $c = 1$ so that we recover the free solution when $g \to 0$.

Substituting in (7.3) we find

$$
(\lambda - M)\chi - \chi \int \frac{g^2(w)dw}{\lambda - w - i\epsilon} = g(\lambda)
$$

Recalling the definition of $\alpha(z)$ this can be written as

$$
\alpha(\lambda - i\epsilon)\chi = g(\lambda)
$$

But now we see that every point in the integration range is a possible eigenvalue. The discrete part of the spectrum is associated with the roots of $\alpha(\lambda)$. We first notice that there can be no complex roots for, from (7.6) the imaginary parts of $z$ and of the integration would add up rather than cancel. Furthermore, we can assume, without loss of generality, that the integration is from 1 to $\infty$; then $\alpha(z)$ has a cut in the $z$ plane that extends from 1 to $\infty$ (Figure 7.2). But on both sides of this cut, $\alpha$ is complex and therefore there can be no real roots there either. If there is a root there, it has to be in the interval $(-\infty, 1)$.* In fact, one can prove that $\alpha$ can have at most one root in this interval. First we notice that $\alpha(z)$ is monotonically increasing in the real axis between minus infinity and one. (Its derivative is everywhere in

*Only if the integration does not extend to $+\infty$, also for value of $\lambda$ larger than those on the cut.
this interval positive.) For large negative values of $z$, $a(z)$ is also large and negative. Therefore, if $a$ is positive near one there is a root, and if $a$ is negative there is no root of $a(z)$ (Fig. 7.3), because the real axis can be crossed at most once without changing the sign of the derivative.* If there is a root $\Lambda$ then

$$\Lambda = M - \int_{1}^{\infty} \frac{g^2(\omega)d\omega}{\omega - \Lambda}$$  \hspace{1cm} (7.9)$$

and therefore the discrete solution is less than $M$. This is of course the expected depression of the ground state.

We consider now the states in the continuum. We have

$$\chi = \frac{g(\lambda)}{a(\lambda - i\varepsilon)}$$  \hspace{1cm} (7.10)$$

and

$$\phi(\omega) = \frac{g(\omega)g(\lambda)}{\lambda - \omega - i\varepsilon a(\lambda - i\varepsilon)} + \delta(\lambda - \omega)$$  \hspace{1cm} (7.11)$$

The solutions are therefore complex. One can think of $\phi(\omega)$ as representing a state in which there is a plane wave ($\delta(\lambda - \omega)$) plus a diverging spherical wave. We shall see below that if on the other hand we had chosen $+i\varepsilon$, the solution would have been a state consisting of a plane wave plus a converging spherical wave. At this point it is

*It is clear, on the other hand, that $a(z)$ is a continuous function everywhere in the cut plane.
clear that if we had been interested in real solutions we could have chosen the principal value for $1/(\lambda - \omega)$. The eigenvalue spectrum would have been the same but the physical interpretation of the eigenstates would have been different.

For the bound state solution $\lambda = \Lambda$ we have

$$
\phi_{\Lambda}(\omega) = \frac{\sigma(\omega)\chi}{\Lambda - \omega} \Lambda
$$

(7.11)

where $\chi_{\Lambda}$ is, for the moment, any complex number. We can specify it further by the normalization condition

$$
1 = |\chi|^2 \left[ 1 + \int \frac{\sigma^2(\omega) d\omega}{(\Lambda - \omega)^2} \right] = |\chi|^2 a'(\Lambda)
$$

(7.12)

where $a'(\Lambda)$ is the derivative of $a(z)$ for $z = \Lambda$. The normalized solution is then given by

$$
\phi_{\Lambda}(\omega) = \frac{\sigma(\omega)}{\Lambda - \omega} \frac{1}{\sqrt{a'(\Lambda)}} ; \quad \chi_{\Lambda} = \frac{1}{\sqrt{a'(\Lambda)}}
$$

(7.13)

For the normalization of the continuum we shall give a proof at the end of this section that with the definitions (7.10) and (7.11) the solutions form an orthonormal and complete set, i.e., we have both

$$
\int \phi_{\Lambda}(\omega) \phi_{\Lambda}^*(\omega) d\omega + \chi_{\Lambda} \chi_{\Lambda}^* = \delta(\lambda - \lambda')
$$

(7.12)

and

$$
\int \phi_{\Lambda}(\omega) \phi_{\Lambda}^*(\omega') d\lambda + \chi_{\Lambda} \chi_{\Lambda}^* d\lambda + \phi_{\Lambda}(\omega) \phi_{\Lambda}(\omega') + \chi_{\Lambda} \chi_{\Lambda} = \delta(\omega - \omega')
$$

(7.13)
i. The Physical Identification of Scattering States: The Scattering Matrix

Strictly speaking this is a one dimensional problem and therefore we have to be careful in the interpretation of the "scattering" solution. Analyzed in terms of the "free" particle states, which are the interesting ones in a scattering process we see that $\phi(\omega)$ does not have a well defined energy for finite times. The analysis of the scattering process requires however the description in terms of states having a simple plane wave structure either for times well in the past (the "in" states) or for times well in the future (the "out" states).

The time dependent form of $\phi(\omega)$ is

$$
\phi_\lambda(t) = \int \frac{g(\omega)g(\lambda)e^{-i\omega t}d\omega}{\alpha(\lambda - i\varepsilon)(\lambda - \omega - i\varepsilon)} + e^{-i\lambda t} \quad (7.14)
$$

We shall now show that it represents a state which for $t \to -\infty$ contains only the term $e^{-i\lambda t}$, and therefore can be identified with an incoming plane wave, and for $t \to +\infty$ contains an additional term, which is also of frequency $\lambda$, whose magnitude depends on $g(\omega)$, and which can therefore be identified with the scattered wave. To prove this we first notice that

$$
\frac{-i}{\lambda - \omega - i\varepsilon} = \int_{-\infty}^{\infty} e^{ix(\lambda - \omega - i\varepsilon)}dx \quad (7.15)
$$

then we define the Fourier transform of $g(\omega)$.
\[ g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(z) e^{i\omega z} \, dz \quad (7.16) \]

We now have
\[ \frac{\tilde{g}(\omega) e^{-i\omega t}}{\lambda - \omega - i\epsilon} = \frac{e^{i\omega t}}{2\pi} \int_{-\infty}^{\infty} dz \int_{-\infty}^{0} dx \, e^{ix(\lambda - \omega - i\epsilon)} e^{-i\omega t} \tilde{g}(z) e^{i\omega z} \]

Integration over \( \omega \) gives a \( \pi \delta(-x - t + z) \) which can be used to integrate over \( z \) giving
\[ \int \frac{\tilde{g}(\omega) e^{-i\omega t} dw}{\lambda - \omega - i\epsilon} = \frac{e^{i\omega t}}{2\pi} \int_{-\infty}^{0} dx e^{ix(\lambda - i\epsilon)} \tilde{g}(t + x) \]

or
\[ \int \frac{\tilde{g}(\omega) e^{-i\omega t} dw}{\lambda - \omega - i\epsilon} = e^{-i\lambda t} \int_{-\infty}^{t} dx e^{i\lambda x} \tilde{g}(x) \quad (7.17) \]

The righthand side of (7.17) clearly goes to zero when \( t \) is large and negative. For \( t \to +\infty \) we have
\[ \int \frac{\tilde{g}(\omega) e^{-i\omega t} dw}{\lambda - \omega - i\epsilon} \approx +i e^{-i\lambda t} 2\pi g(\lambda) \]

which indicates that the process is energy conserving, i.e. during the collision the energy is not well defined, but if one waits long enough, the system evolves in such a way as to allow only the components of the right energy to survive. The asymptotic form of \( \phi_{a}(t) \) is then given by
\[ \tilde{\phi}_{a}(t) \sim \frac{+i2\pi g^{2}(\lambda)}{\alpha(\lambda - i\epsilon)} e^{-i\lambda t} + e^{-i\lambda t} \]

But from (7.6) we have
\[ \alpha(\lambda + i\epsilon) - \alpha(\lambda - i\epsilon) + 42g^2(\lambda) \]  

(7.18)

So we finally get

\[ \tilde{\phi}_\lambda(t) \sim \left[ \frac{\alpha(\lambda + i\epsilon) - \alpha(\lambda - i\epsilon)}{\alpha(\lambda - i\epsilon)} + 1 \right] e^{-i\lambda t} \]

or

\[ \tilde{\phi}_\lambda(t) \sim \frac{\alpha(\lambda + i\epsilon)}{\alpha(\lambda - i\epsilon)} e^{-i\lambda t} \]  

(7.19)

Since \( \alpha(\lambda + i\epsilon)/\alpha(\lambda - i\epsilon) \) is a unimodular complex number, the effect of the scattering is a net change in the phase as compared with the free propagation (phase shift).*

We have indicated the \( \phi(\omega) \) as defined by (7.11) is the appropriate wave function for an "in" state. The "out" states are obtained by simply replacing \( +i\epsilon \) for \( -i\epsilon \) in all the expressions. The statements about the behavior in time are now simply reversed. In fact, since this is essentially a spinless model, the "out" states are the time reversed of the "in" states, since they are obtained from the latter by complex conjugation.

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*In fact we could define a "phase shift" \( \delta \) such that

\[ e^{i2\delta} = \frac{\alpha(\lambda + i\epsilon)}{\alpha(\lambda - i\epsilon)} \]

If we consider "small" values of \( g^2(\omega) \) we see from (7.6) that \( \alpha(\lambda + i\epsilon) \) has a small positive imaginary part. Then \( \delta \) would be positive corresponding to the attractive characters of the force.
The matrix elements of the S operator for the problem can now be defined as usual in terms of the overlap of the "in" and "out" states. We have

$$<\lambda, \text{out}|\lambda', \text{in}> = \int \phi^{(\text{out})\ast}_\lambda(\omega)\phi^{(\text{in})}_\lambda(\omega)\,d\omega + \chi^{(\text{out})\ast}_\lambda\chi^{(\text{in})}_\lambda$$

But from the previous arguments

$$\phi^{(\text{out})\ast}_\lambda(\omega) = \phi^{(\text{in})}_\lambda(\omega) = \phi_\lambda(\omega);$$

$$\chi^{(\text{out})\ast}_\lambda = \chi^{(\text{in})}_\lambda = \chi_\lambda$$

Then

$$<\lambda, \text{out}|\lambda', \text{in}> = \int \phi_\lambda(\omega)\phi^{\ast}_{\lambda'}(\omega)\,d\omega + \frac{g(\lambda)g(\lambda')}{\alpha(\lambda - i\epsilon)\alpha(\lambda' - i\epsilon')}$$

We consider now the integral on the righthand side

$$\int \phi_\lambda(\omega)\phi^{\ast}_{\lambda'}(\omega)\,d\omega = \frac{g(\lambda)g(\lambda')}{\alpha(\lambda - i\epsilon)\alpha(\lambda' - i\epsilon')} \int \frac{g^2(\omega)\,d\omega}{(\lambda - \omega - i\epsilon)(\lambda' - \omega - i\epsilon')} + \frac{g(\lambda)g(\lambda')}{(\lambda - \lambda' - i\epsilon)\alpha(\lambda - i\epsilon)} + \frac{g(\lambda)g(\lambda')}{(\lambda' - \lambda - i\epsilon')\alpha(\lambda' - i\epsilon')} + \delta(\lambda - \lambda')$$

Using the identity

$$\frac{1}{(\lambda - \omega - i\epsilon)(\lambda' - \omega - i\epsilon')} = \frac{1}{(\lambda' - \lambda - i(\epsilon' - \epsilon))} \left[ \frac{1}{\lambda - \omega - i\epsilon} - \frac{1}{\lambda' - \omega - i\epsilon'} \right]$$

We finally get

(7.20)
\[
\int \phi_\lambda(\omega) \phi_{\lambda'}(\omega) d\omega = \frac{g(\lambda)g(\lambda')}{a(\lambda-\lambda')a(\lambda'-\lambda')}(\frac{\alpha(\lambda'-i\varepsilon')-\alpha(\lambda-i\varepsilon)+\lambda-\lambda'+i(\varepsilon-\varepsilon')}{\alpha(\lambda-\lambda')a(\lambda'-\lambda')(\lambda'-\lambda-i(\varepsilon'-\varepsilon))}) \\
+ \frac{g(\lambda)g(\lambda')}{a(\lambda-i\varepsilon)a(\lambda'-i\varepsilon')}[\frac{\alpha(\lambda'-i\varepsilon')}{\lambda'-\lambda'-i\varepsilon'} + \frac{\alpha(\lambda'-i\varepsilon')}{\lambda'-\lambda'-i\varepsilon'}] + \delta(\lambda-\lambda')
\]

