Some fundamental considerations of the equation of radiative transfer

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Abstract

The radiation transfer of the vector electromagnetic field was first formulated by Chandrasekhar while deriving the polarization characteristics of a sunlit sky. There are two subtle problems underlying this treatment.

The first concerns the crucial identification of a Stokes parameter with the specific intensity of radiation. While both depend on position in 3-D space, the latter has, intrinsic to it, an additional angular dependence defining the flow of the radiation field. How can this inadequacy be remedied without damaging the results obtained heretofore from Chandrasekhar's formalism?

The second problem arises from the fact that the radiative transfer equation describes the transport of an incoherent radiation field through space. This, however, seems to contradict the results of the Van Cittert-Zernike-Wolf theorem which implies that an incoherent field develops coherence as it passes through free space implying, of course, that the radiative transfer equation must involve not incoherent but partially coherent fields.

In this paper the vector transfer equation of the direct beam (Beer's law) is derived from first principles. The analysis of this equation provides a satisfactory resolution of these two problems. The result also shows that the Beer's law will have to be modified to a matrix law to accommodate systems that are not spherically symmetric.

1. Introduction

In the discussions of the equation of radiative transfer, derived from energy balance considerations along a pencil of radiation traversing an interacting medium (Schuster 1905, Milne 1930), the radiation field is characterized entirely by the specific intensity. A more complete description of the radiation field, however, requires the consideration of three additional quantities with different transformation properties--the ellipticity, the orientation of the axes of the polarization ellipse and the degree of polarization. The inclusion of these polarization characteristics of the radiation field in the transfer equation, so as to account for their change, as the radiation field propagates through a medium, was not at all obvious. It was apparent to Chandrasekhar, when confronted by this problem in 1945, that an alternate description of the polarized radiation was necessary and, contrary to current notions, this was not well known at that time. The density matrix formalism of Landau and Von Neumann were familiar to nuclear physicists but its relationship to radiative transfer was unknown. It was a chance remark in an old text book of optics by Walker that led Chandrasekhar to the paper of Sir George Stokes, a paper that was as obscure then as it is famous now.

Stokes showed that four bilinear quantities (that have become known as Stokes parameters) constructed from the components of the electric field, afforded a complete and equivalent description of a polarized (incoherent) radiation field. From Stoke's' description, therefore, a sense of unity emerged in that the four properties associated with an incoherent polarized radiation field were described by the four Stokes parameters with similar transformation properties. One of the parameters identified as the specific intensity by Chandrasekhar (1950) obeyed the equation of transfer and this, presumably, led

him to postulate that the other three parameters also satisfied an equation similar in form. Contrasted against this, in the earlier formalism, the ellipticity transformed very differently from the specific intensity and, thus the extension of the equation of radiative transfer to include the ellipticity of the radiation field was nontrivial. Chandrasekhar assembled the four Stokes parameters in a column vector, that has since become known as the Stokes vector, and formally transcribed the scalar equation of transfer to a vector equation. The usefulness of this vector equation was firmly established when Chandrasekhar's multiple scattering calculation yielded the first correct description of the polarization of a sun-lit sky.

There are two subtle and basic problems associated with the conventional formulation of radiative transfer that have been ignored. The first problem in the formalism pertains to the identification of a Stokes parameter with the specific intensity. The specific intensity has associated with it not only the coordinate dependence, selecting the point at which the radiation field is defined, but also an angle dependence defining the direction of flow of the radiation. The Stokes parameters, however, have only a coordinate dependence and this inadequacy in the degrees of freedom has not been, heretofore, explicitly stated in the literature.

The second problem arises out of the wave nature of the radiation field. Wolf (1976) has correctly voiced the concern that despite the formulation of the classical theory of the electromagnetic field over a century ago and the quantized theory over fifty years ago, the radiation transfer has been treated on a phenomenological footing that bears little resemblance to either of the two rigorous formulations. This phenomenological basis of the theory of radiative transfer is rarely, if ever, mentioned in the lierature. For instance, the relationship of the specific intensity, that plays a crucial role in the

description of energy flow in the subject of radiative transfer, with the electromagnetic description of energy flow in terms of the Poynting's vector is obscure. The approximate nature of the radiative transfer theory is evident when it is juxtaposed with the theorem due to Van Cittert, Zernike and Wolf (Born and Wolf 1964). This theorem implies that an incoherent field develops coherence by merely propagating through free space. The implication is that a description of the radiation field in terms of Stokes parameters must necessarily be incomplete or approximate. Radiation transfer follows the path of a ray or a pencil of rays and in this sense it is akin to geometrical optics. It is, of course, well known that geometrical optics provides only an approximate description of a restricted class of optical phenomena. A complete description is possible only when proper account is taken of the wave nature of light and the interference and diffraction phenomena associated with them. This suggests that a rigorous theory of radiative transfer must include the wave aspects of the electromagnetic radiation but the implementation of this idea is not straight forward. These two problems raise serious questions on the validity of the conventional radiative transfer formalism.

