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A User's Guide to Exotic Statistics

Tom D. Imbo*‡, Chandni Shah Imbo†,

Rahul S. Mahajan* and E. C. G. Sudarshan*

Center for Particle Theory* and

Department of Mathematics†

University of Texas at Austin

Austin, Texas 78712

ABSTRACT

The inequivalent quantizations of the system of n identical particles on a manifold M, dim $M \geq 2$, are in 1-1 correspondence with the irreducible unitary representations of the braid group $B_n(M)$. The notion of the statistics of the particles is made precise and it is shown that the set of all quantizations breaks up into statistical equivalence classes. Aspects of the structure and representation theory of the braid groups that are relevant for statistics are developed. We give various examples where all the possible statistics for the system are determined, and find instances where the particles obey statistics different from the well studied Bose, Fermi, para- and θ-statistics. We conclude with a brief discussion of cluster properties and of statistics in one dimension.

‡Address after September 14, 1989: Lyman Laboratory of Physics, Harvard University, Cambridge, Massachussetts 02138

I. INTRODUCTION

It has been half a century since the question of the existence of particles obeying exotic (i.e., non-Bose and non-Fermi) statistics was first addressed seriously.[1] Although this work generated a smattering of interest,[2] it wasn't until more than a decade later that a systematic treatment was presented, and the existence of exotic statistics in quantum mechanics and relativistic quantum field theory was rigorously demonstrated.[3,4] The new statistics, later called parastatistics,[5] are associated with higher-dimensional representations of the permutation groups S_n, just as Bose and Fermi statistics are associated with the one-dimensional representations. However, at the time there was no "experimental need" for parastatistics, and as a consequence the remainder of the 1950's saw very little further work in this area.[6] The early 1960's saw a resurgence of interest in both the formal structure and possible phenomenological applications of parastatistics.[5,7] The most historically significant development was the suggestion that the then recently proposed quarks might be paraparticles.[8] This would explain the successes of the constituent quark model which required a symmetric spatial wave function for the (ground state) $\frac{3}{2}^+$ baryon decuplet. This idea soon led to the (essentially equivalent) notion of quark color[9] which quickly gained wide acceptance. In the last 25 years interest in parastatistics has waxed and waned. Some of the more interesting topics addressed have been the cluster properties of paraparticles,[10] parastatistical relativistic string theory,[11] and the "gauging" of parastatistics.[12] However, for the most part, the study of parastatistics has remained outside the mainstream of modern physics. For a comprehensive review of parastatistics, see Ref. 13

More recently, the existence of a second class of exotic statistics has been shown, and received much attention. These statistics can be obeyed by both point particles[14,15] and solitons[16] in certain $(2+1)$-dimensional theories, and are called fractional or θ-statistics. They are labelled by an angle θ, $0 \leq \theta < 2\pi$, which continuously interpolates between the

Bose ($\theta = 0$) and Fermi ($\theta = \pi$) cases. Several years after their "discovery", fractional statistics found various interesting applications in $2D$ condensed matter physics; they provide a good description of the quasiparticle excitations in the fractional quantum Hall effect,[17] high-T_c superconductivity,[18] planar ferromagnets and antiferromagnets[16] and liquid ^3He films in the A phase.[19] Particles obeying fractional statistics are often called *anyons*[20] and many general properties of anyon systems have been derived in the past 5 years.[21,22]

In particle mechanics, the above two classes of exotic possibilities are a consequence of the nontrivial topology of the relevant configuration space.[15,23–25] In Ref. 26, the connection between configuration space topology and statistics was pursued further, and several new families of exotic statistics were presented. More precisely, the inequivalent quantizations of the system of n identical particles moving on a manifold M, $dim\ M \geq 2$ are in one-to-one correspondence with the irreducible unitary representations of the n-string braid group of M, denoted by $B_n(M)$. The notion of the statistics of the particles provided by a given quantization was made precise, and a prescription was given which, in principle, allows one to determine all the possible choices of statistics for any given n and M. Various examples were given where all the possible statistics of the system were determined. These included several (many with multiply connected spaces M) in which the particles could obey statistics different from the well-studied Bose, Fermi, para- and θ-statistics mentioned above.