We have to consider now very carefully the expression

\[
\frac{\alpha(\lambda'-i\varepsilon')}{\lambda'-\lambda'-i(\varepsilon'-\varepsilon)} - \frac{\alpha(\lambda-i\varepsilon)}{\lambda'-\lambda-i(\varepsilon'-\varepsilon)} - \frac{\alpha(\lambda'-i\varepsilon'')}{\lambda'-\lambda'+i\varepsilon'} + \frac{\alpha(\lambda-i\varepsilon)}{\lambda'-\lambda'-i\varepsilon'} = \\
\alpha(\lambda'-i\varepsilon') \left[ \frac{1}{\lambda'-\lambda-i(\varepsilon'-\varepsilon)} - \frac{1}{\lambda'-\lambda+i\varepsilon'} \right] \\
+ \alpha(\lambda-i\varepsilon) \left[ \frac{1}{\lambda'-\lambda-i\varepsilon'} - \frac{1}{\lambda'-\lambda-i(\varepsilon'-\varepsilon)} \right] = A
\]

Using the identity

\[
\frac{1}{x-i\varepsilon} = P \frac{1}{x} + i\pi\delta(x); \quad \varepsilon = \text{real positive} \quad (7.21)
\]

we find

\[
A = \alpha(\lambda-i\varepsilon)2i\pi\delta(\lambda-\lambda')
\]

and therefore, making use of (7.18)

\[
<\lambda, \text{out} | \lambda', \text{in}> = \frac{\alpha(\lambda+i\varepsilon)}{\alpha(\lambda-i\varepsilon)} \delta(\lambda-\lambda') \quad (7.22)
\]

Since the \( |\lambda', \text{in}> \) states form a complete and orthonormal set this implies that the \( |\lambda, \text{out}> \) states differ only by a phase factor from the \( |\lambda, \text{in}> \) states. In particular we would then have
\[ |\lambda, \text{ out}\rangle = S^+ |\lambda, \text{ in}\rangle = \frac{a(\lambda - i\epsilon)}{a(\lambda + i\epsilon)} |\lambda, \text{ in}\rangle \]

or

\[ S(\lambda) = \frac{a(\lambda - i\epsilon)}{a(\lambda + i\epsilon)} \quad (7.23) \]

The bound state is, on the other hand, real and therefore invariant under time reversal. We could say that in this case the "in" and "out" states are the same and the corresponding matrix element of $S$ would have to be equal to unity. One can easily check that in fact (7.23) is consistent with this argument, i.e.,

\[ S(\lambda) = \lim_{\epsilon \to 0} \frac{a(\lambda + i\epsilon)}{a(\lambda - i\epsilon)} = 1 \quad (7.24) \]

ii. **Scattering in the Indefinite Metric Model**

We consider now the situation when instead of the usual oscillators we have the abnormal ones. The Hamiltonian is given by

\[ H = \frac{1+\sigma_z}{2} \ M - \int \omega b^+ b(\omega) d\omega + \int g(\omega) [b^+(\omega) \sigma_- + b(\omega) \sigma_+] d\omega \]

The minus sign in the second term on the righthand side assures us that the excitations will have larger energies than the ground state.* As it should be clear by now, the

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*We restrict the discussion once again to the sector in which there is either zero or one excitations.
algebraic structure that we get is similar to the previous one and most things look like the "analytic continuations" of the results obtained for the normal oscillators. The implications of the changes in sign are however very important and should be carefully studied. We have for the matrix element of $H$

$$
\tilde{H}_{\omega', \omega} = \begin{pmatrix} M & 0 \\ 0 & \omega \delta(\omega - \omega') \end{pmatrix} + \begin{pmatrix} 0 & \sigma(\omega) \\ -\sigma(\omega') & 0 \end{pmatrix}
$$

and therefore the discussion of the solutions can be repeated but in this case we define

$$
\alpha(z) = z - M + \int_1^{\infty} \frac{\frac{\sigma^2(\omega)}{z - \omega} d\omega}{z - \omega}
$$

(7.25)

The existence of discrete solutions is once again centered around the zeros of $\alpha(z)$. The argument for no complex zeros fails, however, because the imaginary parts of the terms on the righthand side of (7.25) have opposite signs and therefore the possibility always exists that they will cancel each other. If there is a complex root for $z = z_0$, then we immediately verify that $z = z_0^*$ is also a root. In fact one can prove that only two cases are possible, either there is one real root $\lambda < 1$ or there are two complex conjugate roots with $\text{Re} \lambda > 1$.

Since the algebraic manipulations did not depend on the sign, we still get the same form for the solutions with the new definition of $\alpha$. The normalization condition looks
slightly different because if we choose the ground (discrete) state to have positive norm then the continuum states have negative norm and we must write

\[- \delta(\lambda - \lambda') = - \int \phi_{\lambda}(\omega') \phi_{\lambda'}^*(\omega') d\omega' + \chi_{\lambda'} \chi_{\lambda}^*\]

However, because of the change in sign in the definition of \(\alpha\), the corresponding phase shift is now negative, indicating the repulsive character of the force generated by the interaction with the abnormal oscillators. On the other hand, \(S(\lambda)\) is a unimodular complex number and therefore unitarity is assured. The question then arises as to what is the meaning of unitarity for negative norm states and in fact, why is the \(S\) operator unitary? The first observation is that it is only the relative sign of the norms of the discrete and continuum states that is important. Second we have shown for the usual case and the proof still holds in the present case, that energy is conserved in the process. Therefore if we start with a given energy, say in the continuum, during the collision we have virtual transitions between states of opposite norm that produce changes in the wave function, but after a sufficiently long time, all components with energies different from the initial disappear and we are left with states having the same energy and therefore norm, as the initial one. This does not mean that there is no effect from the (opposite norm) intermediate states. On the contrary, they produce measurable effects through the shift in phase of
the wave, which is in the end the observable effect. We remark that it is through this "energy quarantine" (i.e., that states of opposite norm appear with different values for the energy) that unitarity is preserved in this model.

Where \( g^2(\omega) \) becomes large, the zero of \( a(z) \) moves along the real axis until it reaches the cut. For larger values of \( g^2(\omega) \) instead of one real zero, \( a(z) \) acquires a pair of zeros for complex conjugate values of \( z \) with \( \text{Re} \{ z \} > 1 \). The corresponding states belong still in the discrete part of the spectrum. They have zero norm but nonvanishing scalar product. The S matrix however is still unitary because the energy quarantine remains unbroken.

iii. Orthonormality and Completeness

We sketch here the proofs of orthonormality and completeness for the case of the indefinite metric.

\[
\int \phi_\lambda^\dagger (\omega) \phi_\lambda (\omega) d\omega = \frac{g(\lambda)g(\lambda')}{\alpha(\lambda-i\varepsilon)\alpha(\lambda+i\varepsilon)} \int \frac{g^2(\omega)d\omega}{(\lambda-\omega-i\varepsilon)(\lambda'-\omega+i\varepsilon)}
\]

\[
= \frac{g(\lambda)g(\lambda')}{(\lambda-\lambda'-i\varepsilon)\alpha(\lambda-i\varepsilon)} + \frac{g(\lambda)g(\lambda')}{(\lambda'-\lambda+i\varepsilon)\alpha(\lambda'+i\varepsilon)}
\]

\[
+ \delta(\lambda-\lambda') = -\frac{g(\lambda)g(\lambda')}{\alpha(\lambda-i\varepsilon)\alpha(\lambda'+i\varepsilon)} + \delta(\lambda-\lambda') \quad (7.26)
\]

Where we have used (7.20) to compute the integral on the righthand side. Replacing (7.26) and (7.10) in (7.12) we obtain the orthonormality relation. For the completeness relation we consider
\[ \int_1^\infty x^*_\lambda x^\lambda d\lambda = \int_1^\infty \frac{\gamma^2(\lambda)d\lambda}{\alpha(\lambda-i\epsilon)\alpha(\lambda+i\epsilon)} \]

\[ = \frac{1}{2\pi i} \int_1^\infty \left\{ \frac{1}{\alpha(\lambda-i\epsilon)} - \frac{1}{\alpha(\lambda+i\epsilon)} \right\} d\lambda \]

\[ = \frac{-1}{2\pi i} \int_\Gamma \frac{1}{\alpha(z)} dz \]

where the contour \( \Gamma \) is indicated in figure (7.4). The function \( \frac{1}{\alpha(z)} \) is analytic in the cut plane, except for a pole at \( z = \Lambda \). The residue at the pole is \( 1/\alpha'(\Lambda) \). By an appropriate shift of the contour we can pass from \( \Gamma \) of Figure 7.4 to \( \Gamma' \) and \( \Gamma'' \) of Figure 7.5. The integral can be then computed by standard residue methods to give

\[ \int_1^\infty x^*_\lambda x^\lambda d\lambda = 1 - \frac{1}{\alpha'(\Lambda)} \]

The rest of the proof was essentially the same tricks for the other integrals.

VIII. TWO-LEVEL SYSTEM COUPLED TO BOTH NORMAL AND ABNORMAL OSCILLATORS

The results of the previous section are easily generalized to include more independent oscillators. For example, we could think that the fermion is coupled to one continuous set of oscillators whose frequencies go from a value \( \mu \) to infinity and also to another independent set with frequencies going from \( \mu' \) to infinity.
The corresponding Hamiltonian would be:

\[ H = \frac{1+\alpha}{2} M + \int_{\mu}^{\infty} \omega a^+ (\omega) a (\omega) d\omega + \int_{\mu}^{\infty} g(\omega) \left[ a^+ (\omega) \sigma_+ a (\omega) \sigma_+ \right] d\omega \]

\[ + \int_{\mu}^{\infty} \nu a^+ (\nu) a (\nu) d\nu + \int_{\mu}^{\infty} h(\nu) \left[ a^+ (\nu) \sigma_+ a (\nu) \sigma_+ \right] d\nu \]  \hspace{1cm} (8.1)

where \( a(\omega) \) and \( a_1(\nu) \) describe independent oscillators.

As a physical example of such a situation one could consider the \( \Lambda^0 \) particle. It is coupled to the \( \bar{K}p \) system in the reaction

\[ \bar{K}p + \Lambda^0 \rightarrow \bar{K}p \]

but we also have the reaction

\[ \bar{K}^0 \eta + \Lambda^0 \rightarrow \bar{K}^0 \eta \]

The solution of the eigenvalue-eigenstate problem for (8.1) can be obtained by a simple generalization of the methods of section 7. Again we concentrate our attention on the simplest non-trivial sector. This contains three types of states: the discrete state with energy \( M \) and no excitations, the states with just one excitation of the first type and the states with just one excitation of the

*We do not consider terms giving a direct interaction between the oscillators as it does not add to the physics of the problem.
second type. In the process we would have to define a function

\[ a(z) = z - M - \int_{\mu}^{\infty} \frac{g^2(\omega)}{z-\omega} \, d\omega - \int_{\mu}^{\infty} \frac{h^2(\omega)}{z-\omega} \, d\omega \quad (8.2) \]

An eigenstate of \( H \) would then be characterized by three functions: \( \chi, \phi_\lambda(\omega) \) and \( \psi_\lambda(\nu) \). The discrete state is again related to the zero of \( a(\lambda) \), but we are more interested in the solutions in the continuum. For reasons that will become clear later we will be interested in describing a state that for large negative values of \( t \) contains just one excitation with well defined energy in the channel that we have designated by \( \phi \). The corresponding wave functions are

\[ \chi_\lambda = \frac{g(\lambda)}{a(\lambda - i\varepsilon)} \]

\[ \phi_\lambda(\omega) = \delta(\lambda - \omega) + \frac{g(\lambda)g(\omega)}{a(\lambda - i\varepsilon)(\lambda - \omega - i\varepsilon)} \quad (8.3) \]

\[ \psi_\lambda(\omega) = \frac{g(\lambda)h(\omega)}{a(\lambda - i\varepsilon)(\lambda - \omega + i\varepsilon)} \]

For very large positive \( t \), the time dependent form of \( \phi_\lambda \) will be

\[ \tilde{\phi}_\lambda(t) = e^{-i\lambda t} \left( 1 + \frac{2i\pi g^2(\lambda)}{a(\lambda - i\varepsilon)} \right) \quad (8.4) \]

Previously, the factor in parenthesis was a unimodular quantity. But now since
\[ a(\lambda + i\varepsilon) - a(\lambda - i\varepsilon) = 2\pi \text{ig}^2(\lambda) + 2\pi \text{ih}^2(\lambda) \] (8.5)

the factor in question is a complex number of modulus less than one unless \( h^2(\lambda) \) happens to be zero. This result in fact was to be expected, since for any energy \( \lambda \) such that \( h^2(\lambda) \neq 0 \) there is a coupling to the second type of oscillators and therefore there is the possibility of observing a corresponding excitation in the final state. Since overall probability (i.e., normalization) is conserved, this net outgoing flux of excitations of the second type has to be balanced by a decrease in the observed flux for those of the first type. This is equivalent to stating that the phase shift for \( \phi_{\lambda} \) is not real, i.e., there is absorption due to the inelasticity of the problem.

