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TIME REVERSAL FOR SPACETIME AND INTERNAL SYMMETRY

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ABSTRACT

The standard time reversal transformations have to be generalized when we deal with systems with internal symmetry. This generalization is formulated and adapted to higher symmetries involving the angular momentum.

Time reversal in quantum mechanics is a discrete transformation which associates with each motion a "time reversed motion" determined *kinematically*. When the reversed motion obeys the *same* equations of motion as the direct equations we state that there is *time reversal invariance*. Since the choice of the time origin is arbitrary, time reversal invariance automatically implies *time translation invariance*, whether it is reversible or irreversible.

THE WIGNER TIME REVERSAL

In quantum mechanics the time reversal transformation is realized anti-unitarily:

$$\psi(t) \longrightarrow e^{i\Theta}\,\psi^*(-t) \tag{1}$$

where the phase Θ maybe chosen at our convenience. We recognize that if we make a global phase change

$$\psi \longrightarrow e^{i\alpha}\,\psi \tag{2}$$

for all wavefunctions the time reversal transformation changes; but then α can be exploited to restore the time reversal to the standard form. On the other hand

$$C_1\,\psi_1(t) + C_2\,\psi_2(t) \longrightarrow C_1^*\,\psi_1^*(-t) + C_2^*\,\psi_2^*(-t)\,. \tag{3}$$

When the particles have spin this is no longer sufficient. Since all $SU(2)$ representations are either real (bosons) or pseudo real (fermions)[2] we have to define

$$\psi(t) \longrightarrow \exp(i\pi J_2)\psi^*(-t)\,. \tag{4}$$

Rotations are unaffected by this transformation but the generators of rotation change sign:

$$\boldsymbol{J} \longrightarrow \exp(i\pi J_2)\boldsymbol{J}^* \exp(-i\pi J_2) = -\boldsymbol{J} \tag{5}$$

in accordance with the behaviour expected of angular momentum under time reversal. All this is well known and introduced here for establishing the formalism.

SEQUENCES OF ROTATIONS UNDER TIME REVERSAL

We have seen that angular momentum $\boldsymbol{J}$ undergoes a change of sign under time reversal. What happens to a finite rotation? If

$$\psi_2 = e^{i\boldsymbol{J}\cdot\boldsymbol{\theta}}\psi_1 \tag{6}$$

we find

$$\tilde{\psi}_2 = e^{i\pi J_2}\, e^{-i\boldsymbol{J}^*\cdot\boldsymbol{\theta}}\, e^{-i\pi J_2}\,\tilde{\psi}_1 = e^{i\boldsymbol{J}\cdot\boldsymbol{\theta}}\,\tilde{\psi}_1\,. \tag{7}$$

That is, the time reversed states are related by the *same* rotation.

This is to be contrasted with a *sequence* of rotations at various time intervals. For example if we write

$$\psi(t_3) = e^{iH(t_3-t_2)}\, e^{i\boldsymbol{J}\cdot\boldsymbol{\theta}}\, e^{-iH_2(t_2-t_1)}\, e^{i\boldsymbol{J}\cdot\boldsymbol{\theta}'} e^{-iH(t_1-t_0}\psi(t_0) \tag{8}$$

the time reversed states will satisfy

$$\tilde{\psi}\,(-t_3) = e^{+iH(t_3-t_2)}\, e^{i\boldsymbol{J}\cdot\boldsymbol{\theta}} e^{+iH(t_2-t_1)} e^{i\boldsymbol{J}\cdot\boldsymbol{\theta}'} e^{+iH(t_1-t_0)}\,\tilde{\psi}(-t_0)\,. \tag{9}$$

But if we choose

$$t_3 > t_2 > t_1 > t_0\,, \tag{10}$$

the second equation is better re-expressed in the form

$$\tilde{\psi}(-t_0) = e^{-iH(-t_0+t_1)} e^{-i\boldsymbol{J}\cdot\boldsymbol{\theta}'} e^{iH(-t_1+t_2)} e^{-i\boldsymbol{J}\cdot\boldsymbol{\theta}} e^{-iH(-t_2+t_3)}\tilde{\psi}(-t_3) \tag{11}$$

with

$$-t_0 > -t_1 > -t_2 > -t_3\,. \tag{12}$$

So when the time reversed evolution unfolds the rotations are *inverses* of the original rotations in the *reverse* order. We recognize that these also furnish a representation of the rotation group:

$$\begin{aligned} R_1\, R_2 \;\;\longrightarrow\;\; & \mathcal{D}^{-1}(\mathcal{R}_2)\,\mathcal{D}^{-1}(\mathcal{R}_1) = (\mathcal{D}(\mathcal{R}_1)\,\mathcal{D}(\mathcal{R}_2))^{-1} \\ & = \mathcal{D}^{-1}(\mathcal{R}_1\,\mathcal{R}_2)\,. \end{aligned} \tag{13}$$

