Stochastic Dynamics of Quantum Mechanical Systems II [1]

E.C.G. Sudarshan,
Physics Department and the Center for Particle Physics
University of Texas, Austin, Texas 78712

Abstract

Stochastic quantum dynamics is studied both with regard to the convex set of dynamical maps of density matrices and their generic construction as contractions of extended systems. The analytic continuation of this dynamics to dual analytic spaces is carried out and the dominant metastable modes are identified. The behavior of dissipative systems under Galilean and Lorentz transformations are also studied. The various generalized spaces, often used indiscriminately in the literature, are defined and distinguished. The concept of the age of a decaying system obeying a semigroup is introduced and illustrated.

1 Introduction: Quantum Kinematics, Convex Set of States

The generic state of a quantum system is specified by its density distribution which may be viewed as a linear nonnegative normalized number valued linear functional [2] on the operators. In the more restrictive form in which it is identified as a trace class operator [3] in the Hilbert space, it has the canonical decomposition

$$\rho = \sum_1^\infty \mathcal{C}_n \psi_n \tilde{\psi}_n \quad ; \quad \mathcal{C}_n > 0; \; \sum_1^\infty \mathcal{C}_n = 1 \, . \tag{1.1}$$

with

$$\tilde{\psi} = \psi^\dagger \, . \tag{1.2}$$

The set of density distributions can be enlarged provided the dynamical variables for which expectation values are sought are restricted. Conversely, if the set of density distributions is restricted a wider set of dynamical variables may be constructed [2].

The conditions of positivity and normalization still allows the formation of normalized convex combinations:

$$\rho = \rho_1 \cos^2\theta + \rho_2 \sin^2\theta \, . \tag{1.3}$$

They *do not* form a vector space. For several purposed including that of the stochastic dynamics of density distributions, it is advantageous to consider the vector space generated by the density distributions; the additional distributions so obtained may not satisfy either the positivity or the normalization or both. We will see that metastable (decaying) "states" are realized by such pseudodensity distributions.

Given the convex set of density distributions we could seek the boundary elements and more specifically the generating extremal elements. The latter are those density distributions whose (normalized) convex combinations generate all the density distributions but which themselves have no nontrival decomposition. Trace class operators in Hilbert space are a compact set under the Hilbert-Schmidt norm and the extremals are one-dimensional projections. For more general definitions of the density distributions we should investigate the question in each case.

If the dynamical variables undergo a unitary transformation their expectation values change: this is equivalent to a linear transformation on the density distributions:

$$\rho_U(A) = \rho(\vec{U}AU) \; . \tag{1.4}$$

If the unitary transformations concerned form a group so do the linear transformations

$$\rho \rightarrow \rho_U \tag{1.5}$$

furnish a realization of the same group. On the other hand if we consider the linear transformations on the density distribution we have new possibilities. This is particularly true for time evolutions. For simple Hamiltonian systems there is a one-parameter group of transformations on the density distributions; but we have the more general possibility of non-invertible dissipative transformations. Whenever such a dissipation is involved the inverse transformations cannot act on all density distributions. Instead of a time translation group the best we can obtain is a semigroup of dissipative evolutions. In either case we refer to the generator of the group or the semigroup as the Liouvillian.

The generic time evolution is a study in stochastic dynamics [1], and has been the subject of systematic study over the last four decades [4]. Special cases like the relaxation of a spin systems in an external magnetic field was studied in terms of the Bloch equations half a

century ago [5]. But the convex set of dynamical maps has an intricate structure even for 2×2 density matrices [6].

One way of arriving at stochastic dynamics is by considering the system as being embedded in a larger system with a time translation group and then contracting out the extraneous degrees of freedom. When one does this one arrived at a subclass of stochastic dynamical maps, namely those of the completely positive type [7]. Conversely given a completely positive dynamical map we can realize it constructively in terms of the contraction of an extended Hamiltonian time translation [8].

Stochastic dynamics thus involves a *sense of time* and this is a *breaking of time symmetry*. In the contraction procedure the time symmetry breaking is explicit. It has been of continuing interest for more then a century whether the breaking could occur spontaneously, without an explicityly assymmetric procedure and without a restriction on the initial states. On the other hand if the dynamical laws are time symmetric it would be expected that the time reversed states would have a time reversed semigroup. What, then, selects the forward semigroup for time evolution? In other words what is the ingredient that is implicit in dynamics or involved in the choice of physical states that assures the time symmetry breaking second law of thermodynamics? With the increasing recognition of the role of deterministic chaos and the relevance of the Poincaré catastrophe for Large Poincaré Systems [9] the simple objections of Loschmidt and Zormelo to the Boltzmann H-theorem appear less compelling.

This paper is an attempt to clear up this issue by a careful characterization of the various families of states, of the varieties of dual pairs of states and dynamical variables. It appears from this analysis that the states exhibiting dissipation and hence break time symmetry are to be selected from an extended set of states. There are the time reversed states which exhibit negative dissipation and are hence not acceptable as physical states obeying the second law of thermodynamics.

2 Stochastic Quantum Dynamics

For a finite dimensional system the density distributions are nonnegative density matrices of unit trace:

$$\rho \neq 0 \quad ; \quad \mathrm{tr}\,\rho = 1 \quad ; \quad \dim \rho = N \,. \tag{2.1}$$

The extremal generating elements of the convex set of density matrices are projections of rank one:

$$\rho = \Pi_\psi = \psi\tilde{\psi} \quad , \quad \tilde{\psi} = \psi^\dagger \,. \tag{2.2}$$

There are an infinite number of such external elements. For a time-independent (hermitian) Hamiltonian H the time evolutions form a unitary one parameter group:

$$\rho(t) = \exp\left(-i\mathcal{L}t\right)\rho \equiv e^{-itH}\,\rho\, e^{+itH} \,. \tag{2.3}$$

Here $\mathcal{L}$ is the Liouville superoperator

$$\mathcal{L}\rho = H\rho - \rho H \,. \tag{2.4}$$

A much more general evolution is given by a parameterized map:

$$\rho \to \mathcal{A}(t)\rho \quad ; \quad \rho_{rs}(t) = \sum_{r's'} \mathcal{A}_{rs,r's'}(t)\,\rho_{r's'} \,. \tag{2.5}$$

Clearly, since the properties of a density matrix must be preserved by the mapping we must have

$$\sum_r \mathcal{A}_{rr,r's'} = \delta_{r's'} \quad ; \quad \mathcal{A}_{sr,s'r'} = \mathcal{A}^*_{rs,r's'} \,. \tag{2.6}$$

If we define the new $N^2 \times N^2$ matrix $\mathcal{B}$ with elements

$$\mathcal{B}_{rr',ss'} = \mathcal{A}_{rs,r's'} \tag{2.7}$$

then these properties of $\mathcal{A}$ may be used to deduce [1]

$$\mathcal{B}_{ss',rr'} = \mathcal{B}^*_{rr',ss'} \quad ; \quad \sum_r \mathcal{B}_{rs',rs'} = \delta_{r's'} \,. \tag{2.8}$$

If the matrix $\mathcal{B}$ is non-negative then we say that the dynamical map $\rho \to \mathcal{A}\rho$ is strictly positive. Not all maps need be strictly positive [7]: the simplest, not strictly positive map, is the map

$$\rho \to \rho^* \,. \tag{2.9}$$

In all cases, since $\mathcal{B}$ is hermitian and finite dimensional, it can be diagonalized; for the strictly positive maps all eigenvalues are non-negative. In this case we obtain

$$\rho \to \sum_{\mu} V(\mu)\rho V^{\dagger}(\mu) \quad ; \quad \sum_{\mu} V^{\dagger}(\mu)V(\mu) = 1, \tag{2.10}$$

where the sum over μ in general runs from 1 to N^2.