We can now ask what happens if the second oscillator is of the abnormal type. First \( H \) would be written as

\[
H = \frac{1+\sigma^3}{2} M + \int_{\mu}^{\infty} \omega \sigma^\dagger(\omega)a(\omega)d\omega + \int_{\mu}^{\infty} g(\omega) [a^\dagger(\omega)\sigma_+ + a(\omega)\sigma_-]d\omega
\]

\[- \int_{\mu}^{\infty} \psi^\dagger(\nu)b(\nu)d\nu + \int_{\mu}^{\infty} h(\nu) [b^\dagger(\nu)\sigma_+ + b(\nu)\sigma_-]d\nu \] (8.5)

The sector considered is similar to the one just discussed, but because of the indefinite metric associated with the \( b \)-type oscillators, we would have

\[
a(z) = z - M - \int_{\mu}^{\infty} \frac{g^2(\omega)}{z-\omega} d\omega + \int_{\mu}^{\infty} \frac{h(\nu)}{z-\nu} d\nu \] (8.6)

and the solutions would be
\[ x_\lambda = \frac{g(\lambda)}{a(\lambda-i\varepsilon)} \]

\[ \psi_\lambda(\omega) = \delta(\lambda-\omega) + \frac{g(\lambda)g(\omega)}{a(\lambda-i\varepsilon)(\lambda-\omega-i\varepsilon)} \]

The functions \( \psi_\lambda \) correspond to the contribution of the negative norm states. The energy quarantine fails for \( h^2(\lambda) \neq 0 \), and this has some startling effects. We consider again the behavior for \( t \to \infty \) and find again (8.4).

But now

\[ a(\lambda+i\varepsilon) - a(\lambda-i\varepsilon) = 2\pi i g^2(\lambda) - 2\pi ih^2(\lambda) \quad (8.7) \]

and therefore the factor

\[ 1 + \frac{2\pi i g^2(\lambda)}{a(\lambda-i\varepsilon)} \]

is a complex number of modulus larger than one! Viewed as a whole the situation is this; we imposed boundary conditions at \( t \to -\infty \) such that only positive norm states appeared in the preparation of the experiment. When we measure the outgoing flux at \( t \to +\infty \) we find more than what we put. Since the overall normalization is conserved this excess flux must be balanced by the "negative probability flux" contribution of the states with negative norm. Of course this only means that there is nothing wrong in the
mathematics, but the results are not physically acceptable.
Since the problem seems to be in the failure of the
quarantine we could think of several quick solutions.
For instance we could shift $\mu'$ to very large values.* A
much more clever solution is obtained by considering two
abnormal oscillators instead of one. If we make the in-
tegration ranges complex conjugate† of each other (see
Figure 8.1), the hermiticity of $H$ will not be destroyed
but now we have again "energy quarantine". The paths of
integration need not be straight lines, just complex conju-
gate of each other. This solution gives the right results
in this sector. However, it cannot work in a quantum
field theory (or in fact in higher sectors), because there
we can have states that contain parts of excitations with
complex conjugate energies. For these states the cor-
responding energy is real (equal to twice the real part of
any one of the excitations energy) and the quarantine is
again broken. We see then that in all these attempts at
looking only at the positive norm part of the space of
states we fail to produce a scheme that could be used in
the construction of a consistent quantum field theory. It
will be shown in the next section that a solution to this
problem can be found by simply giving up some of the usual
ideas about the analyticity of the $S$-matrix.

*A value say equal to ten times the largest energy acces-
sible to experiment at any given time.

†Although the idea of an oscillator with a complex frequency
may be startling there is clearly nothing wrong with it.
In the application of indefinite metric concepts to quantum mechanics, we are interested in the states with negative norm as intermediate states because, as we have discussed, they give contributions of the right sign to cancel the divergences. But if they are allowed to appear as final states, then problems arise in the probabilistic interpretation of the results. We have seen that simply excluding the negative norm states from the initial state is not the answer because the interaction can create a net flux for the negative norm excitations. If we look back at (8.3) we see that the net flux of the \( \psi_\lambda \) states could be made zero if we changed, for that portion of the solution, the \(-i\varepsilon\) prescription into a principal value because this would be accomplished by considering instead of the \( a(z) \) function

\[
\beta(\lambda) = \lambda - M - \int_\mu^\infty \frac{g^2(\omega)}{\lambda - \omega - i\varepsilon} + \int_\mu^\infty \frac{h^2(\omega)d\omega}{\lambda - \omega}
\]

(9.1)

We would then get

\[
\chi_\lambda = \frac{g(\lambda)}{\beta(\lambda)}
\]

\[
\phi_\lambda(\omega) = \delta(\lambda - \omega) + \frac{g(\lambda)g(\omega)}{\beta(\lambda)(\lambda - \omega - i\varepsilon)}
\]

(9.2)

\[
\psi_\lambda(\nu) = P \frac{g(\lambda)h(\nu)}{\beta^*(\nu)(\lambda - \nu)}
\]
With this choice we now have

\[ \beta^*(\lambda) - \beta(\lambda) = 2\pi i g^2(\lambda) \quad (9.3) \]

and therefore for large positive \( t \), \( \tilde{\phi}_\lambda(t) \) has the form

\[ \tilde{\phi}_\lambda(t) = e^{-i\lambda t} \frac{\beta^*(\lambda)}{\beta(\lambda)} \equiv S(\lambda)e^{-i\lambda t} \quad (9.4) \]

In other words, since \( \beta^*(\lambda)/\beta(\lambda) \) is a unimodular complex number, the flux is conserved for the positive norm states and the unitarity of the S matrix is consistent with the probability interpretation. The negative norm states are there, they contribute to the interaction, but the only fluxes observed are those corresponding to the normal excitations.

However, in setting up these kinds of boundary conditions we have departed rather drastically from the usual assumptions concerning the analyticity of the scattering amplitude. To clarify this we examine the properties of the function \( \beta(\lambda) \). We first notice that this function is actually defined only for real values of \( \lambda \). For \( \lambda \) less than \( \mu' \), \( \beta(\lambda) \) can be thought of as the boundary value of the analytic function

\[ A(z) = z - M - \int_{\mu}^{\infty} \frac{g^2(\omega)}{z - \omega} \, d\omega + \int_{\mu'}^{\infty} \frac{h^2(\omega)}{z - \omega} \, d\omega \quad (9.5) \]

i.e., for \( \lambda < \mu' \) (\( \lambda \) real)

\[ \beta(\lambda) = A(\lambda - i\epsilon) \]
But since

\[ \frac{1}{\lambda - \omega - i \epsilon} = P \frac{1}{\lambda - \omega} + i \pi \delta(\lambda - \omega) \]

we find for \( \lambda > \mu' \)

\[ \beta(\lambda) = \Lambda(\lambda - i \epsilon) - i \pi h^2(\lambda) \]  

(9.6)

If \( h^2(\lambda) \) is analytic for \( \lambda > \mu' \) then the right-hand side of (9.6) is the boundary value of the function

\[ B(z) = \Lambda(z) - i \pi h^2(z) \]

What we find there is that \( \beta(\lambda) \) cannot be written as the boundary value of a single analytic function, but rather we have to consider several functions (which could eventually be analytic in the whole cut plane) in such a way that \( \beta(\lambda) \) is the boundary value of one of these functions in one segment of the real-\( \lambda \)-axis and of another in a different segment. From phase space considerations \( h^2(\lambda) \) should vanish at the threshold \( \lambda = \mu' \), and therefore \( \beta(\mu) \) (and correspondingly the scattering amplitude) varies in a smooth fashion with energy (Figure 9.1). At the point \( \lambda = \mu' \) there will be a discontinuity in the first derivative of \( \beta(\lambda) \) (cusp).* Experimentally this cusp would be

*This is true for an S-wave. For higher waves the discontinuity appears in higher derivatives.
small and difficult to distinguish from a completely smooth behavior.

Going back to the $S$-matrix we find

$$S(\lambda) = \begin{cases} \frac{A(\lambda+i\epsilon)}{A(\lambda-i\epsilon)} & \text{for } \mu > \lambda > \mu' \\ \frac{B(\lambda+i\epsilon)}{B(\lambda-i\epsilon)} & \text{for } \mu' < \lambda \end{cases}$$

but now we see that the functions $A$ and $B$ are not related by analytic continuation, it is no longer possible to decide the high energy behavior from the low energy data. This clearly will have an important bearing in the theory of strong interactions which we will discuss later on.

We note that the new scattering amplitude introduced here is consistent with our general rule for obtaining it from a standard theory by making the second continuum channel a shadow channel.

X. **INTRODUCTION TO QUANTUM FIELD THEORY WITH INDEFINITE METRIC SUPERCONVERGENT PROPAGATORS**

So far we have discussed only simple models. We wanted to learn as much as possible about the structure of indefinite metric theories. Let us now aim at the construction of convergent quantum field theories. To proceed in our discussion we consider the different aspects of field theory that are relevant to physical problems. There are three uses for fields. The simplest appears in
connection with the quantum mechanical description of collections of free identical particles. A correspondence is found there between the number of particles present and the degree of excitation of (a continuum of) oscillators. Furthermore, according to whether the particles are of the Bose or the Fermi type the corresponding creation and annihilation operators for the oscillators obey commutation or anticommutation relations, giving rise to the symmetry or antisymmetry of the related wave functions. In this sense, fields are very useful, because they automatically give the correct bookkeeping for the number of states available to a collection of identical particles of a given kind. A second use is the construction of the dynamical variables for free particles. This allows passage from a static description to a kinematical one, since it is now possible to construct equations of motion according to the rules of quantum mechanics. No problems arise in implementing these two uses for fields.

The difficulties appear when one tries to use fields as the carriers of interactions, in the manner indicated by the pioneering work of Yukawa. In setting up the theory one finds that emission and absorption (either real or virtual) are no longer independent. Particles get related to antiparticles in a simple way. But to outweigh these nice features one finds also that the moment an interaction is introduced, all measurable quantities become infinite! The origin of these infinities can be found in the self-interaction of the sources of the field with the
field that they produce through the emission and reabsorption of virtual quanta. If we compute the field produced by a particle as a function of the distance $r$, we find typically a $1/r$ behavior as we approach the particle, and it is this fast increase in the strength of the field that leads to divergences. Or, expressed in terms of the Fourier transform of the field amplitude we find a $1/k^2$ behavior when $k^2$ ($\vec{k}$ being the momentum) goes to infinity, and this implies to slow a decrease of the large energy components. To make this more clear we take the example of a point source located at the origin of coordinates. Then

$$ (\Box^2 + m^2)\phi(x) = \delta(x) \quad . \tag{10.1} $$

The solution in momentum space looks like

$$ \hat{\phi}(p) = \frac{1}{m^2 + p^2} \quad , $$

or we could also say that the corresponding Green's function has the behavior

$$ G(p) \propto \frac{1}{p^2} \text{ when } p^2 \to \infty \quad . \tag{10.2} $$

Clearly, something must be done about the large momentum components if we want to have finite measurable quantities. Since we do not really want to change the structure of (10.1) we must ask what can be done to change the sign of $1/p^2$ in (10.2). The sign of the $m^2$ term is not important
for $p^2 \rightarrow \infty$ so we must change the sign of the D'Alembertian. This in fact amounts to a change in sign of the Hamiltonian and this, as we have seen, is precisely the effect obtained by introducing an indefinite metric in the theory. In order to get the right answers the corresponding indefinite metric field would have to have the same space-time structure as the original positive metric one. Furthermore, it should be coupled to the same sources and with the same coupling constant. In this sense, the physical field, i.e. the carrier of interactions, would be a linear combination of a part corresponding to particles and another corresponding to antiparticles. The masses for these fields should be chosen different, for the interactions of the physical field (and in particular the self-interaction) would be characterized by a propagation function of the form

\[
\frac{1}{m^2 - p^2} - \frac{1}{M^2 - p^2}
\]

Clearly for $p^2 \rightarrow \infty$ their combination decreases as $1/p^4$. We call this a "superconvergent propagator" because of its improved convergence properties. In particular we now find that the self energy associated with virtual emission and reabsorption by a point source is finite. In configuration space the associated field has nice properties in the vicinity of the source. It should be noticed that this is not a perturbation type of argument. It is rather a general statement about the possibility of constructing
field theories in which the self-energy of point sources is finite.