All the rotations that happened are reversing themselves in the time reversed picture.

TIME REVERSAL FOR SYSTEMS WITH INTERNAL SYMMETRIES

We now seek the behaviour of particles transforming as *isospin* (flavor $SU(2)$) multiplets. In this case neither the simple conjugation

$$\psi(t) \longrightarrow \psi^*(-t) \tag{14}$$

nor the augmented conjugation

$$\psi(t) \longrightarrow \exp(i\pi I_2)\psi^*(-t) \tag{15}$$

would be appropriate. The first one would not commute with the $SU(2)$ group, while in the second one the isospin $\boldsymbol{I}$ would be reversed and along with it the electric change, say of the pion triplet. We need a new kind of time reversal transformation.

Since we want the time reversed pion of a definite charge to be a pion state of the same charge, we write:

$$\begin{pmatrix} \psi_+(t) \\ \psi_0(t) \\ \psi_-(t) \end{pmatrix} \longrightarrow \begin{pmatrix} \psi_+^*(-t) \\ \psi_0^*(-t) \\ \psi_-^*(-t) \end{pmatrix} . \tag{16}$$

This means that the internal symmetry labels are unaffected by time reversal. For the nucleon doublet we have

$$\begin{pmatrix} p(t) \\ n(t) \end{pmatrix} \longrightarrow \exp\left(\frac{i\pi}{2}\sigma_2\right) \begin{pmatrix} p^*(-t) \\ n^*(-t) \end{pmatrix} . \tag{17}$$

These transformations point out that the time reversal transformation is not a straight forward antiunitary transformation but requires some delicate handling.

For a system with internal symmetry we proceed as follows: Choose a complete set of commuting symmetry operators $\{H_\ell\}$ and their normalized eigenfunctions $\eta_{\lambda(\ell)}$. Consider these quantitites as "real," that is unchanged under time reversal. The generic state may be written in the form

$$\psi(t) = \sum_{\{\lambda(\ell)\}} \psi_{\lambda(\ell)}(t) \cdot \eta_{\lambda(\ell)} . \tag{18}$$

The time reversal transformation is defined as

$$\psi(t) \longrightarrow \tilde{\psi}(-t) = \sum_{\{\lambda(\ell)\}} \exp(i\pi J_2)\psi^*_{\lambda(\ell)}(-t) \cdot \eta_{\lambda(\ell)} \tag{19}$$

that is,

$$\begin{aligned} \eta_{\lambda(\ell)} &\longrightarrow \eta_{\lambda(\ell)} \\ \psi_{\lambda(\ell)} &\longrightarrow \exp(i\pi J_2)\psi^*_{\lambda(\ell)}(-t) . \end{aligned} \tag{20}$$

In the Cartan notation[4] $E_\alpha \longrightarrow E_\alpha$, $H_\ell \longrightarrow H_\ell$. It follows that internal quantum numbers like hypercharge, baryon number or isospin are *not* changed so that for example the $SU(3)$ decuplet goes into the decuplet (and note the $\overline{10}$); but the angular momentum changes to its negative.

SEQUENCE OF INTERNAL SYMMETRY TRANSFORMSTIONS

For time reversal the Cartan-Weyl basis of the Lie algebra $(\{H_\ell\}, \{E_\alpha\})$ is considered invariant. Hence an arbitrary element of the Lie algebra undergoes the transformation

$$A \longrightarrow \tilde{A} \tag{21}$$

with

$$A = \sum C_\ell H_\ell + \sum C_\alpha E_\alpha \ , \ \tilde{A} = \sum C_\ell H_\ell + \sum C_\alpha^* E_\alpha = \sum C_\ell H_\ell + \sum C_{-\alpha} E_\alpha \tag{22}$$

with the understanding that

$$C_\ell^* = C_\ell \ \ , \ \ C_{-\alpha} = C_\alpha^* . \tag{23}$$

For a finite transformation

$$\exp(iA) \longrightarrow \exp(-i\tilde{A}) . \tag{24}$$

The finite transformations generated by the any linear combination of H_ℓ, $E_\alpha + E_{-\alpha}$ goes into its inverse, while any transformation generated by $i(E_\alpha - E_{-\alpha})$ remains unaffected. (For example the isospin rotation around the second axis is a real orthogonal transformation between the pions and is invariant.)