The dynamical maps themselves form a convex set since

$$\mathcal{B} = \mathcal{B}_1 \cos^2 \phi + \mathcal{B}_2 \sin^2 \phi \tag{2.11}$$

is also an acceptable map if $\mathcal{B}_1$ and $\mathcal{B}_2$ are. If $\mathcal{B}_1, \mathcal{B}_2$ are both strictly positive so is $\mathcal{B}$. The question naturally arises of finding all generating extremal elements of all dynamical maps and of all strictly positive dynamical maps. The first problem is very complicated and has been done completely [6] only for $N = 2$. For $N > 2$ we know many external maps like unitary, antiunitary and "pin" maps but a complete characterization is still to be done.

For completely positive maps these have been completely done. The extremal cases may be separated into families [7] of rank R, where $1 \leq R \leq N$. The $R = 1$ case corresponds to unitary maps while $R = N$ correspond to the pin maps. Moreover we have a simple construction algorithm for finding all the external maps [10].

The strictly positive maps obtain by a unitary evolution of an extended system consisting of the $N-$ dimensional system of interest and an auxiliarly $R-$ dimensional system which is then contracted by taking partial trace with respect to the auxiliary system. Moreover it can be shown that any strictly positive map may be displayed as a contraction of a rank one map of an extended system.

In this context we note that these maps can be multiplied by performing them in sequence. The result is again a dynamical map. The dynamical maps therefore form a forward semigroup. In particular the strictly positive dynamical maps form a forward semigroup. However these maps are in general not invertible to form a group; they will take the set of density matrices into indefinite matrices (of unit trace!).

Given a unitary map we could consider it being generated by a Liouvillian derived from a Hamiltonian. We can then talk of a continuous group with a generator $\mathcal{L}$. We could ask for a continuous parameter semigroup for the generic semigroup of completely positive maps.

In view of the fact that any such map can be obtained by contraction of a unitary map we may look for clues to the structure of the generator of a dissipative semigroup in such a procedure. By expanding the unitary matrix to second order where dissipation first obtains we get

$$\rho \to \rho - it[H,\rho] + \frac{(it)^2}{2!}\,[H,[H,\rho]] + \cdots . \tag{2.12}$$

Applying it to the extended system and then taking partial trace we obtain

$$\rho \to \rho - it\,[h,\rho] + it\,\left[L_\alpha, \left[L_\beta^\dagger, \rho\right]\right] . \tag{2.13}$$

It is possible to show that this is the generic generator of a completely positive semigroup [?, 11] It is noteworthy that for a finite dimensional system there is no self adjoint Hamiltonian which could lead to dissipation for an isolated system and the Liouvillian eigenvalues are the differences of the Hamiltonian eigenvalues, but if it is coupled to an auxiliary finite dimensional system the contraction map can exhibit dissipation. For the time evolution to obtain as a continuous one-parameter semigroup we may have to take limiting cases of weak coupling and scaling of time.

When the number of dimensions of the vector space becomes denumerably infinite but the Hamiltonian has still a discrete spectrum the situation is not changed dramatically. The only essential change is that there are dynamical maps of arbitarily high rank; and that these can be obtained from not only unitary but also *isometric* operators in the extended space.

3 Liouville Dynamics with Continuous Spectrum

The Liouvillian dynamics of a system with a continuous spectrum furnishes richer possibilities. If ν is a point in the continuous spectrum, $o < \nu < \infty$, then the density matrix may be parameterized by ν_1, ν_2

$$\begin{aligned} H\,\rho(\nu_1,\nu_2) &= \nu_1\,\rho(\nu_1,\,\nu_2 \\ \rho(\nu_1,\,\nu_2)H &= \nu_2\,\rho(\nu_1,\,\nu_2)\,. \end{aligned} \tag{3.1}$$

Then

$$\mathcal{L}\,\rho(\nu_1,\nu_2) = (\nu_1 - \nu_2)\,\rho\,(\nu_1,\,\nu_2)\,. \tag{3.2}$$

and we could relabel the density matrix in the form

$$\rho(\nu; E) \equiv \rho\left(E + \frac{1}{2}\nu,\, E - \frac{1}{2}\nu\right) \quad , \quad -2E < \nu < 2E\,. \tag{3.3}$$

The unitary time evolution is

$$e^{-i\mathcal{L}t}\rho(\nu; E) = e^{i\nu t}\rho(\nu; E)\,. \tag{3.4}$$

The trace is invariant under this evolution

$$\mathrm{tr}\,\left(e^{-i\mathcal{L}t}\rho(\nu; E)\right) = \int dE\,\rho(0; E) \tag{3.5}$$

and the positivity is preserved

$$\rho(\nu; E) \geq 0 \rightarrow e^{-i\mathcal{L}t}\,\rho(\nu; E) \geq 0\,. \tag{3.6}$$

These are equivalent to the statement

$$\int^{\infty}\int_{|\nu|}^{\infty} \rho\,(\nu; E) f^{*}\left(E + \frac{1}{2}\nu\right) f\left(E - \frac{1}{2}\nu\right)\, dE\, d\nu \geq 0\,. \tag{3.7}$$

For any energy E or any finite range of energy $0 < E < E_0$ the time dependent density matrix is an entire function of t and always obeys a group rather than a semigroup. But under suitable conditions the survival probability [12]

$$\begin{aligned} P(t) &= \mathrm{tr}\,(\rho e^{i\mathcal{L}t}\rho) \equiv \int\int \rho(-\nu, E) e^{-i\nu t}\rho(\nu, E) d\nu dE \\ &= \int\int \rho^{\dagger}(\nu, E) e^{-i\nu t}\rho(\nu, E)\, d\nu\, dE \end{aligned} \tag{3.8}$$

may exhibit appropriate exponential behavior. Clearly $P(t)$ is real and bounded by unity¿

Since the density distribution ρ may be expressed in the form

$$\rho(\nu_1,\, \nu_2) = \sum_{\alpha} r_{\alpha}\psi_{\alpha}(\nu_1)\psi_{\alpha}^{\dagger}(\nu_2) \tag{3.9}$$

with $0 \leq \nu_1,\, \nu_2$ and $0 < r_{\alpha} < 1$, it follows that

$$\begin{aligned} \left(e^{-i\mathcal{L}t}\rho\right)(\nu_1, \nu_2) &= \sum_{\alpha} r_{\alpha} e^{-i(\nu_1 - \nu_2)t}\psi_{\alpha}(\nu_1)\psi_{\alpha}^{\dagger}(\nu_2) \\ &= \sum_{\alpha} r_{\alpha}\psi_{\alpha}(\nu_1) e^{-i\nu_1 t} \quad \left(\psi_{\alpha}(\nu_2) e^{-i\nu_2 t}\right)^{\dagger}\,. \end{aligned} \tag{3.10}$$

Hence, the survival probability has the decomposition

$$P(t) = \sum \alpha r_\alpha^2 \, P_\alpha(t) \quad ; \quad P_\alpha(t) = \left| \int_0^\infty \psi_\alpha^\dagger(\nu) e^{-i\nu t} \psi_\alpha(\nu) \right|^2 , \tag{3.11}$$

so that $P_\alpha(t)$ are the absolute values squared of functions of t analytic in the lower half plane. Then by Paley-Weiner theorem

$$\left| \int \frac{\log \, P_\alpha(t)}{1+t^2} \, dt \right| < \infty \, . \tag{3.12}$$

But this is not possible if their convex sum exponentially decreases with t for $t > 0$. This is a slight generalization of a result derived four decades ago by Khalfin [13].