In a quantum field theory however, the propagation functions appear in relation to vacuum expectation values of time ordered products, i.e., one is interested in expressions of the form,

$$<0|T(\phi(x)\phi(y))|0>$$

In computing this quantity the only non-vanishing contributions* come from terms that have an annihilation operator to the left and creation operator to the right. From the commutation relations this can be written as the commutator plus the product of the same operators in the reversed order. But again, since the latter is the vacuum expectation value of a normal ordered product, its contribution vanishes. So the only non-vanishing contribution comes from the commutator itself. But then we immediately notice that the only difference between a usual field and an indefinite metric field is precisely the sign of the commutator. Therefore the contraction function for these fields will be similar in structure but of opposite signs. We can define a field

$$\phi = \phi^{(1)} + \phi^{(2)}$$

(10.3)

*In perturbation theory.
where $\phi^{(1)}$ is of the usual and $\phi^{(2)}$ of the abnormal type. If we consider now the same contraction function but for $\phi(x)$, we have

$$\langle 0 \mid T \left( \phi(x), \phi(y) \right) \mid 0 \rangle =$$

$$\langle 0 \mid T \left[ \left( \phi^{(1)}(x) + \phi^{(2)}(x) \right) \left( \phi^{(1)}(y) + \phi^{(2)}(y) \right) \right] \mid 0 \rangle$$

The fields $\phi^{(1)}$ and $\phi^{(2)}$ commute for equal times (and in general for space-like separation) and therefore (for equal times)

$$\langle 0 \mid T \left( \phi(x) \phi(y) \right) \mid 0 \rangle =$$

$$\langle 0 \mid T \left( \phi^{(1)}(x) \phi^{(1)}(y) \right) \mid 0 \rangle + \langle 0 \mid T \left( \phi^{(2)}(x) \phi^{(2)}(y) \right) \mid 0 \rangle$$

(10.4)

But the leading singularities of the two-point functions are known to be independent of the mass of the particles, and therefore they will cancel on the righthand side of (10.4) because of the change in sign that we indicated before. This decrease in the singularity of the two-point function must be related in momentum space to a faster decrease of the corresponding Green's function. For the scalar field, it turns out that (10.3) is all that is required to obtain a superconvergent propagator. In particular, this Fourier transform at $x = 0$, related to the vacuum expectation value
both $\psi_2$ and $\psi_3$ to be of the indefinite metric type with $c_1$, $c_2$ and $c_3$ and $m_1$, $m_2$ and $m_3$ related by

\[
\frac{c_2}{c_1} = \frac{m_1 - m_2}{m_3 - m_2}; \quad \frac{c_3}{c_1} = \frac{m_1 - m_2}{m_2 - m_3}
\]

The $1/k^3$ behavior leads to a $\log r$ behavior near a point fermion source. This divergence is, however, mild and does not introduce difficulties.

i. **Relation to Bhabha's Proposal for the Use of Higher Order Equations**

Recognizing the basic lack of consistency of standard local interaction quantum field theories Homi Bhabha had proposed the use of higher order field equations. He suggested that we make use of fields satisfying free equations of the form

\[
(\Box + m^2)^2 \phi = 0
\]

or

\[
(\Box + m_1^2)(\Box + m_2^2) \phi = 0.
\]

The corresponding Green's functions show a more rapid convergence of the form

\[
G(k) \sim \frac{1}{(k^2 - m_1^2)(k^2 - m_2^2)}
\]
is now well defined, and therefore in this theory the product of the field operators at any pair of space-time points is a well defined expression.

For fermion fields, the propagator is of the form

$$\frac{1}{\gamma \cdot p - m}$$

and it behaves as $1/p$ for large values of $p$. If we added another field (of the indefinite metric type) the asymptotic behavior would change to $1/p^2$, but this is still too slow. So we have to consider more fields. The simplest useful choice is a combination of three fields, at least one of which is of the abnormal type. The relative weights of the fields are not completely determined but we find that to get improved convergence we have to impose constraints relating the masses of the fields and their relative weight. If we define the (physical) field by

$$\bar{\psi}(x) = c_1 \psi_1(x) + c_2 \psi_2(x) + c_3 \psi_3(x) \quad (10.5)$$

we get an effective propagator of the form

$$S_{\text{eff}}(k) \propto \frac{|c_1|^2}{\gamma \cdot k - m_1} \pm \frac{|c_2|^2}{\gamma \cdot k - m_2} \pm \frac{|c_3|^2}{\gamma \cdot k - m_3}$$

where the minus sign corresponds to indefinite metric.

We can arrange to have a $1/k^3$ behavior if we choose
so that integrals would tend to converge. This is in fact
the kind of superconvergence requirement that we have
pointed out was necessary to obtain finite quantum field
theories but we had to use an indefinite metric. How does
this come about?

Now the higher order equation is satisfied by a
generalized free field

\[ \phi = \phi_1 + \phi_2 \]

\[ (\Box + m_1^2)\phi_1 = 0; \ (\Box + m_2^2)\phi_2 = 0 \]

independent of whether \( \phi_1 \) and \( \phi_2 \) are both positive metric
fields or one of them has a negative metric. However, if
we try to write down a Lagrangian in terms of \( \phi \) alone to
get the higher order equation as the Euler-Lagrangian equa-
tion then the indefinite metric is inevitable.\(^{(13)}\)

Given such a field we have a number of free quanta
that emerge from it, one corresponding to each mass
(suitably renormalized by finite amounts). So equipped the
Bhabha fields are the carriers of interaction some of which
is carried by shadow quanta and some by physical quanta.
If we so choose the masses may be complex.

The squared Klein-Gordon equation is less satis-
factory. It has been used by Heisenberg in his dipole
ghost theory, but I believe that the method has nothing to
endear it to us.
ii. Perturbation Theory

We have shown how the introduction of fields with an associated indefinite metric into the framework of the usual quantum theory of fields leads to unambiguous expressions for the products of field operators as well as to a finite amount of self-interaction for a point source, these statements being of a very general nature. Equipped with this knowledge, it is natural to attempt to establish a perturbation theoretic framework in which relevant matrix elements can be computed to the desired degree of accuracy. After all, perturbation theory is the only known method for actually solving the dynamical problem. Since, as we have indicated, there are no intrinsic ambiguities or infinities in the theory, there is no a priori reason for disregarding a series expansion of the scattering amplitude in powers of the coupling constant as a means of solving the problem.

We notice that, although we have no assurance that the perturbative series will converge, whether it is convergent or not is not something that can be discussed on the basis of arguments about the possible analyticity of the resulting function at the point in which the coupling constant is zero. As an example we take the equation

\[
\frac{1}{|1-x|} = \sum_{n=0}^{\infty} x^n
\]

which is valid for \(x\) positive and less than one. The left-hand side is clearly not analytic at \(x = 0\), but that does
not mean that we cannot write a power series representation valid in appropriately chosen intervals.

We shall base the perturbation calculation on the usual Feynman formalism. A line here represents the propagation of the physical field, since the coupling schemes of different fields must necessarily maintain their structure. In other words, when we talk, let us say, about a Yukawa type of coupling, we mean the expression

\[ g \bar{\psi} \psi \phi \]  \hspace{1cm} (10.6)

where \( g \) is the coupling constant and \( \psi \) and \( \phi \) have the structure given by (10.5) and (10.3) respectively. The graphical representation of (10.6) is then simply as given in Figure 10.1(a). Figure 10.1(b) represents a correction to the fermion propagator and Figure 10.1(c) a "box" diagram in the scattering amplitude for the process

\[ \phi + \psi \rightarrow \phi + \psi \]

The problem can be stated in more general terms. The contribution from a graph with \( r \) circulating momenta is in general of the form

\[ M = \int \frac{N}{D} d^4k_1 d^4k_2 \cdots d^4k_r \]  \hspace{1cm} (10.7)

In the usual formulation we have in \( D \) a factor \((\phi - m)\) for every internal fermion line and a factor \((p^2 - m^2)\) for
every internal boson line. For Yukawa couplings, we may consider \( N \) as a constant. If we apply these rules to graph (b) of Figure 10.1, we have one circulating momentum \( k \), one fermion propagator and one boson propagator. For large \( k \) the integrand goes as \( 1/k^3 \) and the prescription (10.7) is meaningless. Other divergent graphs are indicated in Figure 10.2. For graph (a), we would have

\[
\int \cdots \frac{d^4k}{k^2} \nu \int \cdots kd\kappa
\]

and therefore it diverges quadratically. The "degree" of divergence of graph (b) is even higher.

It turns out that one can formulate general expressions using, which one can decide from the topology of the graph, what is its degree of divergence.\( ^{(14)} \) When we replace the usual propagators by the "effective" propagators of the theory with an indefinite metric, we increase the number of powers of \( k \) in the denominator and therefore improve the convergence of the integrals. In fact with the choices indicated and for Yukawa coupling there are no divergent graphs in the theory. To prove this assertion we use the relations

\[
\Gamma = F_i + P_i - (n - 1)
\]

\[
D = 3F_i + 4P_i
\]

where \( \Gamma \) is the number of independent circulating momenta,
$F_i$ and $P_i$ the number of internal fermion and boson lines respectively, $n$ is the number of corners and $D$ is essentially the number of powers of $k$ in the denominator. The integral in (10.7) will be convergent then if

$$4T < D \quad (10.7)$$

But for Yukawa couplings we have

$$F_i + \frac{1}{2}F_e = n$$
$$2P_i + P_e = n$$

where $F_e$ and $P_e$ correspond to the numbers of external lines.

XI. APPLICATION TO QUANTUM ELECTRODYNAMICS AND TO WEAK INTERACTIONS

In the previous section we have investigated the implications of the improved behavior of the superconvergent propagators for large momenta. To obtain this improved behavior we had to introduce new parameters into the theory. These parameters are by no means arbitrary or devoid of physical interpretation. The satellite terms appearing in (10.3) and (10.5) give rise to singularities (poles) of the scattering amplitude for values of $k$ equal to the mass squared of the corresponding (indefinite metric) field and these will certainly result in a
Distributive behavior that can be observed experimentally.

Undoubtedly, the first place where one should check any field theory is in quantum electrodynamics, because there is where we have the most accurate data. Consider for example Compton scattering. The Klein-Nishina formula is obtained from the graphs of Figure 11.1. But since the propagator contains, besides the electron mass, the masses of the satellite terms, we must make sure that we regain the observed values for the cross sections, etc. Clearly, if $m_1$ and $m_2$ are sufficiently large, nothing much happens for small energies.* However, if the energy is of the order $m_1$ (or $m_2$) we should expect to see a resonance. This would be a lepton resonance, intrinsic to the theory and not the result of binding. As a consequence, the corresponding poles would appear on the physical sheet. Therefore a phase shift analysis of the resonance would show that in the Argand diagram, the phase decreases with energy instead of increasing as for a pole on the second (unphysical) sheet. If the masses $m_1$ and $m_2$ are real we expect the interaction to shift their values by an imaginary amount of the order of $\alpha$, the fine structure constant, and therefore for real satellite masses the experimental data would have to show rather sharp peaks. Another possibility which follows from our discussion about unitarity and

*A calculation shows that there is agreement with experimental evidence on Compton scattering if $m_1$ and $m_2$ are of the order or larger than about ten electron masses.
"energy quarantine", is that $m_1$ and $m_2$ be complex conjugate of each other to begin with. Certain advantages of using such complex masses are discussed in a subsequent section. In that case, the observed imaginary part will be much larger and correspondingly the resonance much broader. In the case of weak interactions we have the four-fermion V-A type of coupling. Typical graphs are shown in Figure 11.2. Graph (a) gives the lowest order contribution to lepton-lepton scattering; while graph (c) gives a second order correction to the process. Since the propagators behave as $1/k^3$ this correction is finite. Graph (c) does not contribute in the case of V-A coupling.