When we consider a sequence of such transformations

$$\psi(t_3) = e^{-iH(t_3-t_2)} e^{iA_2} e^{-iH(t_2-t_1)} e^{iA_1} e^{-iH(t_1-t_0)} \psi(t_0) \tag{25}$$

with

$$t_3 > t_2 > t_1 > t_0 , \tag{26}$$

the time reversed states obey

$$\tilde{\psi}(-t_3) = e^{iH(t_3-t_2)} e^{-i\tilde{A}_2} e^{iH(t_2-t_1)} e^{-i\tilde{A}_1} e^{iH(t_1-t_0)} \tilde{\psi}(-t_0) . \tag{27}$$

In the natural temporal sequence

$$-t_0 > t_1 > t_2 > t_3 \tag{28}$$

we may write

$$\tilde{\psi}(-t_0) = e^{-iH(-t_0+t_1)} e^{i\tilde{A}_1} e^{-iH(-t_1+t_2)} e^{i\tilde{A}_2} e^{-iH(-t_2+t_3)} \tilde{\psi}(-t_3) . \tag{29}$$

Therefore for the "real" generators (that is real combinations of H_ℓ, $E_\alpha + E_{-\alpha}$) the corresponding group elements are *not* inverted but the exponentials of the multiples

of $E_\alpha - E_{-\alpha}$ are inverted. This behaviour is distinctly different from that of angular momentum and finite rotations.

With the standard phase conventions for representations with $\{H_\ell\}$ diagonal and the $\{E_\alpha\}$ having only real matrix elements

$$\tilde{A} = A^T \quad , \quad e^{i\tilde{A}} = (e^{iA})^T . \tag{30}$$

so that

$$\tilde{\psi}(-t_0) = e^{-iH(-t_0+t_1)}\, e^{iA_1^T} e^{-iH(-t_1+t_2)} e^{iA_2^T} e^{-iH(-t_2+t_3)} \tilde{\psi}(-t_3) . \tag{31}$$

Thus the *transposed* internal symmetries are unfolding in the *reverse* order. These furnish a representation of the internal symmetry transformation group:

$$\begin{aligned} \mathcal{S}_1 \cdot \mathcal{S}_2 \quad \longrightarrow \quad & \mathcal{D}^T(\mathcal{S}_2)\mathcal{D}^T(\mathcal{S}_1) = (\mathcal{D}(\mathcal{S}_1)\mathcal{D}(\mathcal{S}_2))^T \\ & = \mathcal{D}^T(\mathcal{S}_1\, \mathcal{S}_2) . \end{aligned} \tag{32}$$

DISCUSSIONS

The time reversal transformation has a "split" action. The spacetime groups and internal symmetries are not treated in the same fashion. The internal symmetry algebra undergoes a transformation which maybe realized as a transposition in the standard phase conventions where the Cartan-Weyl basis is realized by real matrices. In contrast for rotations we have to make the additional unitary transformation of rotation through 180° about the second axis. Consequently the isospin transformations and the spin transformations behave differently under time reversal.

Two further remarks are in order. First: if an $SL(3,R)$ or $SU(3)$ spectrum generating group is generated the quadrupole generators must be odd under time reversal rather than even. So we cannot use the moment of inertia[5] but may use its time derivative[6]. Second the nuclear p shell $SU(3)$ as formulated[7] seems to violate time reversal invariance. As far as CPT invariance of local field theory is concerned both C and CPT would affect the internal symmetry labels the same way.

The aim of this work has been to complete the proposal by Wigner of the time reversal transformation in quantum mechanics.

The extension of time reversal for the Galitli/Poincaré groups in addition to the internal symmetry groups is straightforward; and so is the discussion for CPT.

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