To allay these fears, a rigorous derivation of the vector equation of transfer will be provided from first principles. While interactions will be included, the effect on the direct beam alone will be considered, so as to simplify the task. This is nothing but the vector form of the Beer's law. This specific, albeit simple, example will provide a frame of reference for studying the modifications that are necessary to include the wave aspects of the polarized radiation field. Evident as it is that this analysis has not addressed itself to the important problem of multiple scattering and the diffuse radiation field, it may still be viewed as a useful starting point for such discussions.

This paper is arranged as follows. The two problems associated with radiative transfer are identified in section 2. In section 3 the vector Beer's law is derived from first principles. The new definition of specific intensity that resolves one of the problems as well as providing the means of reconciling this vector equation with the Van Cittert-Zernike-Wolf theorem is given in section 4. The vector Beer's law has associated with it a matrix structure which is crucially dependent on the symmetry of the scattering system. A summary of the forms of this matrix for various systems is also given in section 4 while the details of the analysis are deferred to the appendix. The paper concludes with a discussion of the results.

Section 2. Problem in the radiative transfer formalism.

It has been known for a long time that the concepts of complete coherence and incoherence are inadequate for the description of many interesting physical phenomena. The demonstration by Verdet that sunlight, customarily viewed as an incoherent light, can produce Young's interference fringes when the pin holes are less than about 0.05 mm apart suggested the consideration of partially coherent radiation field. The recent advances in spectroscopy, radio astronomy and laser technology have emphasized the importance of the concept of partial coherence. The theory of partially coherent light deals with the statistical aspects of the electromagnetic radiation field with the amplitude of the light wave being treated as a random variable.

In the coherence theory as introduced by Wolf it is the analytic signal associated with the real electromagnetic field that enters the discussion (Born and Wolf 1964, p. 496). From the analytic signal representation of the electric field $\underset{\sim}{E}(\underset{\sim}{x}_1, t)$ it is possible to construct the 2 x 2 partial coherence matrix whose elements are

$\Gamma_{k\ell}(\underset{\sim}{x}_1, \underset{\sim}{x}_2, t_1 - t_2) = \langle E_k(\underset{\sim}{x}_1, t_1)\, E_\ell(\underset{\sim}{x}_2, t_2)\rangle$ where $*$ stands for complex conjugate, $\langle\ \rangle$ represents an ensemble average and the subscripts k, ℓ assume values 1 or 2. The

appearance of the relative time coordinate is an implicit acknowledgement of the assumption of a stationary field. When $\underset{\sim}{x}_1 \neq \underset{\sim}{x}_2$ then $\Gamma_{k\ell}$ is called the two point function or, as Wolf calls it, the mutual coherence function (Born and Wolf 1964). When $\underset{\sim}{x}_1 = \underset{\sim}{x}_2$ and $t_1 = t_2$ then $\Gamma_{k\ell}$ is called the coherency matrix. The coherency matrix elements $\Gamma_{k\ell}$ are very simply related to the Stokes parameters. When they are assembled into a 4 x 1 vector and subjected to an appropriate linear transformation, they become the Stokes vector [I, Q, U, V] considered by Chandrasekhar. That is

$$I = \Gamma_{11} + \Gamma_{22}; \quad Q = \Gamma_{11} - \Gamma_{22}; \quad U = \Gamma_{12} - \Gamma_{21}; \quad V = i\,(\Gamma_{21} - \Gamma_{12}) \tag{1}$$

This identification of $(\Gamma_{11} + \Gamma_{22})$ with the specific intensity of the radiation field raises a subtle problem that has escaped notice. The specific intensity of radiation I was introduced as a measure of the amount of radiant energy transported across an elemental area in a particular direction. That is to say, I is expected to vary from point to point (coordinate dependent) and also with direction through every point. The Stokes parameters $\Gamma_{k\ell}$ are coordinate dependent but have no further directional dependence. Thus the conventional identification of the Stokes parameter to the specific intensity of radiation is manifestly inadequate. This defect not withstanding, the Chandrasekhar vector radiative transfer formalism has yielded results that have been verified by experiment (Kuriyan 1974). Therefore, it is desirable that any modification introduced in the theory to remedy the above inadequacy does as little damage to, and perhaps even preserve intact, the results derived earlier.