It is important to notice that once we have established the form of the propagator from say Q.E.D., the same form has to be used for that lepton field in weak interactions. If a resonance is found in experiments in Q.E.D. we should find the same resonances in experiments involving weak interactions.\(^{(15)}\)

XII. QUANTUM FIELD THEORIES OF STRONG INTERACTIONS; LOW ENERGY PION-NUCLEON SCATTERING

The general theoretical framework that we have been discussing would be equally valid in the case of strong interactions also. If we want a local field theory of strong interactions then we must use an appropriate indefinite metric to make the theory finite and thus not inconsistent.
The significant difference comes in when we have to make quantitative calculations in such a theory. For electromagnetic interactions and for weak interaction, we could make perturbation calculations, which are now term-by-term finite, and then compare these results with experiment. In the case of strong interaction this is no longer a satisfactory method. We must make use of other methods.

One method that we can use is to make a model of strong interactions that can be explicitly solved. In the mid-nineteen fifties such a program was initiated by Chew and Low in the study of low energy pion-nucleon scattering. They did the following mutilations of the theory.

(1) The nucleon is treated as a nonrelativistic particle which is infinitely heavy: its only degrees of freedom are then related to its (nonrelativistic) spin and isospin.

(2) The meson is treated relativistically as far as its energy and momentum are concerned but only states with one or zero mesons are considered.

(3) An essential feature of a relativistic field theory is its "crossing symmetry", stemming basically from the symmetry between emission and absorption. This is translated into their model by requiring the scattering amplitude to have the crossing property:

\[ T_{\alpha\beta}(z) = T_{\beta\alpha}(-z) \]
the amplitude for scattering from meson of type \( \alpha \) to type \( \beta \) at complex energy \( z \) is the same as the amplitude for scattering of meson of type \( \bar{\beta} \) to type \( \bar{\alpha} \). (Here \( \bar{\beta}, \bar{\alpha} \) involve reversal of charge etc. corresponding to "crossing". The meson of type \( \bar{\beta} \) transforms as the complex conjugate of meson of type \( \beta \).) This crossing symmetry may be thought of as a contribution of Bose statistics and of the emission-absorption symmetry.

Restricting themselves to such a model the orbital angular momentum of the pion is a constant of motion: so one can talk about s-wave scattering, p-wave scattering, etc. At low energies only the lowest partial waves will contribute. Because of the odd intrinsic parity of the pion the simplest parity conserving interaction that can be written is of the form

\[
g \mathbf{\bar{g}} \cdot \nabla (\mathbf{\bar{r}} \cdot \mathbf{\bar{\phi}})
\]

where \( g \) and \( \mathbf{\bar{g}} \) refer to the spin and isospin of the nucleon. Thus for the Yukawa coupling only p-waves are coupled. We can of course write down bilinear s-wave interactions of the form

\[
h_1 (\mathbf{\bar{\phi}} \cdot \mathbf{\bar{\phi}}) + h_2 (\mathbf{\bar{r}} \cdot \mathbf{\bar{\phi}} \times \mathbf{\bar{\phi}})
\]

The s-wave and p-wave scattering can be treated independently.

Chew and Low restricted their attention to p-wave
stem from the structure of the p-wave interaction, but since they were unable to solve for the scattering amplitude they sought to determine an analytic scattering amplitude satisfying all the above requirements from a knowledge of its analytic structure.

They argued as follows: The conservation of probability implies that the imaginary part of the $l$th section for the $l$th partial wave. Hence the partial wave amplitude, considered as a real analytic function of the (complex) energy must have a branch cut for all (real) physical energies $\mu < v < \alpha$. By crossing symmetry we must also have a branch cut for the range $-\infty < v < -\mu$. There should be a pole at $v = 0$ due to the nucleon "bound state" of the pion and the nucleon. They then sought to find such a function. Now we know that such a function is not uniquely determined by these criteria, but in a certain sense there is a "simplest solution". In their calculation Chew and Low showed that the problem did not make any sense
unless the meson-nucleon interaction was "cut off", that is
to say that the interaction was weaker for higher momentum
mesons. Accordingly one replaces the p-wave interaction by

\[ g f(-\nu^2) \sigma \cdot \overline{V}(\tau \cdot \phi) \]

with \( f(k^2) \) acting as the "form factor" for the interaction.
It may be thought of as an "extended nucleon". In terms of
such a model Chew and Low were able to obtain considerable
quantitative success in explaining the gross features of
low energy pion-nucleon scattering including the \( \frac{3}{2} \) iso-
spin \( \frac{3}{2} \) resonance in the neighborhood of 200 Mev.

Within the framework of our insights into the
structure of convergent local field theories we observe the
following: (i) The crossing symmetry automatically fol-
lows, in any properly formulated relativistic theory. A
Klein-Gordon field has both positive and negative frequency
solutions! (ii) The nucleon pole comes automatically in
the "Lee" models that we have showed at the earlier lec-
tures. (iii) Unitarity is automatically satisfied in a
suitably formulated shadow state theory with indefinite
metric. (iv) Our theories are automatically finite without
cut-offs; instead of cut-offs we have shadow satellite
fields.

We also know that the scattering amplitudes are not
analytic globally but are piecewise analytic; so we cannot
go the way of Chew and Low. But why should we? We can, in
the approximation of no more than one meson solve the
problem completely in closed form. The general mathematical apparatus is the same as the one developed for the model of a 2-level system coupled to a continuum of oscillators (the "Lee" model).

Such a theoretical model has been constructed and computations on it carried out by Nelson and Sudarshan.\(^{(11)}\)

The interaction is of the form:

\[
g g \cdot \nabla \tau \cdot (\phi + \phi')
\]

where \(\phi'\) is an isotopic spin one pion field with positive metric and \(\phi'\) a satellite field with negative metric. A detailed presentation of this work would take up too much time; I shall content myself by saying that a good quantitative understanding of low energy pion-nucleon scattering can be obtained in terms of the explicit solution of this model. This is quite encouraging since this model contains no arbitrary cut-off functions or ambiguities. For more details including a discussion of cusps (see below), you are referred to the papers of Nelson and Sudarshan.

XIII. SOME CONSEQUENCES OF A PIECE-WISE ANALYTIC FUNCTION

Not all the time can we rely on completely solved models. [It is sometimes said: "Anything completely solvable is trivial"!] What can we say in general? We already know the following: Local (polynomial)
interactions of fields, which are operators in a Hilbert space are meaningless because of intrinsic infinities. We shall see later that any attempt to impose geometrical (nondynamical) cut-offs is also inconsistent. The natural way out is to make use of spaces with indefinite metric which naturally provide a dynamical cut-off. To restore the quantum-mechanical probability interpretation in such a theory one must allow this vector space to have a positive metric "physical subspace". The other states are "shadow states" which contribute to dynamics and thus occur in virtual states but which do not enter the complete set of physical states. The notion of virtual quanta, exchanges, poles etc. remain valid in this theory; but since the "ghost" quanta cannot enter any physical state the scattering amplitude in different physical regions must be boundary values of different analytic functions. We must therefore automatically have piecewise analytic transition amplitudes in any consistent field theory.

If this is so, even if we cannot solve explicitly for the scattering amplitude we should be able to find experimental evidence for or against such a behavior. If our ideas of local interactions between quantized fields are correct such evidence must be there.

Let us consider a general elastic process described by the four momenta:

\[ P_1 + P_2 + P_3 + P_4 \]

\[ P_1^2 = P_3^2 = m_a^2; \quad P_2^2 = P_4^2 = m_b^2 . \]
\[ s = (p_1 + p_2)^2 \]
\[ t = (p_1 - p_3)^2 ; \quad u = (p_1 - p_4)^2 \]
\[ s + t + u = 2m_a^2 + 2m_b^2 . \]

Let \( G(s,t) \) denote the elastic scattering amplitude, assumed to be piecewise analytic. For simplicity we consider a single shadow channel threshold in each of \( s,t,u \) channels. We unite

\[ G(s,t) = F(s,t) + \Delta(s,t) \]

where \( F(s,t) \) is a globally analytic function and

\[ \Delta(s,t) = h_1(s,t)\theta(s-s_0) + h_2(s,t)\theta(t-t_0) + h_3(s,t)\theta(u-u_0). \]

Then inside the Mandelstam triangle labelled by \( s_0, t_0, u_0, \)
\( G(s,t) \) and \( F(s,t) \) coincide, but in the domain outside \( \Delta(s,t) \) represents the correction. We shall assume that \( F(s,t) \) is the analytic function that has the analyticity, crossing, and asymptotic properties of the Mandelstam representation with Regge pole asymptotics; and that it is \( F(s,t) \) which satisfies the usual dispersion relations attributed to the elastic scattering amplitude.

We now investigate several questions within this modified framework.

i. The Modified Dispersion Relations

The unsubtracted dispersion relation for \( F(v) = \)
\[
\text{Re } F_0(v) = \frac{1}{2}(1 \pm \frac{v}{\mu}) D_+(\mu) + \frac{1}{2}(1 \pm \frac{v}{\mu}) D_-(\mu)
+ \frac{\varepsilon^2 \mu^2}{m \nu B} \frac{q^2}{(\nu_B^2 - \mu^2)(\nu_B \pm \nu)}
+ \frac{q^2}{4\pi^2} \int_0^\infty \frac{d\nu'}{\nu'} \left\{ \frac{\Sigma(\nu')}{\nu' - \nu} + \frac{\Sigma(\nu')}{\nu' + \nu} \right\}.
\]

where

\[
D_+(\nu) = \text{Re } G_+(\nu).
\]

We can repeat Pomeranchuk's proof\(^{16}\) to obtain in the present context

\[
\Sigma_+(\infty) = \Sigma_-(\infty)
\]

which, in turn, implies

\[
\sigma_+(\infty) - \sigma_-(\infty) = \mu_-(\infty) - \mu_+(\infty)
\]

where

\[
\mu_-(\infty) = \lim_{\nu \to \infty} \frac{4\pi}{q} \text{Im } h_-(\nu).
\]

Hence unless these limits are the same, there is no reason to expect that particle and antiparticle cross sections would approach the same limit. As mentioned before, the experimental evidence at the present time is inconclusive, but suggests a distinct difference between particle and
antiparticle cross sections.

iii. Tests of Forward Dispersion Relations in High Energy Experiments

We said already under i that the dispersion relations are different for our amplitude as compared with the standard assumption. Yet the standard dispersion relations have been tested by many people, most recently by Lindenbaum.\(^{(17)}\) What about it?

In Lindenbaum's test the \(D^+(\nu)\) dispersion relations were once subtracted and the \(D^-(\nu)\) were unsubtracted but the Pomeranchuk theorem was assumed as a constraint. The largest disagreement occurred in \(D^-(\nu)\) dispersion relation with better agreement for \(D^+(\nu)\). However both the errors in the Serpukhov values for the cross sections and the errors in \(D_\pm(\nu)\) for large \(\nu\) make it difficult to consider this work to be either a proof of the standard relations or as evidence of piecewise analyticity.

For details of the analysis you should consult the paper of Gündzik and Sudarshan\(^{(18)}\) but the main conclusions are as follows:

(i) If the \(\pi^+p\) and \(\pi^-p\) cross sections are not equal at infinite energy then the dispersion relations for \(D_-(\nu)\) without subtractions cannot be tested, nor can \(\xi_-(\nu)\) be determined since the latter is very sensitive to the precise value of the assumed coupling constant \(f^2\).
$$f(v,0)$$ is given by

$$\text{Re } F(v) = \frac{\gamma}{\nu_B^2 - \nu^2} + \frac{1}{\pi} \int_{\nu_S}^{\infty} \frac{\text{Im } F(v')}{v' - \nu} \, dv' + \frac{1}{\pi} \int_{-\infty}^{\nu_U} \frac{\text{Im } F(v')}{v' - \nu} \, dv'. $$

Let us write functions simplifying the structure,

$$G(v) = F(v) + h(v) \{ \theta(v - v_0) + \theta(-v - v_0) \}$$

Then the elastic scattering amplitude satisfies the modified dispersion relations

$$\text{Re } G(v) = \frac{\gamma}{\nu_B^2 - \nu^2} - \frac{1}{\pi} \int_{\nu_S}^{\infty} \frac{\text{Im } G(v')}{v' - \nu} \, dv'$$

$$- \frac{1}{\pi} \int_{-\infty}^{\nu_U} \frac{\text{Im } G(v')}{v' - \nu} \, dv' = \xi(v)$$

where

$$\xi(v) = h(v) \{ \theta(v - v_0) + \theta(-v - v_0) \}$$

$$- \frac{q^2}{\pi} \int_{\nu_0}^{\infty} \frac{dv'}{(q')^2} \frac{v'}{v'^2 - \nu^2} \frac{\text{Im } h(v')}{v'^2 - \nu^2}.$$

The righthand side would normally have been expected to vanish.

ii. Modifications of the Pomeranchuk Theorem

Based on the forward (twice-subtracted) dispersion...
relations and the assumption that the ratio of the real to imaginary part of the forward amplitude does not grow as fast as the logarithm of the energy, Pomeranchuk demonstrated that if the cross sections at high energies approached a limit, it had to be the same for particle and for antiparticle scattering. This seemed to be in agreement with high energy scattering data, but recent experimental work at Serpukhov has raised questions about this conclusion. It would be unsatisfactory to give up the less-than-logarithmic limit on the growth of the ratio of real to imaginary part of the forward amplitude as it is closely related to the ratio of exponential fall off of the strong interaction between hadrons for large distances.