It would at this stage be useful to clarify the kinds of density distributions that we may consider and their analytic continuation and extension of the set of density distributions. But preliminary to this we note that given any set of dynamical variables we may consider the density distributions as their duals. If we consider too wide a class of distributions we restrict the set of observables and vice versa. Finally if we consider analytic continuations of the dynamical variables the density distributions themselves should be analytically continued [14] initially. The dual correspondence should be maintained.

4 Varieties of Statistical State Spaces

Given the density distribution $\rho(\nu; E)$ the time evolution can be displayed as

$$\left(e^{-i\mathcal{L}t} \rho \right) (\nu; E) = e^{-i\nu t} \, \rho(\nu; E) \tag{4.1}$$

and the survival probability in the form

$$P(t) = \int_0^\infty dE \int_{-2E}^{2E} d\nu e^{-i\nu t} \rho^*(\nu, E) \; \rho(\nu, E) \, . \tag{4.2}$$

The integration over the finite segment $-2E < \nu < 2E$ may be deformed to run along some path in the complex plane provided the function $\rho(\nu, E)$ is analytic in ν in a suitable domain in which the new open contour C from $-2E$ to $2E$ lies

$$\begin{aligned} P(t) &= \int dE \; P(t, E) \, ; \\ P(t, E) &= \int_C \rho^*(z^*, E) \, e^{-izt} \rho(z, E) \, dz \, . \end{aligned} \tag{4.3}$$

We may now define various spaces associated with density distributions $\rho(z, E)$. As a preliminary we observe that the density distributions constitute a convex set, not a vector space. We can however relax the positivity condition and define the vector space spanned by the density distributions. We distinguish the following:

1. The space $\mathcal{B}$ of integrable distributions where the ν integration goes over the bounded range $-2E < \nu < 2E$.

2. The space $\mathcal{C}$ of square integrable (and integrable) distributions. This is the analogue of the classical Coopman phase space densities.

3. The space $\mathcal{D}$ of distributions which are boundary values of functions analytic in a domain providing the analytic continuations for complex contours in the variables ν and E (with 0 and ∞ being the limits of integration in $\mathcal{E}$).

4. the space $\mathcal{E}$ of distributions extended from $-\infty$ to $+\infty$ for the variable E.

5. The space $\mathcal{F}$ of distributions analytic in a half plane for ν except for an essential singularity at infinity and suitable analyticity in the variable $\mathcal{E}$ so that we can develop a forward semigroup in time.

6. The space $\mathcal{A}$ of distributions analytic in a half plane for ν and suitable analytical in the variable E.

That these spaces are different is clear. The space $\mathcal{C}$ contains the space $\mathcal{B}$; and $\mathcal{A}$ is contained in the space $\mathcal{F}$. The space $\mathcal{B}$ is contained in the space $\mathcal{E}$. As sets $\mathcal{D}$ and $\mathcal{B}$ are dense in each other but there are elements in $\mathcal{B}$ which have no counterpart in $\mathcal{D}$.

For an isolated system the total energy $\mathcal{E}$ is bounded from below; but *for a system which is open to dynamical interaction with other systems this may not be an essential requirement*; it is only under this provision that the spaces $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{A}$ are relevant.

A piecewise analytic function, or any general measure which belongs to the space can be arbitrarily closely approximated by boundary values of analytic functions. Similarly a distribution along a complex contour in $\mathcal{C}$ can be approximated by arbitrarily closely by functions in $\mathcal{B}$ though there is no one-to-one correspondence between the vectors [15].

A specially interesting case is the complex delta distribution which assigns, to a function representing a vector in the dual space which is analytic in a domain containing the particular complex point, the value of the function at the complex point. There is *no vector in* $\mathcal{B}$ which corresponds to this vector in $\mathcal{C}$, but in $\mathcal{E}$ there is a vector which also lies in $\mathcal{E}$ which corresponds to this vector. This vector would be appropriate for describing the simplest metastable excitations.

If we take a physical state in $\mathcal{D}$ and analytically continue it we can value it as a function along a complex contour together with one or more isolated poles (or, more generally, branch cuts). The pole terms control the behavior of the survival probability *but it is always accompanied by a background integral.* This background integral is essential and it reproduces the correct behavior at short (Zeno) times and long (Khalfin) times. The isolated pole (or poles) by itself would not have a corresponding state in the space $\mathcal{D}$ of physical states.

In the space $\mathcal{F}$ the situation is quite different. There exist states in $\mathcal{F}$ which correspond precisely to a discrete complex point (or points). For a single complex point these are the familiar Gamow-Breit-Wigner states with a unique exponential dependence of the survival probability. These correspond to unique vectors in $\mathcal{F}$. The correspondence between $\mathcal{F}$ and its analytic continuation is a correspondence of complete spaces, not merely dense sets.

A special subset of these functions are analytic in an entire half plane. Such function constitute the Hardy class functions [16] with many interesting properties and are often taken to represent *nascent* metastable states. But the Hardy class property is *not* preserved by time evolution since

$$\rho(\nu; E) \to e^{-i\nu t}\rho(\nu; E) \tag{4.4}$$

which has an *essential (exponential) singularity at infinity.* So after any finite time has elapsed, a nascent state evolves into a non-Hardy class function. These non-Hardy class functions are labeled by the index of exponential growth at infinity.

5 Need for Extended Space: Breaking of Time Symmetry

If the states having a pure exponential survival probability are to be included as natural ("physical") states, the spectrum of energies E are to be extended from $0 < E < \infty$ to $-\infty < E < \infty$. This is the space $\mathcal{E}$ and the space $\mathcal{F}$ of functions which are the boundary values of analytic functions, analytic in a half plane except perhaps for an exponential type singularity at infinity. The extension of the energy spectrum for $0 < E < \infty$ to $-\infty < E < \infty$ is equivalent to lifting the restriction

$$-2E < \nu < 2E \tag{5.1}$$

and allowing the ν integration to run from $-\infty$ to $+\infty$. So if the survival amplitude is calculated for positive and negative times we will get two distinct functions.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(\nu_i E)\, e^{-i\nu t} \cdot \rho^*(\nu, E)\, d\nu\, dE = P(t)\,. \tag{5.2}$$

For $t > 0$, $e^{-i\nu t}$ is a convergence factor for the lower half plane (and an exponential increase for the upper half plane. Hence,

$$P(t) = \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\nu\, e^{-i\nu t} \rho(\nu; E)\, \rho^*(\nu^*, E) \tag{5.3}$$

can be considered as a closed contour integration with the integration closed in the lower half plane by an infinite semicircle contribution vanishes. The result is, then,

$$P(t) = \int_{-\infty}^{\infty} dE \cdot 2\pi i \sum_{\substack{\text{Residues in the} \\ \text{upper half plane}}} \rho(z; E)\rho^*(z^*, E)\, e^{-izt} \tag{5.4}$$

while for $t < 0$

$$P(t) = \int_{-\infty}^{\infty} dE \cdot 2\pi i \sum_{\substack{\text{Residues in the} \\ \text{upper half plane}}} \rho(z, E)\, \rho^*(z^*, E)\, e^{-izt(z+z^*)} \tag{5.5}$$

which furnish two distinct functions of t. In particular if there are no poles in the upper half plane then

$$P(t) = \sum a_j\, e^{-\gamma_j t} \qquad , \qquad t \leq 0\,. \tag{5.6}$$

For the density functions in the extended space $\mathcal{F}$ with no poles in the lower half plane but poles in the upper half plane $P(t)$ is exponentially decreasing with $|t|$ for the past ($t < 0$).