After having identified the physical quantities associated with the characteristics of the radiation field, it is necessary to describe its transport through space. This is

contained in the equation of radiative transfer that defines the change of the specific intensity from point to point in space. Chandrasekhar obtained the corresponding equation for the polarized radiation field by replacing the specific intensity by the Stokes vector. In this formalism, therefore, it is the coherency matrix that describes the radiative field at all times. This, however, violates a fundamental theorem due to Van Cittert, Zernike and Wolf (Born and Wolf 1966; Beran and Parrent 1964; Klauder and Sudarshan 1968) that the intensity of radiation at any point in space can be constructed in terms of the two-point function (mutual coherence matrix elements) defined on a surface and not simply in terms of the intensity across the surface. An equivalent statement of this theorem is that a field generated by an incoherent source acquires partial coherence by virtue of propagation through free space and thus a complete treatment of the transfer of radiation necessitates the consideration of the partially coherent field and its transport. To elaborate further, conventionally it is the positive frequencies in the time dependence of the fields $\underset{\sim}{E}(\underset{\sim}{x},t)$ that are chosen and these are the analytic signals of Gabor (Born and Wolf 1964, p. 496). In this case the fields obey first order differential equations in time and by virtue of their defining equation, $\Gamma_{k\ell}(\underset{\sim}{x}_1,\underset{\sim}{x}_2,t_1-t_2)$ obeys two differential equations (Sudarshan 1969) one in the set of variables $(\underset{\sim}{x}_1, t_1)$ and the other in the set $(\underset{\sim}{x}_2,t_2)$. These equations imply that if $\Gamma_{k\ell}(\underset{\sim}{x}_1,\underset{\sim}{x}_2,t_1-t_2)$ is known for any one value of t_1-t_2 over <u>all</u> $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ then $\Gamma_{k\ell}$ is known for all values of t_1-t_2. The equation of motion in free space can also be used, with the aid of Green's theorem, to propagate $\Gamma_{k\ell}$, that is to say, to evaluate $\Gamma_{k\ell}$ on 'later' surfaces $\underset{\sim}{x}_1,\underset{\sim}{x}_2$ from its knowledge on an earlier surface. The construction here is analogous to the construction of solutions to wave equations and leads to the Van Cittert

Zernike-Wolf theorem.

It is evident that the Van Cittert-Zernike-Wolf theorem is a manifestation of the wave nature of the radiation field and the requirement imposed by the equation of radiative transfer that the intensity at any one point be constructed in terms of intensity at other points forces the exclusion of these wave aspects of the radiation. This suggests that the relationship of radiative transfer to the theory of partial coherence is akin to the relationship of geometrical to physical optics.

To incorporate the wave aspects into radiative transfer Wolf (1976) defined a scalar and vector quantity in terms of invariants of certain new correlation tensors of the electromagnetic field and showed that the equation obeyed by the new quantities for a statistically homogeneous field reduces to the equation of radiative transfer for a free field (Zubairy and Wolf 1977, Wolf 1978). The extension of this discussion to the transport of radiation in an interacting medium is, however, not obvious.

Further clarificationwill be obtained from a study of the simple problem of the passage of a beam of radiation through a scattering medium, ignoring the diffuse radiation field generated by the scattering process. The solution to this problem in radiative transfer is the Beer's law. The scattered wave will be used to construct the transmitted beam and an explicit check performed on the validity of the Beer's law. This analysis, contained in the next section, will be used to establish the compatibility of the derived results with the Van Cittert-Zernike-Wolf theorem. This example will also serve as a vehicle to re-examine the inadequacy associated with the identification of the specific intensity with a Stokes parameter.

Section 3. Vector Beer's law from first principles.