What does our theoretical framework say?

We could rewrite the dispersion relations for the extrapolated amplitude \( F(\nu) \) in the forms

\[
\text{Re } F_\pm(\nu) = \frac{f^2 \mu^2}{m v_B} \left( \frac{1}{v_B} \pm \frac{1}{\nu} \right) + \frac{1}{4\pi^2} \int_0^\infty \, d\nu' q' \left\{ \frac{\Sigma(\nu')}{\nu' - \nu} + \frac{\Sigma(\nu')}{\nu' \pm \nu} \right\}
\]

where

\[
\Sigma_\pm(\nu) = \sigma_\pm(\nu) - \frac{4\pi}{q} \text{Im } h_\nu(\nu)
\]

and \( \sigma_\pm(\nu) \) denotes the cross sections for \( \pi^\pm \) on protons with low energy amplitude \( F_\pm(\nu) \) at energy \( \nu \). The twice-subtracted form is
(ii) The tests for $D_\pm(\nu)$ which are less sensitive to the coupling constant indicate that it is somewhat more probable that $\varepsilon_+(\nu)$ is not zero but the large errors on this prevent any definite conclusion.

iv. Finite Energy Sum Rules

The finite energy sum rules, (19) are another test of analyticity for a function with Regge asymptotic behavior. If we have

$$f(\nu, t) = \frac{2\nu}{\pi} \int_0^\infty \frac{\text{Im} \, f(\nu', t)}{\nu'^2 - \nu^2} \, d\nu'$$

and

$$f(\nu, t) + f_R(\nu, t) = \frac{\beta_i(t) \cdot \{t \pm i \pi \alpha_i(t)\} \nu^\alpha(t)}{\sin \pi \alpha_i(t) \, (\alpha_i(t) + 1)}$$

with $\alpha_i < -1$, then

$$\frac{1}{N} \int_0^N \text{Im} \, f(\nu', t) \, d\nu' = \sum_i \frac{\beta_i N^{\alpha_i(t)}}{i \, \Gamma(\alpha_i + 2)}$$

where for $\nu > N$ the Regge form is taken to be more or less equal to the actual amplitude. More generally we may have

$$\frac{1}{N^{n+1}} \int_0^N \nu^n \text{Im} \, f(\nu, t) \, d\nu = \sum_i \frac{\beta_i(t) N^{\alpha_i(t)}}{i \, (\alpha_i + n + 1) \, \Gamma(\alpha_i + 1)}$$

So much for the usual case.
In our theory these forms have to be modified. If we take \( P(s,t) \) to be the amplitude that satisfies the Regge asymptotic behavior, the physical amplitude \( G(s,t) \) will satisfy the modified relation

\[
\frac{1}{n^{n+1}} \int_0^N v^n \text{Im} \ G(v,t) dv = \frac{1}{n^{n+1}} \int_{v_0}^N v^n \text{Im} \ h(v,t) dv + \sum_i \frac{\beta_i(t) N^{\alpha_i(t)}}{(\alpha_i + n + 1) \Gamma(\alpha_i + 1)}
\]

The first term is the correction, and it is zero for \( N \leq v_0 \). Hence as the upper limit \( N \) increases we should expect to get poorer agreement since the correction term grows with \( N \).

v. Local Tests. Cusps and Resonances

The dispersion tests are tests of global behavior. In addition to these we could look for the behavior of the function at special points or their vicinity. These require, of course, more precise data for comparison.

At the point of joining of the pieces of the piecewise analyticity there is expected to be continuity of the scattering amplitude. But for the s-wave amplitudes we would expect a discontinuous derivative. [For the higher partial wave we would have to look for higher derivatives to see the discontinuity.] We should therefore expect to
see a "cusp" in the scattering amplitude. In solvable models we do see such behavior. The direct test of such a behavior in a measured amplitude is difficult since even in the best experiments it could easily be missed. But if other analyses point to a particular shadow threshold at a specific value, then the search for cusps would give added confirmation.

We also expect that in such theories we have resonances due to complex energy states. These show the typical enhancement of cross sections along the real axis of the energy variable. But in this case the complex energy states are part of the complete set of mathematical states; they therefore appear on the physical sheet. This means that as we go along real values of energy, we are going around the pole in a counterclockwise direction; this is to be contrasted with the usual case of a resonance pole being on the unphysical sheet in which case we circle the pole in the clockwise direction as the energy increases. Hence in these two cases the phase shifts rotate in opposite directions. For usual resonances the phase increases with energy so that the amplitude rotates counterclockwise in the Argand plane but for our case we expect some resonances for which the reverse is the case.

These resonances ought to be sought.

XIV. RESONANCE POLES ON PHYSICAL SHEETS

Given such "contrary" resonances we might ask two
other questions: First, given the existence of such a resonance would we not expect to find a phase advancement of the scattered wave which will violate causality? Second, while poles on the unphysical sheet correspond to unstable particles that die down with distance from their origin, what is the physical interpretation of poles on the physical sheet?

Now, phase advance is typical of a repulsive potential and we would therefore tend to identify a physical region resonance with a repulsive interaction. The "range" of the repulsive interaction is estimated by computing the derivative of the phase shift with respect to the momentum. In the case of an amplitude with just one pair of poles this quantity is basically the reciprocal of the imaginary part of the pole. So if we wish to see it as a repulsive (momentum dependent) potential of reasonably small range we should choose a reasonably large imaginary part for these resonances. In other words these resonances are expected to be broad.

The association of repulsive interactions with physical sheet poles is strengthened by considering the time behavior of a resonance on the physical sheet. For a resonance on the unphysical sheet we have the simple interpretation of a decaying state. This is obtained as follows: The time behavior is given by

\[ u(t) = \int_{1}^{\infty} e^{-iEt} T(E) dE \quad (14.1) \]

where \( T(E) \) is the scattering amplitude. If we have a
resonance pole on the unphysical sheet sufficiently near the real axis we may approximate \( u(t) \) by extending the integration from \(-\infty\) to \(+\infty\). For \( t < 0 \) we close the contour in the upper half plane and get nothing in this approximation but for \( t > 0 \) we close the contour in the lower half plane picking up the contribution from the resonance pole reached by deformation of the path. We thus obtain

\[
\begin{cases}
  0 & t < 0 \\
  \Re e^{-i(E_0 - \frac{i\Gamma}{2})t} & t > 0
\end{cases}
\]

(14.2)

where

\[
R = (E - E_0 + \frac{i\Gamma}{2})T(E) \bigg|_{E = E_0 - \frac{i\Gamma}{2}}
\]

is the strength of this pole. This may be viewed as an unstable particle created at \( t = 0 \) and decaying by a rate given by \( \Gamma \).

If we proceed in the same manner for a physical sheet pole we choose the contours as before but now we seem to get

\[
\begin{cases}
  \Re e^{-i(E + \frac{i\Gamma}{2})t} & t < 0 \\
  0 & t > 0
\end{cases}
\]

(14.3)

It appears as if we have a particle that dies at \( t = 0 \) but
formed at an earlier time \( t < 0 \) with a probability proportional to \( e^{-\Gamma |t|} \). This sounds quite unreasonable.

Quiet reflection shows that this must be viewed as a repulsive interaction. The collision producing the resonance and then leading to its decay has a pictorial representation as depicted in Figure 14.3. But if we have a physical sheet pole the calculation we have done seems to display something like shown in Figure 14.4. Actually the correct interpretation is shown in Figure 14.5 where we see that the "collision" producing the pole never gets to take place. Instead the reaction takes place when the (centers of the) particles are separated by the range of their repulsive interaction.

We can attempt to make this qualitative consideration more precise by constructing a simple model bringing about such physical sheet poles. Consider a separable non-local interaction with form factor \( f(\omega) \). The scattering amplitude in this case is of the form

\[
T(\omega) = \frac{|f(\omega)|^2}{\gamma(\omega - i\epsilon)} \tag{14.4}
\]

\[
\gamma(z) = 1 - \int_1^\infty \frac{|f(\omega')|^2 d\omega'}{\omega' - z} \tag{14.5}
\]

If \( |f(\omega)|^2 \) defined for real \( \omega \) is the boundary value of an analytic function

\[
|f(\omega)|^2 = \frac{\xi \xi^*}{(\omega - \xi)(\omega - \xi^*)} \tag{14.6}
\]
then $\gamma(z)$ will be analytic in the cut plane. Hence $T(\omega)$ will now be analytic and will have poles on the physical sheet at $\omega = \xi$ and $\omega = \xi^*$ in addition to the branch cut from 1 to $\infty$. But if

$$f(\omega) = \frac{\xi}{\xi - \omega}$$

for a nonrelativistic particle $\omega = k^2/2m$ so that

$$P(k) = \frac{\xi}{\xi - k^2/2m} = -2m \xi \int \frac{\sin kr}{kr} \cdot \frac{e^{-\sqrt{2\xi m}r}}{r} r^2 dr.$$  

Hence the coordinate space form factor is

$$\tilde{F}(r) = \frac{e^{-\sqrt{2\xi m}r}}{r}$$  \hspace{1cm} (14.7)

which has the Yukawa form with $\sqrt{-\xi}$ as the inverse "mass". For a resonance at

$$\xi = E + \frac{i\Gamma}{2} = \frac{(\mu + i\lambda)^2}{2m}$$  \hspace{1cm} (14.9)

the form factor falls off as

$$\exp\{(-\lambda + i\mu)r\}.$$  \hspace{1cm} (14.10)

The resonance energy determines the oscillatory factor while the damping factor is essentially dependent on the imaginary part of the resonance energy. As the imaginary part becomes small the form factor is nonvanishing for a
larger distance and we have a long range force. For small values of $\lambda/\mu$ we may write

$$\lambda = \frac{m\Gamma}{2\sqrt{2m\varepsilon}} \quad (14.11)$$

and the range $\lambda^{-1} = \frac{2\sqrt{2E/m}}{\Gamma}$ may be thought of as the distance an unstable particle of lifetime $\Gamma^{-1}$ can travel at the resonance velocity.

We have therefore two possibilities: The first is to deal with small range interactions and therefore to require that all physical sheet poles have a "strong interaction width", i.e., to have imaginary parts corresponding to lifetimes comparable to nuclear periods. The other possibility is to allow small or even vanishing imaginary parts and look for large impact parameter collisions. It is good to keep this possibility in mind.

XV. RELATION TO FORMAL AND AXIOMATIC THEORIES

Much of the study of general field theories, particularly of the mathematical structure of local quantum theories was based on the logical ground work of the earlier formal field theories. These assumed, amongst other things, that: (1) the fields are operator-valued distributions in a Hilbert space, (2) the spectrum was real time-like, (3) the amplitudes were globally analytic, (4) the physical states were asymptotically complete.

All these assumptions are unsatisfactory since we
see that the price we pay for a finite local field theory is to have negative metric and the formalism of shadow states. We cannot therefore use most of the conventional axiomatic field theory. But then axiomatic field theory results to date do not imply that there is any theory of interacting fields in the usual space time consistent with the axioms of the theory!

The most celebrated results are concerned with the spin-statistics theorem and with the TCP theorem. We should be then able to derive them without the constraint of a positive metric space etc. A beginning in this direction is made in a paper of Sudarshan. (20)

Much of axiomatic field theory relies on existing mathematical literature on Hilbert space theory and the theory of C* algebras. Since no corresponding mathematical literature exists on indefinite metric space, the purely mathematically inclined axiomatics are not likely to embrace these new developments, however, inevitable the physics is!

We must, therefore, pay more attention to the semi-rigorous but more elementary developments of formal field theory. In this area the use of indefinite metric, complex masses and shadow states make many modifications but mostly in the direction of releasing unpleasant constraints.