These two classes of functions are disjoint except for the constant function, but the constant function in ν leads to an unphysical survival "probability"

$$P(t) = \delta(t) \,. \tag{5.7}$$

In the extended space $\mathcal{F}$ there are thus two disjoint set of states, the forward evolving states with

$$1 \geq P(t) = \text{tr}\,(\rho(t)\rho(0)) > 0 \quad , \quad t > 0 \,. \tag{5.8}$$

These are the states consistent with the second law of thermodynamics [17]. There is a time reversed set of backward regressing states with

$$1 \geq P(t) > 0 \quad ; \quad t < 0 \,. \tag{5.9}$$

These states are not suitable for a system that obeys the second law. The choice of physical states as forward evolving is the breaking of time symmetry. It is not dependent upon objective information or the act of isolated measurements but it is a property of thermodynamically adapted states and is picked automatically and universally. Open systems must then have this time symmetry breaking of the second law if thermodynamics is to be generally valid.

We now consider in detail the correspondence between the states of open systems in the space $\mathcal{F}$ with the states of closed systems in the spaces $\mathcal{C}$ and $\mathcal{D}$. Clearly given any element of $\mathcal{F}$ we can restrict it to the domain

$$-2E < \nu < 2E \quad , \quad 0 < E < \infty \tag{5.10}$$

and this yields an element of $\mathcal{D}$. However given an element of $\mathcal{D}$ we cannot automatically extend it to $\mathcal{F}$ since analytic continuation to the negative real axis may not be possible.

Despite this *there is a natural splitting of any vector in* $\mathcal{C}$ or in $\mathcal{D}$ into two vectors in F with differing domains of analytic continuation. Given the function $F(\nu, E)$ for a vector in $\mathcal{C}$ which vanishes outside the range $-2E < \nu < 2E$, $0 < E$ we define [14]

$$g(\nu, E) = \frac{+1}{\pi i} \int_{-2E}^{2E} d\nu' \, \frac{f(\nu'; E)}{\nu - \nu' i\epsilon} \,. \tag{5.11}$$

This integral, if it exists, defines a function for all values of ν and is analytic in the lower half plane and hence it is a suitable member of $\mathcal{F}$ appropriate for describing an open system with forward (dissipative) evolution. A companion state with backward (dissipative) evolution is given by

$$h(\nu, E) = \frac{1}{\pi i} \int_{-2E}^{2E} \frac{f(\nu', E)}{\nu' - \nu - i\epsilon} \,. \tag{5.12}$$

Clearly

$$g(\nu, E) + h(\nu, E) = f(\nu, E) \tag{5.13}$$

and hence

$$h(\nu, E) = -g(\nu, E) \quad , \quad |\nu| > 2E \,. \tag{5.14}$$

The functions $g(\nu, E)$ obtained here belong to the space $\mathcal{A}$ of functions analytic in the upper half plane, more restrictive than the space $\mathcal{F}$ where we could have essential singularities of exponential type at infinity. This class of functions are the Hardy class and are sometimes used for describing these states.

However if $f(\nu, E)$ behaves like $\exp(-\nu\tau)$ at infinity for $\tau > 0$ the definition of $g(\nu, E)$ is unaltered, except that it too will behave like $e^{i\nu t}$ at infinity. These dependences are therefore quite appropriate for describing forward dissipative evolution. But these are not in the Hardy space $\mathcal{A}$ but in the space $\mathcal{F}$ discussed above.

This behavior of the density distribution is automatic with temporal evolution. Given $\rho(\nu, E)$ at time $t = 0$, at time t, the density distribution function becomes

$$\rho(\nu, E) e^{-i\nu t} \tag{5.15}$$

which belongs in the space $\mathcal{F}$ but not in the Hardy space $\mathcal{A}$. Each such state is labeled by a parameter τ labeling the rate of exponential growth at infinity. This parameter increases linearly with time evolution and may therefore be called the age of the state [18]. The nascent state introduced before corresponds to states of zero age.

6 Dissipative Semigroups and Moving Frames

When we pass from one inertial frame to another inertial frame the dynamical operators of energy (H), momentum ($\boldsymbol{P}$) and angular momentum ($\boldsymbol{J}$) corresponding to time translation,

space translation and rotation change. The operation of changing to a moving frame is the "boost". For nonrelativistic systems these boosts ($\boldsymbol{G}$) together with seven operators H, $\boldsymbol{P}, \boldsymbol{J}$ and the neutral element mass M constitute the extended Galilei Lie algebra with fundamental commutation relations [19],

$$\begin{aligned}
[H, P_j] &= 0 \;\; ; \;\; [H, J_j] = 0 \;\; , \;\; [H, G_j] = iP_j \, ; \\
[P_j, P_k] &= 0 \;\; ; \;\; [P_j, J_k] = -i\epsilon_{jk\ell} P_\ell \, ; \; [P_j, G_k] = i\delta_{jk} M \\
[J_j, J_k] &= i\epsilon_{jk\ell} J_\ell \;\; ; \;\; [J_j, G_k] = i\epsilon_{jk\ell} G_\ell \, ; \; [G_j, G_k] = 0 \, ; \\
[H, M] &= 0 \;\; ; \;\; [P_j, M] = 0 \;\; ; \;\; [J_j, M] = 0 \, ; \; [G_j, M] = 0 \, .
\end{aligned} \tag{6.1}$$

This is an eleven-parameter algebra which can be integrated to an eleven-parameter extended Galilei group. All its representations are known [20].

Just as the Hamiltonian generator H has associated with it the Liouvillian vector field $\mathcal{L}$ we may introduce the eleven Liouvillians (vector fields) associated with the extended Galilei algebra.

$$\begin{aligned}
&\left[\mathcal{L}_{P_j} \, , \mathcal{L}_{P_k}\right] = 0 \, ; \; [\mathcal{L}_H, \mathcal{L}P_j] = 0 \, ; \; \left[\mathcal{L}_H, \mathcal{L}_{J_j}\right] = 0 \, ; \; \left[\mathcal{L}_{J_j}, \mathcal{L}_{P_k}\right] = i\epsilon_{jk\ell}\mathcal{L}_{P_\ell} \, ; \\
&\left[\mathcal{L}_{J_j}, \mathcal{L}_{J_k}\right] = i\epsilon_{jk\ell}\mathcal{L}{+}_{J_\ell} \, ; \; \left[\mathcal{L}_{G_j}, \mathcal{L}_H\right] = \mathcal{L}_{P-j} \, ; \; \left[\mathcal{L}_{G_j}, \mathcal{L}_{P_k}\right] = 0 \, ; \\
&\left[\mathcal{L}_{J_j}, \mathcal{L}_{G_k}\right] = i\epsilon_{jk\ell}\mathcal{L}_{G_\ell} ; \left[\mathcal{L}_{G_j}, \mathcal{L}_{G_k}\right] = 0 \, .
\end{aligned} \tag{6.2}$$

The vector field corresponding to the neutral element M vanishes identically; so the Liouvillians constitute the ten-parameter Galilei Lie algebra.