Consider a plane electromagnetic wave propagating along the z direction and let it traverse a plane parallel medium homogeneous in the x-y plane. In the ensuing discussion multiple scattering will be ignored. The form of the scattered solution at large enough distances will be

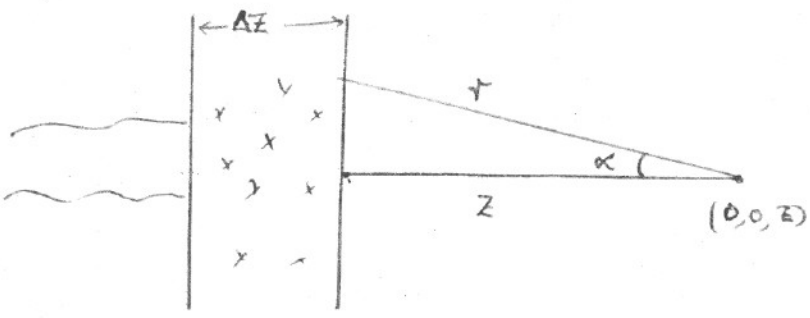
$$\underset{\sim}{E}(\underset{\sim}{r}) = \left[\frac{1}{r} \underset{\approx}{S} \, e^{ik(r-z)} \right] \underset{\sim}{E}\,\text{inc} \tag{2}$$

where $\underset{\approx}{S}$ is the 2 x 2 scattering amplitude matrix whose elements are functions of the angle $\underset{\sim}{r}$ makes with the propagation direction z.

Consider scatterers within a slice Δz of the medium near the origin. The radiation field viewed in the forward direction on a screen at a distance z is obtained by summing the effects due to each of the scatterers in the slice of thickness Δz about the plane z = 0. Let N be the number of scatterers per unit volume. The forward scattered field at the screen will be given by

$$\underset{\sim}{E}(0,0,z) = \left[\mathbb{1} + N\Delta z \int_0^{2\pi} d\varphi \int_0^{\infty} \underset{\approx}{S}(\alpha) \, \frac{e^{ik[\sqrt{V^2+z^2} - z]}}{\sqrt{V^2+z^2}} \, V\,dV \right] \underset{\sim}{E}\,\text{inc} \tag{3}$$

where $\mathbb{1}$ is the unit matrix in 2 x 2 space, (V, φ) define the polar coordinates in the plane (z = const) containing the scatterers, and $\alpha = \tan^{-1} \frac{V}{z}$.



It is possible to obtain an approximation to the above integral as follows.

$$\int_0^\infty \underset{\approx}{S}(\alpha)\,\frac{e^{ik[\sqrt{V^2+z^2}-z]}}{\sqrt{V^2+z^2}}\,V\,dV = \frac{1}{ik}\int_0^\infty \underset{\approx}{S}(\alpha)\,\frac{d}{d}\left[e^{ik(\sqrt{V^2+z^2}-z)}\right]dV$$

$$= -\frac{i}{k}\left[\underset{\approx}{S}(\alpha)\,e^{ik(\sqrt{V^2+z^2}-z)}\right]_{V=0}^{V=\infty}$$

$$+\frac{i}{k}\int_0^\infty e^{ik(\sqrt{V^2+z^2}-z)}\,\frac{d}{dV}\underset{\approx}{S}(\alpha)\,dV$$

$$\simeq \frac{i}{k}\,\underset{\approx}{S}(0) \tag{4}$$

Thus, for a slab of thickness Δz, the forward scattered field is

$$\underset{\sim}{E}(0,0,z) = \left[\mathbb{1} + \frac{i2\pi N\Delta z}{k}\,S(0)\right]\underset{\sim}{E}\,\text{inc.} \tag{5}$$

It is worth noting that since $z \gg t$, in the above expressions and $\sqrt{z^2+V^2} - z \approx V^2/2z$ and $\sqrt{z^2+V^2} \approx z$. It is now evident that the transformation $V \to \mu V$; $z \to \mu^2 z$ will not alter the value of the integral. This scaling law implies that the contribution to the integral in the approximation arises from a circular section whose radius grows as the square root of the linear distance from the screen.

The essential point of the approximation, however, is the neglect of the second and higher order terms in the Taylor series expansion of $\underset{\approx}{S}(\alpha)$. Having calculated $\underset{\sim}{E}$ for an incident plane wave $\underset{\sim}{E}$ inc at the point (0,0,z), the translation invariance of

the plane wave (in the x-y plane) implies that the same expression is valid for $\underset{\sim}{E}(x,y,z)$ and thus the emergent wave is also a plane wave.