This paper is a continuation of the investigations begun in Ref. 26, and proceeds as follows. Section 2 contains a review of the connection between the inequivalent quantizations of a classical system with configuration space Q and the irreducible unitary representations of the fundamental group $\pi_1(Q)$. Particular attention is given to the possible existence of *nonscalar* quantizations, which possess an internal symmetry of topological origin. Some rigorous results are presented concerning the number and type

of quantizations available to a given system. The approach developed in section 2 is then applied in section 3 to the system of n identical particles on M. The determination of the configuration space $Q_n(M)$ and the structure of the braid groups $B_n(M) \equiv \pi_1(Q_n(M))$ are considered. We conclude by defining a "statistical" equivalence relation on the set of all irreducible unitary representations of $B_n(M)$, which allows us to determine when two quantizations provide the same statistics for the n identical particles. In section 4 we develop further those aspects of the theory of braid groups which are of particular relevance to the study of statistics. For example, if $dim\ M \geq 3$ we have shown that $B_n(M) \cong \pi_1(M) \wr S_n$, the *wreath product* of $\pi_1(M)$ with S_n. We demonstrate the existence of parastatistics for any M and $n \geq 2$, and the existence of fractional statistics for any submanifold M of $\mathbb{R}^2$ (for all $n \geq 2$). Further examples show the existence of other exotic statistics. The simplest of these occur for many 2 particle systems and are called *ambistatistics*. The corresponding particles (called ambions) behave, in some sense, like *both* fermions and bosons at the same time. Section 5 contains more general results concerning the number and type of statistics available to a given system; mostly for $dim\ M \geq 3$ using the fact that $B_n(M) \cong \pi_1(M) \wr S_n$. Much useful information on the representation theory of wreath products is provided. In section 6 we demonstrate that the inequivalent quantizations of identical particle systems obey the appropriate cluster decomposition properties required of realistic models, and in section 7 the notion of statistics in one-dimension is investigated. Some novel results for identical particles on S^1 are derived. Finally, in section 8 we sum up, discuss some related work in gravity and string theory, and consider future directions.

We wish to make clear at the outset that the purpose of this treatise is to provide a framework in which one can rigorously discuss the statistics of identical particles on arbitrary manifolds, and to develop techniques which allow one to determine all statistical possibilities for many systems. Only formal aspects are discussed and, in particular, no

attempt is made to apply our results to the description of any realistic situation.

2. INEQUIVALENT QUANTIZATIONS AND CONFIGURATION SPACE TOPOLOGY

When quantizing a classical system with configuration space[27] Q, the standard procedure is to construct the fixed-time quantum mechanical state vectors as functions from Q into the complex numbers $\mathbb{C}$. However, more generally we may choose them to be sections of a $\mathbb{C}^N$-bundle over Q, $N \geq 1$. The classical limit of a quantum theory built as above on a bundle B will differ from the original classical system by the introduction of an external gauge potential, namely, the natural $U(N)$ connection on B. In order to classify the inequivalent quantizations of a *fixed* classical system, we require this connection to be flat so as not to change the classical equations of motion. On each such bundle the holonomy of the flat connection provides an N-dimensional unitary representation of the fundamental group $\pi_1(Q)$. Conversely, given any such representation ρ, one can construct a complex vector bundle whose holonomy realizes ρ.[28] If ρ is reducible, then the corresponding bundle B_ρ breaks up into a Whitney sum of bundles $\{B_{\rho_i}\}$ where the ρ_i's are irreducible components of ρ. A similar decomposition of the Hilbert space of sections of B_ρ occurs. If we let $\mathcal{R}(\pi_1(Q))$ denote the set of all (equivalence classes of) finite-dimensional irreducible unitary representations (IUR's) of $\pi_1(Q)$, then the quantum theories associated with the irreducible bundles B_α, $\alpha \in \mathcal{R}$, represent the "prime" quantizations of the original system.[29] $\mathcal{R}$ always contains at least one element, namely the trivial IUR, and the associated quantum theory has ordinary complex-valued functions as state vectors. However, in general $\mathcal{R}$ will contain more than one element, showing the essential "kinematical ambiguity" in quantizing a classical system.[30]

An alternative way of viewing the members of the above Hilbert spaces $\mathcal{H}_\alpha, \alpha \in \mathcal{R}$, is as "multiple-valued functions" from Q to $\mathbb{C}^N$, $N = dim \quad \alpha$. More specifically $\Psi(q)$

has N-components $\Psi_n(q)$, $1 \leq n \leq N$, and when $q \in Q$ is brought around a loop in the homotopy class $\ell \in \pi_1(Q)$ we have

$$\Psi_n(q) \xrightarrow{\ell} \sum_{m=1}^{N} U_{nm}(\ell)\Psi_m(q) \tag{1}$$

where the unitary $N \times N$ matrices $U(\ell)$ are associated with the IUR α of $\pi_1(Q)$. Note that the probability density $\Psi^\dagger\Psi$ is single valued and that there is a superselection rule between any two state vectors which "transform under loops" according to distinct IUR's of $\pi_1(Q)$.