A dissipative system has an effective nonselfadjoint Liouvillian $\mathcal{L}_H$. If the antiselfadjoint part is invariant under the Galilean vector field then we will have a Galilean system in which the dissipation is the same in all inertial frames.

For an irreducible realization of the Galilei algebra, that is for an elementary system, the dissipative part must be a constant. This is a generalization of the "internal energy" U which is a Galilean invariant that can be added to the Hamiltonian.

$$U = H - \frac{P^2}{2M} \, . \tag{6.3}$$

When this is complex we have dissipation. For a reducible representation, appropriate for a composite system U may be a function of the relative variables. This is what we would

expect as far as the relative coordinates are concerned; but Galilean invariance requires that the momentum dependence should be through the relative velocities

$$v_j^{ab} = \left(P_j^a / M_j\right) - \left(P_j^b / M_j\right) \tag{6.4}$$

only. This is again a minor generalization of the result for Galilean mutual interactions [21]. Thus, in Galilean systems the *dissipation is independent of the frame.* There need not be any dissipative terms in the generators other than $\mathcal{L}_H$.

When the boost transformation is treated as a Poincaré boost the commutation relations involving this boost $\boldsymbol{K}$ are [22]

$$\begin{aligned} [P_j, K_k] &= -i\delta_{jk}\, H \quad , \quad [H, K_k] = -iP_k \; ; \\ [J_j, K_k] &= i\epsilon_{jk\ell} K_\ell \quad ; \quad [K_j, K_k] = -i\epsilon_{jk\ell} J_\ell \; . \end{aligned} \tag{6.5}$$

The Liouvillian vector fields $\mathcal{L}_P, \mathcal{L}_h, \mathcal{L}_{J_j}, \mathcal{L}_{K_j}$ satisfy the same ten-parameter Poincaré algebra. The Hamiltonian vector field cannot have a dissipative additive term without either the boost vector field $\mathcal{L}_{K_j}$ or the displacement vector field $\mathcal{L}_{P_j}$ being modified by virtue of the Lie algebra relation

$$\left[\mathcal{L}_{K_j}, \mathcal{L}_{P_k}\right] = i\,\delta_{jk}\,\mathcal{L}_H \; . \tag{6.6}$$

This complicates the inclusion of dissipation into a relativistic theory.

Following Dirac [22] we may choose the *point form* or the *instant form* fo Liouvillian dynamics. In the first case we will make $\mathcal{L}_{P_j}$ and $\mathcal{L}_H$ have dissipation teams while $\mathcal{L}_{K_j}$ would not have any. This is accomplished as follows: define the quadratic form in the generator [23]

$$\mathcal{M}^2 = P_j \cdot P_j + H^2 \; . \tag{6.7}$$

This form corresponds to the square of the mass and in an invariant under the Poincaré vector fields. A dissipative mass operator is constructed either phenomenologically or form an underlying dynamical theory and identified with the square of the Hamiltonian in the center of mass frame. For any frame which moves with a velocity (in units of the velocity of light) $\boldsymbol{u}$ in relation to the center of mass, the generators are defined by

$$P_j = \frac{M u_j}{\sqrt{1-u^2}} \quad ; \quad H = \frac{M}{\sqrt{1-u^2}} \; . \tag{6.8}$$

The vector fields $\mathcal{L}_{P_j}$ and $\mathcal{L}_H$ may now be constructed for each value of $\boldsymbol{u}$. We could of course retain the relation

$$P_j = u_j H \ . \tag{6.9}$$

So if M is dissipative so would be H *and* P_j. While a complex momentum seems strange it is consistent with the picture of a decaying metastable excitation [24]: if it was created at time 0 at time $t = \tau$ its probability density at the origin would decrease as $\exp(-\gamma\tau)$. But the density at the wave front will be large since they are from the undecayed excitation at $t = 0$. So the spatial dependence should represent an amplitude modulation appropriate to a complex momentum.

The *instant form* treats the center of mass behavior in the same fashion, except that we may, if we choose, construct a dissipative *mass* operator rather than a $(mass)^2$ operator. The momentum in the moving frame is considered as a real quantity with no disspation. This means that the boost must contain some disspation by virtue of the special commutation relation between energy, momentum and boost. The dissipation in the energy is now frame dependent

$$H = \sqrt{P^2 + M^2} \tag{6.10}$$

and hence $\mathcal{L}_H$ depends on P^2. But unlike in the point form, the dissipation does not simply scale.

There is a third possible way of looking at the problem and that is to retain the Lorentz transformations only for their effect on the Liouvillian $\mathcal{L}_H$ and ignore the translation vector fields $\mathcal{L}_{P_j}$. This becomes a reasonable way of approach when we recall that a state approximating a thermodynamic state is infinitely extended and the total momentum (or its vector field the displacement operator) are not relevant in such cases. The dissipation, to the extent that it is intensive (that is proportional to the volume) will change from frame to frame. In the geometrical language of the Minkowski space all these generators are in the future light cone with negative definite dissipative terms. Since these infinitely extended states have a preferred frame of the center of mass and the boost of an infinitely extended object is not possible, global Lorentz invariance itself is broken for these states.

7 Dynamical Processes and Dissipative Evolution

The discussion so far has been about generic systems. We have not yet talked about interactions, scattering and explicit dissipative evolutions. Let us start with a generic system with "total Hamiltonian" H which may be written

$$H = H_C + V \tag{7.1}$$

where H_C is *isospectral* with H and is a simple structure, say a collection of " free Hamiltonians." If $|E, r\rangle$ are a set of (ideal) eigenstates of H_C with degeneracy label r and $|E, r \gg$ a corresponding set of (ideal) eigenstates of H, then there would be, by definition of the isospectral property of H and H_C, a one-to-one correspondence and with the same degeneracies. Apart from normalization this correspondence may be written

$$\begin{aligned} |E \gg &= |E\rangle + (E - H_C + i\epsilon)^{-1} V |E \gg \\ &= \left[1 - (E - H_C + i\epsilon)^{-1} V\right]^{-1} |E\rangle \, . \end{aligned} \tag{7.2}$$

This can be formally expanded in the perturbation series

$$|E \gg = |E\rangle + \sum_1^\infty \{G_C(E) V\}^n \, |E\rangle \qquad ; \quad G_C(E) = (E - H_C + i\epsilon)^{-1} \, . \tag{7.3}$$

This corresponds to the "in" state appropriate for the initial state of scattering; with the energy denominators chosen with $-i\epsilon$ will furnish the "out" states. Both the solution and the perturbation expansion can be extended from the Hilbert spaces to the analytically continued spaces [26]. No substantial change is needed if the spectrum condition is relaxed to include arbitrarily large negative energy continua.