The Chandrasekhar-Stokes vector defined earlier can be obtained from a simple linear transformation of the 4 x 1 vector

$$\underset{\sim}{E}^*(\underset{\sim}{r}) \otimes \underset{\sim}{E}(\underset{\sim}{r})$$

where $\otimes$ denotes an outer product of matrices. Using the scattered field derived above, in matrix notation,

$$(\underset{\sim}{E}^* \otimes \underset{\sim}{E}) = [\mathbb{1} \otimes \mathbb{1} + \frac{i2\pi N}{k} \Delta z \{\mathbb{1} \otimes \underset{\approx}{S}(o) - \underset{\approx}{S}(o)^* \otimes \mathbb{1}\}](\underset{\sim}{E}^*_{inc} \otimes \underset{\sim}{E}_{inc}) \tag{6}$$

The Chandrasekhar-Stokes vector I is obtained by the linear transformation

$I = T(\underset{\sim}{E}^* \otimes \underset{\sim}{E})$ where T is a 4 x 4 numerical matrix that can be derived from (Eq. 1).

It is, therefore, possible to cast the above equation in the form

$$I_{sc} \equiv I_{inc} + \Delta I_{inc} = I_{inc} + \Delta z\, M\, I_{inc} \tag{7}$$

where M is the 4 x 4 matrix

$$M \equiv \frac{i}{k} \cdot 2\pi N \cdot T[\mathbb{1} \otimes S(o) - S^*(o) \times \mathbb{1}]T^{-1} \tag{8}$$

In general M is a matrix that can have non-diagonal elements coupling the various components of the Stokes vector. Therefore, the generalization of the differential form of Beer's law to include polarization of the radiation field is not simply a scalar law extended to all the components of the Stokes vector but a matrix differential law of the form

$$\Delta I = \Delta z\, M I \tag{9}$$

Only when M is a scalar multiple of the unit matrix does the above law becomes the usual differential form of Beer's law.

Chandrasekhar's interest was centered on the effects of multiple scattering and thus his differential vector equation was for the diffuse radiation field, obtained by subtracting the direct beam from the total radiation field. The subtracted term in Chandrasekhar's treatment is nothing but the scalar Beer's law. The analysis provided later will reveal that the adoption of the scalar Beer's law was correct for the problem considered by Chandrasekhar but will require modification for other physical situations.

The incident and scattered fields are 'pure' states of polarization. In nature, however, partially polarized ('mixed') states occur. The coherency matrix or the Stokes parameter formalism permits the consideration of mixed states. The equivalent method in quantum mechanics is the density matrix formalism of Landau and Von Neumann. The transformation induced by scattering (eq. 5) can be formally represented as $\underset{\sim}{E}_{in} \rightarrow \underset{\sim}{E} = Q\,\underset{\sim}{E}_{inc}$ and corresponding to this the coherency matrix is transformed, $\Gamma_{inc} \rightarrow \Gamma = Q^{+}\,\Gamma_{inc}\,Q$. This will, of course, define the development of the Stokes parameters of the field during the passage of the radiation through the interacting medium.

These results will be examined next, so as to clarify the two problems identified in the last section. Analysis of the matrix M will also be provided for various physical situations.

Section 4. Resolution of the problems.

It is now possible to provide a precise identification of specific intensity without perturbing the results derived in the Chandrasekhar formalism of radiation transfer.

In radiative transfer theory the radiation field is assumed to behave much like an ideal gas of corpuscular photons that are defined with arbitrary color and located in narrow pencils. The wave nature of light requires that this narrow pencil diffract and spread. Thus the basic postulates of radiative transfer theory must be carefully interpreted to make it consistent with wave optics.

A similar problem exists when a transition is made from classical to quantum mechanical systems. In a classical system the positions and momenta of particles can be represented by a phase space density that is everywhere non-negative. The situation for the quantum system is very different owing to the restriction placed by the uncertainty principle. Wigner (1932) and Moyal (1949) defined a phase space density for a quantum system as well, but this is a quasi-probability density that is not everywhere non-negative. Nevertheless, as long as the average of this phase space density over phase volumes of a unit cell or more is considered, the resulting expressions are non-negative.