The quantizations corresponding to $N = 1$, the so-called *scalar* quantum theories, are labelled by the character group Ω of $\pi_1(Q)$: $\Omega = Hom(\pi_1(Q), U(1)) \cong Hom(H_1(Q), U(1)) \cong H^1(Q, U(1))$.[24,31,32] The quantum theories associated with irreducible $\mathbb{C}^N$-bundles, $N > 1$, are of a qualitatively different nature. They possess an "internal symmetry" of topological origin associated with the entire system.[33]

In what follows, by a Q-system we will mean a classical system with configuration space Q. One natural question concerning the above classification of inequivalent quantizations of a Q-system is:

(A) When does a Q-system have a unique quantization?

For scalar quantizations, this question is answered in Ref. 31.

<u>Theorem 1</u>: A Q-system has a unique scalar quantization if and only if $\pi_1(Q)$ is a perfect group[34], or equivalently, $H_1(Q)$ is trivial.

There exist many nontrivial perfect groups, both finite and infinite. For example all nonabelian simple groups are perfect. An interesting gravitational Q-system with $\pi_1(Q)$ perfect is discussed in Ref. 31.

More generally, if we call a group with no nontrivial finite-dimensional IUR's a U-inert group, then by definition a Q-system has a unique quantization if and only if

$\pi_1(Q)$ is U-inert. (Clearly U-inert groups must be perfect). A characterization of finitely generated U-inert groups is given in Ref. 35:

Theorem 2: A finitely generated group G is U-inert if and only if G has no nontrivial finite homomorphic images.

The restriction to finitely generated groups is not severe since almost all spaces Q of interest in physics have $\pi_1(Q)$ finitely generated. There are many nontrivial examples of U-inert groups (which must be infinite and nonabelian) and the possibility of finding physically interesting Q-systems with $\pi_1(Q)$ U-inert (and nontrivial) is addressed in Ref. 35 with some success. We therefore see, contrary to the usual intuition, that there exist physical systems with *multiply-connected* configuration spaces which nonetheless have a unique quantization. Another natural question is the following

(B) When are all the quantizations of a Q-system scalar?

Let us call a group U-scalar if it has no finite-dimensional IUR's of degree > 1. Then by definition, all the quantizations of a Q-system are scalar if and only if $\pi_1(Q)$ is U-scalar. Clearly, all abelian groups are U-scalar. In Ref. 35 it is shown that:

Theorem 3: A finitely generated group G is U-scalar if and only if G has no finite nonabelian homomorphic images.

There are many examples of nonabelian U-scalar groups (they must be infinite), and such groups *can* be realized as $\pi_1(Q)$ of interesting Q-systems. For many examples and further properties of U-inert and U-scalar groups, see Ref. 35.

The inequivalent quantizations of many interesting systems have been studied. It is well known that $(2+1)$ and $(3+1)$-dimensional Einstein gravity with compact space manifold Σ[36], and the system of n identical particles moving on a manifold M (the

subject of this paper) almost always possess nonscalar quantizations. However, it has recently been shown that gauge theories and generalized nonlinear sigma models also allow for nonscalar quantization.[37] Further investigations are underway of these very interesting, yet little studied, quantum theories with topological internal symmetry.

3. IDENTICAL PARTICLES, EXOTIC STATISTICS AND BRAID GROUPS

In the above we have seen that to determine the inequivalent (prime) quantizations of a given system one must identify the configuration space Q, calculate $\pi_1(Q)$, and then construct $\mathcal{R}(\pi_1(Q))$. We now carry out this program for the system of n identical particles moving on a smooth, path-connected manifold M (without boundary) of dimension $d \geq 2$.[38] For $n = 1$, Q is simply M and the IUR's of $\pi_1(M)$ label the inequivalent quantizations. If $n \geq 1$ and the particles were *distinguishable*, then $Q = M^n$, the n-fold cartesian product of M with itself. However, since the particles are *identical* we must identify any two points of M^n which differ only by a permutation of the particle labels. The configuration space could then be the orbit space of M^n under this action by S_n, the permutation group on n symbols. We denote this space by M^n/S_n, called the n-fold symmetric product of M. There are two problems with the choice $Q = M^n/S_n$. First, the S_n action on M^n has fixed points and therefore Q is not, in general, a smooth manifold[39]; hence ordinary techniques of qua ntization utilizing the tangent bundle of Q cannot be applied. Second, even if a consistent quantization procedure can be found, one can demonstrate that only theories with Bose statistics will be obtained since we have included points of coincidence of two or more particles in our configuration space. (We will come back to this point shortly.) One may remedy both of the above problems by removing from M^n the subcomplex Δ consisting of all points where two or more particle coordinates coincide. Now S_n acts freely (i.e., without fixed points) on $M^n - \Delta$ and the orbit space $(M^n - \Delta)/S_n \equiv Q_n(M)$ is a smooth manifold. We choose this manifold as our

configuration space. The group $\pi_1(Q_n(M)) \equiv B_n(M)$ is called the n-string braid group of M[40]. (Note that $B_1(M) \cong \pi_1(M)$.) The set $\mathcal{R}(B_n(M))$ of IUR's of $B_n(M)$ labels the inequivalent quantizations. Speaking vaguely for a moment, the different quantizations are related to the different possible "statistics" for the n identical particles ($n \geq 2$), but one must be careful not to overcount. There is, in general, a quantization ambiguity already present for $n = 1$ (and therefore having nothing to do with statistics) which will manifest itself again in $\mathcal{R}(B_n(M))$ for any n. To get the set which labels the different statistics, one must take $\mathcal{R}(B_n(M))$ and "mod out" by $\mathcal{R}(B_1(M))$ in an appropriate way.