In place of discussing the problem of the correspondence between the (ideal)eigenstates of H and H_C we could make the comparison in terms of (ideal) density distributions in relation to the Liouville operators $\mathcal{L}$ and $\mathcal{L}_C$

$$\begin{aligned} \mathcal{L} \, R(\nu; E) &= \nu \, R(\nu, E) \, ; \\ \mathcal{L}_C \, R_C(\nu; E) &= \nu \, R_C(\nu, E) \, . \end{aligned} \tag{7.4}$$

Here $R(\nu, E)$ and $R_C(\nu, E)$ are the (ideal) density distributions

$$R_{rs}(\nu; E) = |E + \frac{1}{2}\nu, r \gg \ll E - \frac{1}{2}\nu, s|$$

$$R_{C_{rs}}(\nu;E) \;=\; |E+\frac{1}{2}\nu,r\rangle\,\langle E-\frac{1}{2}\nu,s|\,. \tag{7.5}$$

They are related by

$$\begin{aligned} R_{\rm in}(\nu;E) &= \left\{1-G_C\left(E+\frac{1}{2}\nu+i\epsilon\right)V\right\}^{-1} R_C(\nu;E)\left\{1-G_C^\dagger\left(E-\frac{1}{2}\nu+i\epsilon\right)V\right\}^{-1} \\ R_{\rm out}(\nu;E) &= \left\{1-G_C\left(E+\frac{1}{2}\nu-i\epsilon\right)V\right\}^{-1} R_C(\nu;E)\left\{1-G_C^\dagger\left(E-\frac{1}{2}\nu-i\epsilon\right)V\right\}^{-1} \end{aligned} \tag{7.6}$$

The scattering probability for ideal states is

$$\begin{aligned} &{\rm tr}\left\{R^\dagger_{{\rm out}_{rs}}(\nu,E)\,R_{{\rm in}_{r's'}}(\nu',E')\right\} \\ &\quad =\ll E-\frac{1}{2}\nu,s,{\rm out}|E'-\frac{1}{2}j',s',{\rm in}\gg\ll E'+\frac{1}{2}\nu',r',{\rm in}|E+\frac{1}{2}\nu,r,{\rm out}\gg \end{aligned} \tag{7.7}$$

But

$$\ll E-\frac{1}{2}\nu,{\rm sout}|E'-\frac{1}{2}\nu',s',{\rm in}\gg=\delta\left(E-E'-\frac{1}{2}\nu+\frac{1}{2}\nu\right)S_{rs}\left(E-\frac{1}{2}\nu\right) \tag{7.8}$$

is the scattering matrix. So the scattering probability can be expressed in the form

$$\delta(E-E')\delta(\nu-\nu')\;S_{rs}\left(E-\frac{1}{2}\nu\right)S^\dagger_{r's'}\left(E+\frac{1}{2}\nu\right)\,. \tag{7.9}$$

With the proper understanding of the adjoints and duals, these considerations apply not only to the real spectrum representations but also to analytic continuations. Of course, as long as one deals with the real spectrum representation, no metastables per se occur in the scattering probabilities; rather, the resonances manifest themselves by characteristic "resonant shapes" of the probability distributions. If we want to consider the role of metastable states and the scattering of metastable excitations we should consider the analytic continuations which would uncover the resonant states as members of complete set of states.

The (unnormalized) states $|E\gg$ can be normalized by suitable state sensitive multiplicative changes. When this is done we denote

$$|E,{\rm in}\gg=\Omega_{\rm in}\,|E\rangle\quad;\quad |E,r;{\rm in}\gg=\Omega_{{\rm in}r,s}|E,s\rangle\,. \tag{7.10}$$

with $\Omega_{{\rm in}r,s}$ as a *unitary* operator. Then, if F_C is any invariant for the Hamiltonian H_C,

$$[F_C,H_C]=0\,, \tag{7.11}$$

then there is an invariant of H given by

$$F = \Omega F_C \Omega^\dagger, \quad [F, H] = 0 \tag{7.12}$$

by virtue of

$$H = \Omega H_C \Omega^\dagger. \tag{7.13}$$

But there is no guarantee that if the matrix elements of F_C are smooth nonsingular functions of E, the matrix elements of F are nonsingular functions. In cases where there is nontrivial scattering the matrix elements of F will definitely be singular functions. F and F_C are unitarily equivalent and are respectively constants of motion for H and H_C. If H and H_C share some symmetry properties, the corresponding operators are regular constants of motion for both H_C and H; these are analogous to the traditional constants of motion for the total Hamiltonian in classical dynamics. But there are additional constants of motion.

Let us now consider time evolution. For the Hilbert space, the time evolution is the exponential of a hermitian Hamiltonian and, as such, is unitary (norm preserving) whether the energy spectrum is bounded below or not. When we generalize to dual spaces, there is no longer a norm for the state. We must rather consider the invariance of the scalar product bilinear in the vector of the two dual spaces. If $\psi, \tilde{\phi}$ are such vectors, we have

$$\tilde{\phi}\,\psi \rightarrow \tilde{\phi}(t)\,\psi(t) = \tilde{\phi}\,e^{iHt}\,e^{-iHt}\psi = \tilde{\phi}\psi\,. \tag{7.14}$$

If H has complex eigenvalues for ψ there are ϕ which the *same* complex eigenvalues, and therefore the product of the two remains constant. But it is no longer true that $\psi(t)$ has the same "length" as $\psi(0)$; the notion of the "length" of a vector is not defined in dual spaces. But if there were complex eigenvalues of H, then it is clear that $\psi(t)$ can be a complex multiple of $\psi(0)$. But $\tilde{\phi}(t)$ would be the inverse multiple of $\tilde{\phi}(0)$.

Similar considerations apply for the spaces spanned by density distributions. In this case, there is always an invariant state with normalized trace; and all the other states are pseudodensities with zero trace. The evolution is "unitary", that is, preserving scalar products between duals.

A measure of this scale change is provided by the survival amplitude

$$A(t) = \tilde{\phi}(0)\,\psi(t) \equiv \tilde{\phi}\,e^{-iHt}\psi\,. \tag{7.15}$$

As ψ change so does $A(t)$. In particular, if the state ψ is dominated by a complex pole at z, then the survival amplitude has the dependence

$$A(t) = A(0)\, e^{-zt} . \tag{7.16}$$

When

$$\Im\, z < 0\ ,\ t > 0\ , \qquad |A(t)| \ < \ |A(0)| \ . \tag{7.17}$$

Thus, *in this sense*, the complex energy state is a *decaying* state.

From the vectors $\psi, \tilde{\phi}$ in the dual spaces we can construct pseudodensity distributions

$$\rho = \psi_1 \psi_2^\dagger \quad , \quad \sigma = \tilde{\phi}_1^\dagger \tilde{\phi}_2 \tag{7.18}$$

which generate dual spaces.[2] With these we can calculate the survival probability

$$\begin{aligned} P(t) &= \mathrm{tr}(\sigma\rho(t)) = \mathrm{tr}\left(\sigma\, e^{-Ht}\rho e^{iHt}\right) \\ &= \left(\tilde{\phi}\, e^{-iHt}\psi\right)\left(\psi^\dagger e^{iHt}\tilde{\phi}^\dagger\right) \\ &= |A(t)|^2 \ . \end{aligned} \tag{7.19}$$

Therefore, if we know the survival amplitude, the survival probability can be computed.

