Notes on Separability of Density Matrices

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Abstract

These are notes from discussions from Prof. Sudarshan, Mims and Prof. Boya.

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1. Introduction

1.1. Separability

A density matrix is said to be separable if it can be written as a convex combination of products of density matrices for subsystems, *viz.* if ρ is separable then it can be written in the following form:

$$\rho = \sum_i \lambda_i \rho_A(i)(\otimes \rho_B(i). \tag{1}$$

1.2. Positivity

$2\times n$ density matrices are separable if their partial transpose is positive. In other words if you transpose one of two couple two state systems in the combined density matrix, and the matrix is still a positive one, then the system is separable. The reason this works is that the positivity of a matrix is determined by the following. If M is a positive matrix then

$$x^T M x \geq 0, \qquad \forall x\ . \tag{2}$$

Since the partial transpose is a (positive, but not completely positive) map which exchanges pairs of elements, it gives the necessary and sufficient condition for the positivity and thus separability for the $2 \otimes 2$ systems.

It is clear that this does not work for 3×3 density matrices because the 2×2 case can be seen to be valid due to the positivity of the minors. In the 3×3 case we must consider the 3×3 minors as well. Since this depends on more than just the interchanging of pairs, we could have a 3×3 minor that has all positive 2×2 minors but is negative itself. For example

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

One may also tell that a matrix is positive if all its sequential upper right minors' determinants are positive. In other words, if the upper right hand element is positive and the upper two by two minor has positive determinant and the upper right three by three minor has positive determinant and ... and the determinant of the matrix is positive.

2. Invariants of Positive Matrices

Consider the characteristic equation

$$\det(M - \lambda \mathbb{I}) = 0 \ .$$

We can write this as (for a 4×4, but it's immediately generalizable).

$$\lambda^4 - I_1\lambda^3 + I_2\lambda^2 - I_3\lambda + I_4 = 0 \ , \tag{3}$$

where the Is can be written as

$$I_1 = \sum_i \lambda_i = TrM, \tag{4}$$

$$I_2 = \sum_{i<j} \lambda_i\lambda_j = \frac{1}{2}\Big((\mathrm{Tr}M)^2 - \mathrm{Tr}(M^2) \Big) \ , \tag{5}$$

$$I_3 = \sum_{i<j<k} \lambda_i\lambda_j\lambda_k \tag{6}$$

$$= (-2\mathrm{Tr}(M^3) + (\mathrm{Tr}M^2)(\mathrm{Tr}M) + 6\mathrm{Tr}M), \tag{7}$$

$$I_4 = \det M \ . \tag{8}$$

It is important to note that: The equation 3 has all positive roots iff all I_i's are positive. Taking a matrix of the form

$$A = \begin{pmatrix} a_1 & x_1 \\ x_1^* & b_1 \end{pmatrix} \otimes \begin{pmatrix} c_2 & y_1 \\ y_1^* & d_1 \end{pmatrix} \tag{9}$$

and a similar matrix B with the subscripts changed to 2s, we form the product

$$\rho_s = \alpha A + (1 - \alpha) B \ . \tag{10}$$

for this very general separable state. This implies not only that the roots are all positive, but that the eigenvalues are unchanged.

3. A Possible Condition for Separability

Conjecture 1 A density matrix ρ is separable if and only if there exists a spectral decomposition of ρ into eigenmatrices which, in turn, have a spectral decomposition consisting of only a single term.

Consider an arbitrary density matrix ρ. This Hermitian matrix has an eigenvector decomposition given by

$$\rho_{ir,js} = \sum_{a} \mu_\alpha \xi_{ir}^{(a)} \xi_{js}^{(\alpha)\dagger} \ . \tag{11}$$

Each 'eigenmatrix'ξ itself has an eigenvector decomposition

$$\xi_{ir} = \sum_{\beta} \lambda_\beta \eta_i^\beta \zeta_r^\beta \ . \tag{12}$$

We wish to establish that the separable density matrices are exactly those where all eigenmatrices ξ have an eigenvector decomposition with only a single term

$$\xi_{ir} = \lambda \eta_i \zeta_r \ . \tag{13}$$

4. The Separation of Density Matrices

Peres [5] and the Horodeckis have stated that a density matrix is separable if it has a positive partial transposition (at least for 2×2 and 2×3 systems they claim this works). Although we have seen many people refer to these two papers, and we have seen extensive discussion in the literature about separable density matrices, we have not yet seen a density matrix separated.