It is possible, by analogy to the Wigner-Moyal phase space density, to provide a new definition of the specific intensity of radiation as the following transform of the two-point (mutual coherence) function

$$I(\underset{\sim}{x},\underset{\sim}{p}) = \frac{c}{4\pi} \int \Gamma(\underset{\sim}{x} - \tfrac{1}{2}\underset{\sim}{y}, \underset{\sim}{x} + \tfrac{1}{2}\underset{\sim}{y}) e^{i\underset{\sim}{p}\cdot\underset{\sim}{y}} \, d\Omega$$

where $d\Omega$ denotes the integration with respect to the angular variables defined by the unit vector $\underset{\sim}{y}$. Here I depends on the intensity at and around the point $\underset{\sim}{x}$. Note that in proceeding

from the wave optics description of the two-point function $\Gamma(\underset{\sim}{x},\underset{\sim}{x}')$ involving the 6 position labels, to the geometrical ray optics quantity $I(x,p)$, the relative coordinate label $\underset{\sim}{x}-\underset{\sim}{x}'$ has been transformed to the 'momentum' label $\underset{\sim}{p}$. The magnitude of $\underset{\sim}{p}$ is identified with the frequency and its angular orientation (θ, φ) refers to the direction along which the specific intensity of radiation is defined at the point $\underset{\sim}{x}$. As stated earlier, scattering imposes the transformation $\Gamma_{inc} \rightarrow \Gamma = Q^{+}\Gamma_{inc}\, Q$ Since the matrix Q has no coordinate dependence the new definition of the specific intensity undergoes the same transformation as before. Thus the results derived from Chandrasekhar's formulation carry over to this new formulation with essentially no change.

Wolf (1976) has an alternate and different identification of the specific intensity of quasi-monochromatic radiation. As a result of this the specific intensity is no longer a real function but, as Wolf points out, the observable averaged quantities are real. As opposed to this, the definition suggested in this paper leads to a specific intensity that is real. Having arrived at a consistent definition of the specific intensity of the radiation field, it is now possible to investigate the compatibility of the matrix Beer's law with the results of the Van Cittert, Zernike and Wolf Theorem.

In the derivation provided in section 3 the propagation equation was arrived at from general considerations that did not include rays or narrow pencils. The scatterers were an assembly of parallel slabs and the beam very broad. The Van Cittert-Zernike-Wolf theorem states that even when the primary radiating surface is fully incoherent, when the wave travels a distance that is very large compared with the linear dimensions of the primary surface then the wave acquires partial coherence. But when the primary surface is infinitely large, as in the case of the plane wave considered earlier, the coherence developed by propagation is such that any two points on any plane parallel to the primary surface have

no correlation, provided the two points are separated by a distance large compared with the wavelength. This is analogous to the situation in optics where the consideration of a wide slit obviates the need for the introduction of diffraction effects. Thus the derivation of the simple radiative transfer equation in the last section side steps the consequences of the Van Cittert-Zernike-Wolf theorem because the consideration of a plane wave is a sufficient condition for the rigorous validity of the transfer equation.

Having derived the generalized version of the Beer's law it is meaningful to investigate the restrictions placed on the result by the symmetry of the scattering system. It is convenient to return to the description of the electric field and the transformation induced on it by the scattering mechanism.

$$\underset{\sim}{E}_{inc} \longrightarrow \underset{\sim}{E} = \left\{ \mathbb{1} + \frac{2\pi i}{k} N\Delta z \; \underset{\approx}{S}(o) \right\} \underset{\sim}{E}_{inc}$$

There are three distinct cases of interest:

Case (i). If $\underset{\approx}{S}(o)$ is a multiple of the unit matrix then the scattering system is spherically symmetric and optically inactive. This corresponds to the case treated by Chandrasekhar and yields the usual scalar Beer's law.

Case (ii). $\underset{\approx}{S}$ is not a multiple of the unit matrix but is invariant under rotations around the propagation direction $\underset{\sim}{k}$ (z axis in this case).

The general form of $\underset{\sim}{E}$ is $f \; \underset{\sim}{E}_{inc} + g \, \underset{\sim}{k} \times \underset{\sim}{E}_{inc}$ where f and g are, in general, complex numbers (see appendix for further details). The medium affects the two circular polarization states differently. This would be due to either optical activity or an asymmetry in the assembly of scatterers.