When the states contain a superposition of eigenvectors of H, the behavior is

$$P(t) = \left|\sum \tilde{\phi}_E \psi_E\, e^{-iEt}\right|^2 \longrightarrow \left|\int \phi^*(E)\psi(E)^{-iEt} dE\right|^2 \ . \tag{7.20}$$

As long as $\psi(E)$ and $\phi^*(E^*)$ are boundary values of functions analytic in the lower half plane except for poles (or "short" branch cuts), we could evaluate the integral over E by closing the contour with an infinite semicircle in the lower half plane (for $t > 0$). If there is only one pole in the lower half plane, the entire survival amplitude is as if there were only one complex "energy" point eigenvalue contributing to the integral. More generally it will the superposition of several such and possibly an integral over them.

Let us consider the single complex eigenvalue in detail. The state vector

$$\psi_0(E) = N_0\, (E - z)^{-1} \quad ; \quad N_0^2 = \frac{z^* - z}{2\pi i} \ ; \tag{7.21}$$

and its dual

$$\tilde{\phi}_0(E) = N_0\, (E - z^*)^{-1} \tag{7.22}$$

give the survival amplitude for $t > 0$

$$\begin{aligned} A_0(t) &= \int \tilde{\phi}_0(E)\, e^{iEt} \phi_0(E) dE \\ &= N_0^2 \int \frac{e^{-iEt}}{(E-z)(E-z^*)}\, dE = -\frac{2\pi i}{z - z*}\, N_0^2\, e^{-izt} = e^{-izt}\,. \end{aligned} \tag{7.23}$$

In this case the analytic continuation of the wavefunction vanishes as z^{-1} at infinity. For $t < 0$ we get similarly

$$A_0(t) = e^{+iz^*t}\,. \tag{7.24}$$

The survival amplitude as a function of t is therefore the join of two *distinct* analytic functions, one for $t > 0$ and another one for $t < 0$.

Now consider the state

$$\begin{aligned} \psi_\tau(E) &= N_0 (E-z)^{-1}\, e^{-iz\tau} \\ \tilde{\phi}_\tau(E) &= N_0 (E-z^*)^{-1} \end{aligned} \tag{7.25}$$

obtained by a multiplicative transformation. Then the survival amplitude is

$$\begin{aligned} A_\tau(t) &= N_0^2 \int (E-Z)^{-1}\, (E-Z^*)^{-1}\, e^{i(z-z^*)\tau}\, e^{-izt} dE \\ &= N_0^2\, \frac{-2\pi i}{z - z^*}\, e^{iz\tau}\, e^{izt} \end{aligned} \tag{7.26}$$

which may be written

$$A_\tau(t) = A_0(t + \tau)\,. \tag{7.27}$$

In other words, the state ψ_τ may be thought of as having been created at time $t = \tau$. If τ is positive, we extrapolate, *for these states*, the semigroup for negative values of t such that

$$t + \tau > 0\,. \tag{7.28}$$

This quantity τ may be called the "age" of the state in the *extended* space.

Having defined the age and the survival amplitude for the states we could define the age and survival probability for density distributions. Analytic density distributions in the space $\mathcal{F}$ can be chosen so that we can define the forward semigroup on them. But after the time

evolution for any finite time is considered, the states are no longer in $\mathcal{F}$ but are in $\mathcal{E}$. If we denote a state in $\mathcal{F}$ by $\rho_0(\nu, E)$, then

$$\rho_\tau(\nu, E) = e^{-i\nu\tau}\, \rho_\tau(\nu, E) \tag{7.29}$$

is not in the space $\mathcal{F}$ but remains in the space $\mathcal{E}$. For them, the forward semigroup can be extrapolated to *negative* values of t such that $t + \tau > 0$. These are the metastable states with an age τ.

In these discussions we have labeled the density distribution $\rho(\nu, E)$ with the labels appropriate to the total energy and total Liouvillian.

In many cases we have a comparison Hamiltonian H_C and an interaction V such that H_C is isospectral with H and

$$H = H_C + V\,. \tag{7.30}$$

Then we could have an alternate labeling of the states by ν_C, E_C appropriate to

$$\begin{aligned} H_C\,\rho &= \left(E_C + \frac{1}{2}\nu_C\right)\rho \\ \rho\, H_C &= \left(E_C - \frac{1}{2}\nu_C\right)\rho\,. \end{aligned} \tag{7.31}$$

To avoid confusion we use the symbol R for the density distribution labeled by ν_C, E_C so that

$$R(\nu_C, E_C) \equiv \rho\,(\nu, E) \tag{7.32}$$

with ν_C having the same range as ν and E_C the same range as E. Then

$$R(\nu, E) = \left(1 - G_C\left(E + \frac{1}{2}\nu\right)V\right)^{-1} R_0 \left(1 - G_C\left(E - \frac{1}{2}\nu\right)V\right)^{\dagger - 1}\,. \tag{7.33}$$

It may be that the creation of the state may be most simply described in the comparison Hamiltonian representation R_C and then the dependence on the variables ν, E would be governed by the dependence of the wave matrix factors preceding and following R_C. In the special can of the Dirac-Friedrichs-Lee model [27] of a discrete (metastable) state coupled to a continuum, the resonant complex pole plus background dependence is immediately realized if the initial state is the discrete state of the comparison Hamiltonian with the discrete energy level lowered by continuing in the "mass" parameter of the model. The time

dependence of the survival amplitude and survival probability have been studied extensively in the literature [28].

More generally, the wave matrix

$$\Omega(\varepsilon) = (1 - G(\varepsilon)V)^{-1} \tag{7.34}$$

has an *analytic dependence*[29] on ε; and as a consequence if $R_C(\nu, E)$ is a simple function of E and ν, then $R(\nu, E)$ will be *analytic* in both ν and E. The singularities of the wave matrix in the complex variable of energy reappears in the survival probability. While both the scattering amplitude and wave matrix depend on both the total Hamiltonian H and the comparison Hamiltonian H_C, it is known that only the singularities of the wave matrix appear in the survival amplitude; the redundant poles of the scattering amplitude, if any, do not contribute. Any way, the *survival amplitude depends on both the total Hamiltonian H and the comparison Hamiltonian H_C.*

The second point to be noted is that when we consider the time evolution, despite the fact that the (norm)^2 of the state is not defined directly, we still can talk of affine scale: that is whether the state gets multiplied by a number e^{izt}. Such states *do not* exist in $\mathcal{C}$ but do exist in the extended space $\mathcal{E}$. They may be realized along the real axis but could equally well be identified as complex discrete energy states. As a consequence, while the product of a state and its dual is invariant under time evolution, the survival amplitude does depend on time. For the special state corresponding to a discrete complex pole, the dependence is purely exponential.

References

[1] Part I was published in Physical Review three decades ago: E.C.G. Sudarshan, P.M. Mathews and J. Rau, *Phys. Rev.* **121**, 920 (1961).

[2] I. Segal, *Annals of Math* **48**, 930 (1947); R. Haag and D. Kastler, *J. Math Phys.* **5**, 848 (1964). Singular linear functionals are used in E.C.G. Sudarshan, *Phys. Rev. Let.* **10**, *277 (1963); C.L. Mehta and E.C.G. Sudarshan, Phys. Rev.,* **B274**, *138 (1965).*

[3] J. von Neumann, "Mathematical Foundations of Quantum Mechanics," Princeton University Press, Princton, NJ (1955).