XVI. RELATION TO THE AXIOMATIC FORMULATION OF WIGHTMAN

In his classic paper on axiomatic field theory
Hightman\(^{(21)}\) posed and solved the problem of defining a quantum field theory in terms of vacuum expectation values of (ordinary) products of field operators. These vacuum expectation values are not functions but distributions, which were postulated to be tempered distributions in Hightman's system. Hightman was also able to show that the spectral conditions imply that these distributions are boundary values of analytic distributions, analytic in the "future tube", i.e., for complex space time points whose ordered differences had imaginary parts within the future light cone. Within the indefinite metric form this future tube analyticity is lost since complex four momenta appear in the spectrum. The usual topologies of operators in a Hilbert space are also not there. But the purely algebraic aspects of the Reconstruction Theorem (that is, the reconstruction of the fields given the vacuum expectation values) remain; in fact it is a special case of the Reconstruction Theorem that we formulated at the beginning to introduce general mechanics. If the vacuum expectation value is taken as the real linear functional and the ordinary product is taken as the associative product, then we have a linear representation with the vacuum serving as the standard state.

XVII. RELATION TO FORMAL FIELD THEORY

Many of the theorems of formal field theory became invalid within the indefinite metric framework. Take for
example the proposition: "If the two-point vacuum expectation value of any field is a multiple of the expectation value for a free field then the field is a multiple of a free field". In symbols,

$$<0 | \phi(x) \phi(y) | 0> = x^2 \Delta^{(+)}(x-y, m)$$

then

$$(\Box^2 + m^2) \phi(x) = 0$$

The proof is simply to note that

$$<0 | (\Box^2 + m^2) \phi(x) (\Box^2 + m^2) \phi(y) | 0> = 0$$

so that, in a positive metric space the result follows. But, clearly, in an indefinite metric space the result is not true.

Similarly, we have the familiar assertion: "The complete two-point function is at least as singular as the free field two-point function". It is based on the heuristic derivation:

$$<0|\phi(x)\phi(y)|0> = \sum_{n} <0|\phi(x)|n><n|\phi(y)|0>$$

$$= \int dx^2 \sum_{n} \frac{d^3 p_n}{n} \frac{\delta(p_n^2 - k^2)}{2p_n^0} e^{ip_n^0 x-i p_n^0 y} e^{-ip_n^0 p_n^0} <0|\phi(0)|n><n|\phi(0)|0>$$

$$= dk^2 \rho(k^2) \Delta^{(+)}(x-y, k)$$

where
\[ \rho(k^2) = \sum \frac{\phi(0) \langle n|\phi(0)|0 \rangle}{2p_n^0} \frac{\delta(p^2 - k^2)}{\Delta^+(x-y;k)} \]

is positive definite. Since the leading singularities of \( \Delta^+(x-y;k) \) is independent of \( k \), the conclusion follows.

But in an indefinite metric theory, particularly the type of coherent combinations of fields we have considered \( \rho(k^2) \) is not positive definite and hence the two-point function is superconvergent.

Yet another result of formal field theory: "If a Bose field has a source that is dynamically independent of the field [i.e., the source commutes with the time derivative of the meson field at equal times] then the boson mass gets depressed by the interactions". The usual proof is as follows:

\[ (\Box^2 + m_0^2)\phi(x) = j(x) \sim g\bar{\psi}(x)\psi(x) \]

with the understanding

\[ \delta(x^0 - x^0') [\phi(x), j(x')] = 0 \]

Then, if

\[ \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = i \int \rho(k^2) dk^2 \cdot \Delta(x-y;k) \]

by acting with the Klein-Gordon operator with bare mass \( m_0 \).
\[\langle 0 | [j(x), \phi(y)] | 0 \rangle = i \int dk^2 (m_0^2 - k^2) \rho(k^2) \Delta(x-y;k) .\]

Take the time derivative with respect to \(y\) and evaluate it at \(x^0 = y^0\). Then the lefthand side vanishes and since \(\Delta(x-y;k) = \delta(x-y)\) we get

\[\int dk^2 (m_0^2 - k^2) \rho(k^2) = 0\]

But if \(m\) is the renormalized boson mass

\[\rho(k^2) = \delta(m^2 - k^2) + \sigma(k^2)\]

where \(\sigma(k^2)\) is a continuous function, we can then unite, provided \(\sigma(k^2)\) is nonnegative,

\[(m_0^2 - m^2) = \int dk^2 \sigma(k^2) (k^2 - m_0^2) > \int dk^2 \sigma(k^2) \cdot (m^2 - m_0^2)\]

\[(m_0^2 - m^2) \{1 + \int dk^2 \sigma(k^2)\} > 0\]

which implies, in turn,

\[(m_0^2 - m^2) > 0 \text{ provided } \sigma(k^2) > 0\]

This result is again not valid in the indefinite metric framework since \(\sigma(k^2)\) is no longer necessarily positive definite. In fact, for the superconvergent propagators that we have employed the spectral function \(\rho(k^2)\) must have
a vanishing integral and the ground state mass may be increased rather than decreased. We saw an elementary example of the phenomenon in the second lecture.

There are other more involved results which depend on nonlinear relations between various vacuum expectation values, like the LSZ result \(^{(22)}\) about the slow approach to zero of the three-point function which is based on the positivity of the underlying space. An analysis of these relations within our extended framework is of interest but beyond the scope of these lectures.

In general the trend is for the "theorems" of formal field theory to become invalid. It appears that these results are lost. However, we must remember that there has been no real loss of any experimentally observed phenomena. The losses are paper losses!

There is a very welcome loss that is the "loss" of singular terms in the local product of fields. One manifestation of these terms is the so-called Schwinger terms \(^{(23)}\) which have been the topic of much study in recent years. The singularities in local products is another symptom of the inconsistencies in the standard formulation of local field theory and can be directly related to the most divergent terms in a perturbation theory and stem from the indefinite number of degrees of freedom of a local field.

Let us first deal with the Schwinger terms and, following Schwinger, consider a conserved current density, say, the electromagnetic current density \(j^\mu(x)\). For a
spinor field like the electron field, the current is given by

\[ j^\mu(x) = \frac{i}{2} [\bar{\psi}(x), \gamma_\mu \psi(x)] \]

A formal calculation for the equal time commutator yields

\[ \delta(x^0 - x'^0) [j(x), j_0(x')] = 0 \]

But this result is inconsistent with general principles. Schwinger proceeds as follows: If \( \delta(x^0 - x'^0) [j(x), j_0(x')] = 0 \), by taking the spatial divergence of both sides we get

\[ \left[ \nabla \cdot j(x), j_0(x') \right]_{x^0 = x'^0} = 0 \]

This means

\[ \left[ j_0(x), j_0(x') \right]_{x^0 = x'^0} = 0 \]

since

\[ \partial^\mu j^\mu(x) = 0 \]

Hence, with \( H \) as the Hamiltonian,

\[ <0 | \left[ [H, j_0(x)], j_0(x') \right] |0> = 0 \]

which implies
meaninglessness of currents built up from local products? The answer is that in a properly superconvergent indefinite metric theory, the coherent superpositions necessary to obtain superconvergence automatically assume the absence of any divergence of the local products constituting the currents.

These remarks are also relevant to the general question of local products of field operators and their expansions. In particular the question of the "inevitable" type of singularities in various orders of the expansion is no longer relevant. Clearly the whole question of "dimension" of interacting fields and their polynomials should be reexamined within the new framework [but not in these lectures!].

XVIII. CONSERVED CURRENTS, CURRENT ALGEBRA AND THE WARD-TAKAHASHI IDENTITIES

In the traditional version of quantum electrodynamics the source of the electromagnetic field is strictly conserved; they have the physical interpretation as the density and flow of electric charge. Generalizing this in the V-A theory of weak interactions, we have the hypothesis of conserved vector currents in which the hadron weak vector current is identified with the charged component of the conserved isospin current. More generally the vector and axialvector currents are identified with the generators of an SU(2) x SU(2) or
SU(3) x SU(3) algebra. These assumptions lead to observable consequences for hadron phenomena. Among these may be mentioned the lack of renormalization of the weak vectors coupling constant and of the electric charge of hadrons, a computation of the renormalization of the axial vector coupling constant in terms of pion-nucleon scattering amplitudes and calculations of the scattering lengths and low energy pion-nucleon scattering in terms of elementary parameters. Altogether the hypothesis that currents are related to elements of suitable "algebra of currents" is a powerful hypothesis. [This observation is still relevant even if most of these results could be derived on a much simpler basis.\(^{(4)}\)]

We have already seen that in finite quantum electrodynamics with indefinite metric the interaction current is

\[
j'_\mu(x) = \frac{e}{2} \sum_{i,j} c^*_i c_j [\bar{\psi}_i(x), \gamma_\mu \psi_j(x)]
\]

which is different from the conserved current

\[
j_\mu(x) = \frac{e}{2} \sum_{i,j} c^*_i \eta_{ij} c_j [\bar{\psi}_i(x), \gamma_\mu \psi_j(x)]
\]

This same difference would come in for the other interactions and currents. Does this mean that we lose all the good results of current algebra?

First, let us note that the algebra of the interaction currents is quite different from what we are used to
From elementary current algebra formulations. For example, with

\[ j_\mu^\alpha(x) = i \frac{1}{2} \sum_{i,j} \bar{c}_i^* \cdot c_j \left[ \bar{\psi}_i(x) \gamma_\mu \tau^\alpha \psi_j(x) \right] \]

we can easily verify that

\[ [j_\alpha^\alpha(x), j_\beta^\beta(x')] = 0 \]

provided the superconvergence relation

\[ \sum_j |c_j|^2 \eta_{jj} = 0 \]

is satisfied. Hence all the interaction current densities commute with each other. We do have an algebra for the interacting currents, but it does not coincide with the SU(2) algebra of the conserved current densities.

\[ [j_\alpha^\alpha(x), j_\beta^\beta(x')] = i \epsilon^{\alpha\beta\gamma} j_\gamma(x) \delta(x-x') \]

Of course, the interaction currents behave as isospin vector operators:

\[ [j_\alpha^\alpha(x), j_\beta^\beta(x')] = i \epsilon^{\alpha\beta\gamma} j_\gamma(x) \delta(x-x') \]

so that the isotopic spin relation between magnetic moments and masses\(^{(27)}\) will continue to hold.
XIX. **FREEDOM IN THE CHOICE OF GREEN'S FUNCTIONS**

Both in the discussion of the elementary theory of scattering by a potential and in the general theory of scattering we have come across a freedom in defining the Green's function and hence the response of a particular source. We made use of this freedom in introducing the formalism of shadow states with its corresponding new probability interpretation. Let us now examine this circumstance in some further detail.

If \( f(t) \) is any function of \( t \) satisfying a linear inhomogeneous differential equation of the form

\[
D(t)f(t) = \rho(t)
\]  
(19.1)

where \( \rho(t) \) is some "source", the general solution to this equation is of the form:

\[
f(t) = f_1(t) + f_0(t)
\]  
(19.2)

where \( f_1(t) \) is any particular solution of this inhomogeneous equation ("the particular integral") and \( f_0(t) \) is the general solution ("the complementary function") to the homogeneous equation

\[
D(t)f_0(t) = 0
\]  
(19.3)

This obtains since the difference between any two solutions
of the inhomogeneous equation is a solution to the homogeneous equation. And thus by choosing any particular integral and the full set of complementary functions we get the full set of solutions of the inhomogeneous equation. The particular integral can be written formally:

\[ f_1(t) = D^{-1}(t) \rho(t) \]

and the freedom in \( f_1(t) \) is precisely the freedom in the choice of the Green's function \( D^{-1} \). If we have two such choices

\[ f_1(t) = D_1^{-1}(t) \rho(t) \]
\[ f_2(t) = D_2^{-1}(t) \rho(t) \]

then

\[ f_1(t) - f_2(t) = (D_1^{-1}(t) - D_2^{-1}(t)) \rho(t) \]  \hspace{1cm} (19.4)

is a solution of the homogeneous equation. We have thus a solution of the "source-less" equation which is however proportional to the source!

This shows, in turn, that it would be too hasty to conclude that in

\[ f(t) = f_1(t) + f_0(t) \]  \hspace{1cm} (19.2)
the part $f_0(t)$ is "free". It does satisfy the "free" equation

$$D(t)f_0(t) = 0 \quad (19.3)$$

but it does not follow that it is functionally independent of the sources. The choice of this dependence is a "boundary condition" and is intimately tied to the physical interpretation of the theory.