Case (iii) $\underset{\approx}{S}$ is not invariant under rotations around $\underset{\sim}{k}$. Let $\underset{\sim}{n}$ be the vector (not coincident with $\underset{\sim}{k}$) that characterizes the orientation of the system of scatterers. In this case,

as shown in the appendix, the form of $\underset{\sim}{E}$ is $f\,\underset{\sim}{E}_{inc} + g\,\underset{\sim}{k} \times \underset{\sim}{E}_{inc}$.

$$+ h\left\{(\underset{\sim}{n}\cdot\underset{\sim}{E}_{inc})\,\underset{\sim}{n} - \frac{(\underset{\sim}{n}\cdot\underset{\sim}{E}_{inc})(\underset{\sim}{n}\cdot\underset{\sim}{k})\,\underset{\sim}{k}}{k^2}\right\}$$

$$+ \ell\,(\underset{\sim}{n}\cdot\underset{\sim}{E}_{inc})\,(\underset{\sim}{n}\times\underset{\sim}{k})$$

5. Conclusions

In attempting to reconcile the theory of transfer of the polarized radiation field with the results of electromagnetic theory two important problems arise, both of which are attributable to the wave nature of the radiation field.

The first problem is the recognition that the conventional identification of the specific intensity with the elements of the coherency matrix is inadequate, because the latter depends solely on a co-ordinate point and cannot accommodate the additional angular dependence necessary to define the specific intensity. By establishing an analogy between this problem and that of defining the phase space density of a quantum mechanical system it was possible to arrive at a new identification of the specific intensity as a Fourier transform (with respect to the relative coordinates) of the two-point function. The transformations induced on the two-point function by the scattering process carry over unscathed to the newly defined specified intensity and, therefore, call for no revisions to the numerical results obtained until now. Next, as a result of the Van Cittert-Zernike-Wolf theorem, the complete description of the propagation of the radiation field requires the consideration of the two-point function (mutual coherence matrix) and not, as in radiative transfer, merely the coherency matrix (Stokes parameters). A related

observation that a narrow pencil of radiation would eventually, due to diffraction, spread also has its roots in the wave aspect of the radiation field. This suggested that a geometrical optics limit may avoid the consideration of diffraction type phenomena rendering the radiative transfer formulation rigorous.

For purposes of illustration of these ideas the transport of a "wide" beam of radiation through a scattering medium was considered. The forward scattered beam was found to obey Beer's law, the violation of the Van Cittert-Zernike-Wolf theorem being attributable to the large width of the beam. The two point functions were calculated and the transformation induced on them by the scattering medium explicitly stated. This transformation carried over to the newly defined specific intensity as well.

Quite unexpectedly the Beer's law exhibited a matrix structure that depended crucially on the symmetry of the scattering system. The special cases corresponding to the diverse forms of the scattering system were analyzed and the general form of the scattered wave was deduced. It was evident that the transmission of radiation through an assembly of non spherical scatterers will necessitate the consideration of a matrix Beer's law with the coupling of the various components of the Chandrasekhar-Stokes vector. This result will find application in extinction measurements through non spherical scatterers such as those present in Cirrus clouds.

Appendix

To systematically examine the implications of the matrix structure of M it is convenient to return to the description of the electric field and its transformation by the scattering medium, eq. (5)

$$E_{inc} \rightarrow \underset{\sim}{E} = \left[\mathbb{1} + \frac{2\pi i}{k} N\Delta z \, \underset{\approx}{S}(o) \right] \underset{\sim}{E}_{inc}$$

Case (i) If S(o) is a multiple of the unit matrix then the scattering system is spherically symmetric and this corresponds to the case considered by Chandrasekhar and yields the usual Beer's law.

Case (ii) S is not a multiple of the unit matrix but is invariant under rotations around the propagation direction $\underset{\sim}{k}$ (z axis in this case).

The most general form of $\underset{\approx}{S}\underset{\sim}{E}_{inc}$ can be deduced from the following conditions: $\underset{\approx}{S}\underset{\sim}{E}_{inc}$ (i) must be linear in $\underset{\sim}{E}_{inc}$; (ii) must be orthogonal to the propagation vector. The first condition is a consequence of the linearity of the transformation and the second condition corresponds to the transversality of the scattered wave.

It is evident that the most general form of the scattered wave is given by $\underset{\sim}{E} = f \underset{\sim}{E}_{inc} + g \underset{\sim}{k} \times \underset{\sim}{E}_{inc}$ with f and g, in general, complex.

If the circular polarization states $E^{(\pm)} = E_x \pm i E_y$ are considered, then the above equation simplifies to $E^{(\pm)} = (f \pm i g k) E^{(\pm)}_{inc}$. Now it is evident that the medium affects the two circular polarization states differently, indicating optical activity. Further, since f and g are in general complex, there is also the possibility of different absorption coefficients for the two polarization states.