[4] E.C.G. Sudarshan, P.M. Mathews and J. Rau, Phys. Rev **121**, *920 (1961);*
E.B. Davies, Comm. Math. Phys. **15**, *277 (1969);* **19**, *83 (1970);* **23**, *51 (1971);*
G. Lindblad, Comm. Math. Phys. **40**, *147 (1975). A. Kossakowski, Rep. Math. Phys.* **13**, *149 (1978)*
V. Gorini, F. Verri and A. Kossakowski, Rep. Math Phys. **13**, *149 (1978);*
V. Gorini, A. Kossakowski and E.C.G. Sudarshan, J. Math. Phys. **17**, *821 (1976);*
V. Gorini, A. Frigerio, F. Verri, A. Kossakowski and E.C.G. Sudarshan, Rep. Math. Phys. **13**, *149 (1978).*

[5] F. Bloch, Phys. Rev. **70**, *460 (1946).*

[6] V. Gorini and E.C.G. Sudarshan, Comm. Math. Phys. **46**, *43 (1976).*

[7] M.D. Choi, Can. J. Math **24**, *520 (2972); Illinois J. Math* **18**, *565 (1974).*

[8] E.C.G. Sudarshan, in "From SU(3) to Gravity," edited by E. Gotsman and G. Tauber, Cambridge Univ. Press, London 1985 p.33.

[9] T. Petrosky and I. Prigogine, Proc. Nat. Acad. Sc. USA, **90**, *933 (1993); Physica,* **A147**, *439 (1988).*

[10] E.C.G. Sudarshan, Phys. Rev **46**, *37 (1992).*

References

[1] Part I was published in Physical Review three decades ago: E.C.G. Sudarshan, P.M. Mathews and J. Rau, *Phys. Rev.* **121**, 920 (1961).

[2] I. Segal, *Annals of Math* **48**, 930 (1947); R. Haag and D. Kastler, *J. Math Phys.* **5**, 848 (1964). Singular linear functionals are used in E.C.G. Sudarshan, *Phys. Rev. Lett.* **10**, 277 (1963); C.L. Mehta and E.C.G. Sudarshan, *Phys. Rev.*, **B274**, 138 (1965).

[3] J. von Neumann, "Mathematical Foundations of Quantum Mechanics," Princeton University Press, Princton, NJ (1955).

[4] E.C.G. Sudarshan, P.M. Mathews and J. Rau, *Phys. Rev* **121**, 920 (1961);
E.B. Davies, *Comm. Math. Phys.* **15**, 277 (1969); **19**, 83 (1970); **23**, 51 (1971);
G. Lindblad, *Comm. Math. Phys.* **40**, 147 (1975). A. Kossakowski, *Rep. Math. Phys.* **13**, 149 (1978)
V. Gorini, F. Verri and A. Kossakowski, *Rep. Math Phys.* **13**, 149 (1978);
V. Gorini, A. Kossakowski and E.C.G. Sudarshan, *J. Math. Phys.* **17**, 821 (1976);
V. Gorini, A. Frigerio, F. Verri, A. Kossakowski and E.C.G. Sudarshan, *Rep. Math. Phys.* **13**, 149 (1978).

[5] F. Bloch, *Phys. Rev.* **70**, 460 (1946).

[6] V. Gorini and E.C.G. Sudarshan, *Comm. Math. Phys.* **46**, 43 (1976).

[7] M.D. Choi, *Can. J. Math* **24**, 520 (2972); *Illinois J. Math* **18**, 565 (1974).

[8] E.C.G. Sudarshan, in "From $SU(3)$ to Gravity," edited by E. Gotsman and G. Tauber, Cambridge Univ. Press, London 1985 p.33.

[9] T. Petrosky and I. Prigogine, *Proc. Nat. Acad. Sc. USA*, **90**, 933 (1993); *Physica*, **A147**, 439 (1988).

[10] E.C.G. Sudarshan, *Phys. Rev* **46**, 37 (1992).

[11] A. Kossakowski, *Rep. Math. Phys.* **3**, 247 (1972; *Bull. Acad. Pol. Soc. Sci. Math. Phys.* **20**, 1021 (1972).

[12] B. Misra and E.C.G. Sudarshan, *J. Math. Phys.* **18**, 756 (1977).

[13] L. Khalfin, *Soviet Physics JETP* **6**, 1053 (1958).

[14] P.L. Kapur and R. Peierls, *Proc. Roy. Soc. Lond* **A166**, 277 (1938);
N. Hu, *Phys. Rev.* **74**, 131 (1948); E.C.G. Sudarshan, C.B. Chiu and V. Gorini, *Phys. Rev.* **D18**, 2914 (1978);
G. Parravicini, V. Gorini and E.C.G. Sudarshan, *J. Math. Phys.* **21**, 2208 (1980).

[15] E.C.G. Sudarshan, *Phys. Rev.* **A46**, 37 (1992); *Phys. Rev.* **A50**, 2006 (1994).

[16] M. Rosenblum and J. Rovyak, "Hardy Classes and Operator Theory," (Oxford University Press, New York 1985);
A. Bohm and N, Gaella, "Dirac Kets, Gamow Vectors and Gelfand Triplets" edited by A. Bohm and J.D. Dollard, Springer Lecture Notes in Physics vol. 348, Springer-Verlag, Berlin (1989).

[17] I. Prigogine, "From Being to Becoming," (Freeman, NY 1980);
G. Nicolis and I. Prigogine, "Exploring Complexily," (Freeman, NY 1989).

[18] A. Ramakrishnan, "Probability and Stochastic Processes," in Handbuch de Physik, S. Flugge (ed), Vol. III/2, Springer, Berlin (1959);
E.C.G. Sudarshan, *Phys. Rev. A46*, 37 (1992).

[19] V. Bargman, *Annals of Math* **59**, 1 (1947);
M. Hamermesh, *Annals of Phys.* **9**, 518 (1960).

[20] E. Inönü and E.P. Wigner, *Nuovo Cimento* **9**, 705 (1952).

[21] L. Eisenbud and E.P. Wigner, *Proc. Nat. Acad. Sc. US* **27**, 281 (1941);
S. Okubo and R.E. Marshak, Annals of Phys, **4**, 166 (1958).

[22] P.A.M. Dirac, *Rev. Mod. Phys.* **21**, 392 (1949);
D.G. Currie, T.F. Jordan and E.C.G. Sudarshan, *Rev. Mod. Phys.* **35**, 350 (1963).

[23] E.C.G. Sudarshan and N. Mukunda, "Classical Dynamics," John Wiley, New York.

[24] R.E. Peierls, *Proc. Roy. Soc.* **A253**, 16 (1959).

[25] E.C.G. Sudarshan, C.B. Chiu and G. Bhamathi, *Physica* **A202**, 540 (1994).

[26] E.C.G. Sudarshan, C.B. Chiu and V. Gorini, *Phys. Rev.* **D18**, 2914 (1978).
E.C.G. Sudarshan, C.B. Chiu, *Phys. Rev.* **D47**, 2602 (1993).

[27] P.A.M. Dirac, "Principles of Quantum Mechanics," Oxford University Press, 3nd edition (1947);
K.O. Friedrichs, *Comm. Pur Appl. Math* **1** 361 (1948).
T.D. Lee, *Phys. Rev.* **95**, 1329 (1954).

[28] C.B. Chiu and E.C.G. Sudarshan, *Phys. Rev.* **D42**, 3712 (1990);
C.B. Chiu, E.C.G. Sudarshan and G. Bhamathi,"Advances in Chemical Physics," (in press) and references contained there.

[29] E.C.G. Sudarshan, V. Gorini, C.B. Chiu, *Phys. Rev.* **D18**, 2914 (1978).