A particular illustration of this circumstance is provided by action-at-a-distance theories and their definition of the "field". If we refer to the direct action-at-a-distance between particles, the field is a construct; for action-at-a-distance electrodynamics the electromagnetic field is a construct. This field satisfies the differential equation

$$\Box A^\mu(x) = \sum_\alpha e_\alpha \int \delta(x - z_\alpha(s)) \frac{\partial A^\mu}{\partial s}(s) \, ds = j^\mu(x) \quad (19.5)$$

and it was originally defined by the explicit expression

$$A^\mu(x) = \sum_\alpha e_\alpha \int \delta((x - z_\alpha(s))^2) \frac{\partial A^\mu}{\partial s}(s) \, ds \quad (19.6)$$

where $\delta((x - y)^2)$ is the time-symmetric (half-advanced half-retarded) Green's function for the d'Alembertian operator. However the general solution to the differential equation is
\[ A^\mu(x) = A^\mu_1(x) + A^\mu_0(x) \]  \hspace{1cm} (19.7)

where \( A^\mu_0(x) \) is any solution of the homogeneous equation

\[ \Box A^\mu_0(x) = 0 \] \hspace{1cm} (19.8)

When all the particles in the universe are taken there is obviously no scope for \( A^\mu_0(x) \) in a pure action-at-a-distance theory; but if we separated the matter into the "nearby matter" system and "distant matter", then \( A^\mu_0(x) \) could be thought of as arising from "distant matter". We may now look for physical situations in which the \( A^\mu_0(x) \) contains a part \( A^\mu_{\text{near}}(x) \) dependent on the "nearby matter" in such a fashion that

\[ A^\mu_{\text{near}}(x) = \sum_{\text{nearby matter}} e_\alpha \int D_1(x - z_\alpha(s)) z^\mu_\alpha(s) ds \] \hspace{1cm} (19.9)

plus another part \( A^\mu_{\text{inc}}(x) \) independent of the nearby matter. In such a case we may say that the electromagnetic potentials satisfy

\[ A^\mu(x) = A^\mu_{\text{inc}}(x) + \sum_{\text{nearby matter}} e_\alpha \int D_\Phi(x - z(s)) z^\mu_\alpha(s) ds. \] \hspace{1cm} (19.10)

where \( D_\Phi \) is the Stuckelberg-Feynman "causal" Green's function. This boundary condition may be called the "dark night sky"; it is called the "absorber" by Wheeler and Feynman.\(^{29}\)

Let us now consider an equation of the homogeneous
\[ \langle 0 | \int j_0(x) f(x) d^3x H + \int j_0(x') f(x') d^3x' | 0 \rangle = 0 \]

But this last result is absurd in a positive metric theory; and therefore it must be false to assume that the current and charge density components commute. [Again we note that the commutativity of the current density components does not lead to any inconsistencies in an indefinite metric theory!]

The traditional method of analysis is to observe that the formal method of computation of the commutator is false since the local product of field operators to produce the current components is inconsistent since such products do not exist as well defined quantities. For example, if we consider the quantity

\[ K_\mu(x, \xi) = e^{[\bar{\psi}(x + \xi / 2), \gamma_\mu \psi(x - \xi / 2)]} \]

and consider its limit as \( \xi \to 0 \) the quantity has no limit at all. One can in fact calculate the expression for free fields and show that it develops a singularity of the form \( \frac{1}{\xi^2} \). Hence the local expression for the current is not correct: in that case the formal derivation of the commutation relations are also not correct; and so on.

In an indefinite metric theory this problem does not necessarily arise. We have already seen that the commutativity of the charge and current compounds does not in any way lead to any inconsistencies. But what about the
\[ D(t)f(t) = F(t)f(t) \quad (19.11) \]

Then we can pretend that the righthand side is a source and proceed in the same manner and arrive at a solution of the form

\[ f(t) = f_0(t) + D^{-1}(t)F(t)f(t) \quad (19.12) \]

which is an integral equation the solution of which clearly depends on the choice of \( f_0(t) \) and of the Green's function \( D^{-1}(t) \). If \( f(t) \) is identified with the "initial state" of the comparison system the precise correspondence will depend on the specific choice of the Green's function. We have seen earlier the possibilities this implies for unitary shadow state theories. This boundary condition and the correspondence of the states with the states of a comparison system are fundamental. It is in a misunderstanding of this freedom that people insist about the inevitability of the usual choice and the corresponding questions of incident and scattered waves, of retarded and advanced responses etc.

I am tempted to quote a passage from a conversation between Don Juan and Carlos regarding the task of "seeing":

"The world is such-and-such or so-and-so only because we tell ourselves that that is the way it is. If we stop telling ourselves that the world is
so-and-so, the world will stop being so-and-so. At this moment I don't think you're ready for such a momentous blow, therefore you must start slowly to undo the world."

"I really do not understand you!"

"Your problem is that you confuse the world with what people do. Again you're not unique at that. Every one of us does that. The things people do are the shields against the forces that surround us; what we do as people gives us comfort and makes us feel safe; what people do is rightfully very important, but only as a shield. We never learn that the things we do as people are only shields and we let them dominate and topple our lives. In fact I could say that for mankind, what people do is greater and more important than the world itself."

"What do you call the world?"

"The world is all that is encased here," he said, and stomped the ground. "Life, death, people, the allies, and everything else that surrounds us. The world is incomprehensible. We won't ever understand it; we won't ever unravel its secrets. Thus we must treat it as it is, a sheer mystery!

"An average man doesn't do this, though. The world is never a mystery for him, and when he arrives at old age he is convinced he has nothing
more to live for. An old man has not exhausted the world. He has exhausted only what people do. But in his stupid confusion he believes that the world has no more mysteries for him. What a wretched price to pay for our shields!

"A warrior is aware of this confusion and learns to treat things properly. The things that people do cannot under any conditions be more important than the world. And thus a warrior treats the world as an endless mystery and what people do as an endless folly." (30)

It seems to me that this dialogue is relevant to the situation in fundamental physics at the present time!

XX. INDECOMPOSABILITY OF A SHADOW SYSTEM

Another essential point to be made is that in shadow state theory the Green's functions and states that we talk about refer to the whole system. In the case of potential scattering the Green's function corresponded to the propagation of the particle. For a multichannel scattering by a potential we have again single particle Green's functions in each channel. Shadow boundary conditions entail principal value Green's functions and standing waves in each channel. But when we consider multiparticle states and quantum field theory we have to remember that the Green's function now refers to the propagation of the whole system and not of individual particles. The Green's
function

\[
G(E) = \frac{1}{E - H_0 + i\varepsilon (1 - \sigma)} = -i \int_{-\infty}^{\infty} \{\theta(t) - \delta(t)e^{-i(E-H_0)t}\} dt \tag{20.1}
\]

refers to the total system. We have seen above that the choice of this Green's function leads to the expression for an amplitude $T$ which is related to the Dyson amplitude $\tau$ by the rule

\[
T^{-1} = \tau^{-1} - i\pi \sigma \tag{20.2}
\]

or

\[
T = \tau (1 + \pi i \sigma)^{-1} \tag{20.3}
\]

There is no simple modification of the single particle propagator which would bring about this new amplitude.

There is a simple physical interpretation to all this. The effect of the shadow particles is to bring about effective nonlocal interactions which cannot be obtained by physical particles. The modification in the propagation of individual particles is dependent on the number and motions of the other particles in the problem and we cannot bring it about by an autonomous modification of the single particle propagation.

It then follows that such simple notions as pole
factorizations, minimal analyticity etc. are not true in shadow state theory. Unitarity is still valid but the unitarity that one must impose is the primitive idea of unitarity:

$$T^\dagger - T = 2\pi i T^\dagger T$$

(20.5)

for the physical amplitude $T$. (We have already shown that for the shadow channels there is no contribution to $T^\dagger - T$. Since $T$ is no longer a globally analytic function we must not interpret the unitarity relation as a discontinuity formula for an analytic function across a Landau surface. However such notions can be used for the primitive amplitude $T$ which has all the familiar analytic properties (with the understanding that complex poles and negative imaginary parts are possible in view of the existence of an indefinite metric).

We must also add that while the invariance properties under the Poincaré group and under suitable internal symmetry operations are still valid for the physical amplitude $T$ as well as for $T$, along with the loss of global analyticity we also lose crossing symmetry. But in the energy domain below the opening of shadow channels crossing symmetry is still valid.

In the myth of creation light was the first to be created. Harmonies abounded in the successive stages of creation and disharmony came only with the coming
together of the serpent, the woman, the man and the apple. But quantum field theory would rewrite the story and have it that the disharmony has its roots in creating anything other than light which could emit or absorb it! There were no divergent radiative corrections until the sun and the moon were created. The coupling constant is the root of all disharmony.

Yet it is said that the Creator found what he had created on the six days to be "Good". All the disharmony and error crept in when the snake persuaded the woman to take an arbitrary independent point of view about the nature of things. Much laborious work by the good Lord was necessary to correct these errors.

Let us draw a moral from this myth and assert that the interacting radiation and matter system is not inconsistent either physically or mathematically. But this consistency is for the complete system of interacting constituents. The traditional version of quantum electrodynamics is not consistent in this sense. We must start afresh, but the theory must not only describe the usual kind of electrons and photons but be in quantitative agreement with the observed interactions.

Some of the divergences of electrodynamics are an inheritance from classical theory. Classical radiation theory in conjunction with the principles of statistical mechanics gave rise to the Rayleigh-Jeans paradox. The classical electron is inconsistent. If it were not for nonelectromagnetic forces the electron should have
exploded. Shall we follow Poincaré and postulate such non-electromagnetic forces of electromagnetic strength? Could these non-electromagnetic forces be? Should we now invent other forces like surface tension or incompressible matter, and then give up the connection between forces and coupled fields? But if we invent new fields to produce these forces would they not have their own quanta? Where are they?

Return to Harmony

In conclusion we remark that harmony can be restored in creation provided we use a consistent formulation without mathematical absurdities from the beginning. This involves a generalization involving satellite fields and an indefinite metric and the theory of shadow states. It is a new theory. The standing waves in the shadow channel are the mathematical counterpart of the physical requirement that stability requires the presence of non-electromagnetic forces at all times.

I have not talked about the mathematical problems of taking products of field operators at the same point. But let me add here that once the indefinite metric is properly used only suitable combinations of the fields and their satellites appear in the interaction; and the mathematical ambiguities disappear.

At the turn of the century, Planck showed us that basic divergences of field theory required the bold
experiment of a new conceptual structure. Bose showed that our notions of what particles are had to be amended to deal with photons. Should we not continue his work?
Figure 7.1  Lowest Hilbert Space Sectors Preserved by H
Figure 7.2  Singularity Structure of \( \alpha(z) \)
Figure 7.3  \( \alpha \) as a Function of \( x \)
Figure 7.4  Integration Contour \( \Gamma \)
Figure 7.5  Integration Contours \( \Gamma' \) and \( \Gamma'' \) Equivalent to \( \Gamma \) of Fig. 7.4
Figure 8.1  Integration Contour for Normal Oscillator and Complex Conjugate Contours for Two Abnormal Oscillators
Figure 9.1  Discontinuity in the First Derivative of \( \beta(\lambda) \)
Figure 10.1  Feynman Diagrams for (a) a Yukawa Type Coupling, (b) a Correction to the Fermion Propagator, and (c) \( \phi-\psi \) Scattering
Figure 10.2  Divergent Graphs with Fermion Loops
Figure 11.1  Compton Scattering Graphs
Figure 11.2  Typical Graphs for the Four-Fermion Weak Coupling
Figure 14.1  The Temporal Behavior of a Resonance State
Figure 14.2  Temporal Behavior of a Physical Sheet Pole

Figure 14.3  Production and Decay of a Resonance

Figure 14.4  Incorrect Picture of Production and "Decay" of a Physical Sheet Pole

Figure 14.5  Collision Process Involving Physical Sheet Pole
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\[
\begin{align*}
\sigma_3 &= -1 & N &= 0 \\
\sigma_3 &= +1 & N &= 0 \\
\sigma_3 &= -1 & N &= 1 \\
\sigma_3 &= +1 & N &= 1 \\
\sigma_3 &= -1 & N &= 2
\end{align*}
\]

FIG. 7.1
FIG. 7.2

FIG. 7.3
FIG. 7.4

FIG. 7.5
FIG. II.1

(a)  (b)  (c)

FIG. II.2