<u>Case (iii)</u> S is not invariant under rotations around $\underset{\sim}{k}$. Let $\underset{\sim}{n}$ be vector not coincident with $\underset{\sim}{k}$ that characterizes the orientation of the system of scatterers.

In this case there are 3 vectors $\underset{\sim}{E}_{inc}$, $\underset{\sim}{k}$ and $\underset{\sim}{n}$ and it is necessary to construct the linearly independent set of vectors that obey the two conditions (of transversality and linearity in E_{inc}).

The vectors that are linear in $\underset{\sim}{E}_{inc}$ are $\underset{\sim}{E}_{inc}$, $\underset{\sim}{k} \times (\underset{\sim}{n} \times \underset{\sim}{E}_{inc})$, $\underset{\sim}{n} \times (\underset{\sim}{k} \times \underset{\sim}{E}_{inc})$, $\underset{\sim}{n} \times (\underset{\sim}{n} \times \underset{\sim}{E}_{inc})$, $\underset{\sim}{n} \times \underset{\sim}{E}_{inc}$, $\underset{\sim}{k} \times \underset{\sim}{E}_{inc}$, $(\underset{\sim}{n} \cdot \underset{\sim}{E}_{inc})\underset{\sim}{n}$, $(\underset{\sim}{n} \cdot \underset{\sim}{E}_{inc})(\underset{\sim}{n} \times \underset{\sim}{k})$. To render the above vectors transverse it is necessary to subtract their projection along $\underset{\sim}{k}$. For instance

$$(\underset{\sim}{n} \times \underset{\sim}{E}_{inc})_{tr} = \underset{\sim}{n} \times \underset{\sim}{E}_{inc} - \left(\frac{\underset{\sim}{n} \times \underset{\sim}{E}_{inc} \cdot \underset{\sim}{k}}{k^2}\right)\underset{\sim}{k}$$

Since the transverse vectors are confined to the x-y plane it is possible to view them as 2 x 2 matrices acting on the 2 x 1 column vector defined by the incident electric field.

It is well known that the 2 x 2 matrix space is spanned by 4 independent 2 x 2 Pauli matrices, usually represented as $\sigma_\alpha (\alpha = 0, 1, 2, 3)$,

$$\sigma_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus it is possible to deduce from general principles that there are, at most, 4 independent transverse vectors in the problem under consideration. To proceed with the analysis, the expansion of both $\underset{\sim}{k} \times (\underset{\sim}{n} \times E_{inc})$ and $[\underset{\sim}{n} \times (\underset{\sim}{k} \times \underset{\sim}{E}_{inc})]$ yields a scalar multiple of $\underset{\sim}{E}_{inc}$. Therefore, these two vectors can be ignored, if $\underset{\sim}{E}_{inc}$ is chosen as the first vector.

$(\underset{\sim}{n} \times \underset{\sim}{E}_{inc})$ and $(\underset{\sim}{k} \times \underset{\sim}{E}_{inc})$ both correspond to the same matrix $-i\sigma_2$ acting on $\underset{\sim}{E}_{inc}$ and thus only one of them say $\underset{\sim}{k} \times \underset{\sim}{E}_{inc}$ is independent and is the second vector.

$[\underset{\sim}{n} \times (\underset{\sim}{n} \times \underset{\sim}{E}_{inc})]$ and $[(\underset{\sim}{n} \cdot \underset{\sim}{E}_{inc})\, \underset{\sim}{n}]$ can both be expressed in the form $[a\,\sigma_o + b\sigma_1]$ acting on $\underset{\sim}{E}_{inc}$. Again the third independent vector is chosen to be one of the two say, $(\underset{\sim}{n} \cdot \underset{\sim}{E}_{inc})\underset{\sim}{n} - (\underset{\sim}{n} \cdot \underset{\sim}{E}_{inc})\, \frac{(\underset{\sim}{n} \cdot \underset{\sim}{k})}{k^2}\, \underset{\sim}{k}$. The vector $(\underset{\sim}{n} \cdot \underset{\sim}{E}_{inc})\,(\underset{\sim}{n} \times \underset{\sim}{k})$ gives rise to the matrix expressible as $a\,\sigma_o + b\sigma_2 + c\sigma_3$. Thus this is the fourth independent vector for the problem. The four vectors defined thus span the 2 x 2 matrix space completely and every other matrix or vector can be expressed in terms of this basis.

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