In this section we give the explicit decomposition of some density matrices.

4.1. Separation One

This is the first of the density matrices for which we provide a decomposition into separated components. This matrix is an example discussed by Peres in [5]. The density matrix is given by

$$\rho = \begin{pmatrix} \frac{1-x}{4} & 0 & 0 & 0 \\ 0 & \frac{1+x}{4} & -\frac{x}{2} & 0 \\ 0 & -\frac{x}{2} & \frac{1+x}{4} & 0 \\ 0 & 0 & 0 & \frac{1-x}{4} \end{pmatrix} \tag{14}$$

To separate this into components so that we obtain the form (1), we write the following.

$$\rho = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \otimes \begin{pmatrix} A(1-x) + C(1+x) & B \\ B^* & A(1+x) + C(1-x) \end{pmatrix}, \tag{15}$$

where $a,\ b, c, A,\ B$, and C are 4-vectors. They have the following properties

$$\begin{aligned} a \cdot C = c \cdot A = b \cdot B^* = 0 \\ b \cdot B = 2x,\ a \cdot A = c \cdot C = 1\ . \end{aligned} \tag{16}$$

Except for normalization, one can immediately see that these conditions ensure that we obtain (14). To satisfy these conditions, we may choose:

$$\begin{aligned} a &= (\alpha,\ 0,\ 0,\ \beta), \qquad A = (\beta',\ 0, 0, -\alpha'), \\ b &= (0, u, v, 0), \qquad B = (0,\ iv^*, -iu^*, 0), \\ c &= (\alpha', 0, 0, \beta'), \qquad C = (-\beta, 0, 0, \alpha). \end{aligned} \tag{17}$$

Thus in separated components this becomes

$$\rho = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \otimes \begin{pmatrix} \beta'(1-x) - \beta(1+x) & 0 \\ 0 & \beta'(1+x) - \beta(1=x) \end{pmatrix} \tag{18}$$

$$+ \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -iv \\ iv^* & 0 \end{pmatrix} \tag{19}$$

$$+ \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & iu \\ -iu^* & 0 \end{pmatrix} \tag{20}$$

$$+ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \otimes \begin{pmatrix} -a'(1-x) + \alpha(1+x) & 0 \\ 0 & -\alpha'(1+x) + \alpha(1-x) \end{pmatrix}\ . \tag{21}$$

Now we take linear combinations of these to form legitimate density matrices and then enforce normalization. This gives

$$\rho = \frac{1}{4}\begin{pmatrix} \frac{3}{4} & u \\ u^* & \frac{1}{4} \end{pmatrix} \otimes \begin{pmatrix} \frac{1-2x}{2} & -iv \\ iv^* & \frac{1+2x}{2} \end{pmatrix} \tag{22}$$

$$+\frac{1}{4}\begin{pmatrix} \frac{3}{4} & -u \\ -u^* & \frac{1}{4} \end{pmatrix} \otimes \begin{pmatrix} \frac{1-2x}{2} & iv \\ -iv^* & \frac{1+2x}{2} \end{pmatrix} \tag{23}$$

$$+\frac{1}{4}\begin{pmatrix} \frac{1}{4} & v \\ v^* & \frac{3}{4} \end{pmatrix} \otimes \begin{pmatrix} \frac{1+2x}{2} & -iu \\ iu^* & \frac{1-2x}{2} \end{pmatrix} \tag{24}$$

$$+\frac{1}{4}\begin{pmatrix} \frac{1}{4} & -v \\ -v^* & \frac{3}{4} \end{pmatrix} \otimes \begin{pmatrix} \frac{1+2x}{2} & -iu \\ iu^* & \frac{1-2x}{2} \end{pmatrix} \tag{25}$$

with the condition that $iuv^* - ivu^* = -2x$. This is insensitive to $u \to ue^*v \to v_e^*$. So only phase differences matter. So $iu(v^* - v) = 2uImv = -2x$. This leaves Rev completely arbitrary. The best choice is to make Rev=0. Therefore $|u| = |v| = \sqrt{x}, \frac{v}{u} = i$

4.2. Separation Two

5. Ideas for Further Study

- Use the density matrix for two two-state systems to study the question of partial transposition and positivity.
- Boya, et. al.: efficiency of measurement.
- Horodeckis' proof. pos. partial trans. $\Longleftrightarrow 2 \times n$.
- Pos. maps generate separable states, hull exensions (to generate all separable states).
- Perez's Chapter 9.

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