An element of $B_n(M)$ can be thought of as a homotopy class of paths in $M^n - \Delta$ whose (fixed) initial and final points are related by a permutation of the particle coordinates. It is straightforward to identify a set of such paths which generate $B_n(M)$. (In what follows we will speak of an element of $M^n - \Delta$ as an ordered set of n distinct points in one copy of M. We also identify a path with its homotopy class.) Fix n distinct points $m_1, m_2, ..., m_n$ in M which together will represent the initial point of our paths. First, consider a path which takes a given point, say m_1, around a noncontractible loop in M and fixes the points m_2 through m_n. Call the set of all such paths $\mathcal{L}$. Next, let D be an open d-disk in M containing $m_1, \ldots, m_n$. For each $i < n$ consider a path σ_i which interchanges m_i and m_{i+1} in D, not enclosing any of the other m_j's which all remain fixed. Denote the set of all $\sigma_i, 1 \leq i \leq n-1$, by $\mathcal{P}$. $\mathcal{L}$ and $\mathcal{P}$ generate $B_n(M)$. For example $\sigma_1^{-1} \circ \ell \circ \sigma_1$, $\ell \in \mathcal{L}$, is (homotopic to) a path which takes m_2 around a loop in M, fixing and avoiding all other particles, while the path $\ell \circ \sigma_1$ interchanges m_1 and m_2 in a non-simply connected region of M, etc.. A nice property of this set of generators is that it decouples into those pertaining to the loop topology of M (namely $\mathcal{L}$) and those associated with permutations alone ($\mathcal{P}$). The relations among these generators may be very complicated and M-dependent. In particular for $d = 2$ they can mix the $\mathcal{L}$ and $\mathcal{P}$ generators in a highly nontrivial way (see below).

Let $\Sigma_n(M)$ be the subgroup of $B_n(M)$ generated by $\mathcal{P}$. It is clear that the statistics of the n identical particles on M provided by an IUR ρ of $B_n(M)$ is determined by $\rho \downarrow \Sigma_n(M)$, the restriction of ρ to $\Sigma_n(M)$. ($\rho \downarrow \Sigma_n$ is, in general, reducible.) We recall the following definition from Ref. 26:

Two IUR's ρ_1 and ρ_2 of $B_n(M)$ are *statistically equivalent* (written $\rho_1 \sim \rho_2$) if for some positive integers s and t

$$\mathbb{1}_s \otimes (\rho_1 \downarrow \Sigma_n) \simeq \mathbb{1}_t \otimes (\rho_2 \downarrow \Sigma_n). \tag{2}$$

(Here the symbol "$\simeq$" means equivalence as representations, $\otimes$ denotes the inner tensor product, and $\mathbb{1}_s$ and $\mathbb{1}_t$ are the trivial representations of Σ_n of dimensions s and t respectively.) The presence of $\mathbb{1}_s$ and $\mathbb{1}_t$ in the above equality accounts for differences which only pertain to the distinct dimensionalities of ρ_1 and ρ_2. It is easy to check that "$\sim$" is an equivalence relation on $\mathcal{R}(B_n(M))$. Therefore, $\mathcal{R}(B_n(M))$ breaks up into equivalence classes, each containing only IUR's whose corresponding quantizations yield the same statistics for the n identical particles. If M is simply connected then $\Sigma_n(M) = B_n(M)$ and distinct quantizations give distinct statistics as expected. Our definition provides a natural generalization to the case $\pi_1(M) \neq \{e\}$.

Two final points need to be addressed. First, the subgroup $\Sigma_n(M)$ which we used to define statistical equivalence, is defined with respect to a disk D in M containing $m_1, \ldots, m_n$. If we start with a different disk D' containing $m_1, \ldots, m_n$, then the corresponding subgroup $\Sigma'_n(M)$ of $B_n(M)$ is, in general, not the same as $\Sigma_n(M)$. However, it is straightforward to show that the subgroups $\Sigma_n(M)$ and $\Sigma'_n(M)$ are conjugate to each other, and hence the partitioning of $\mathcal{R}(B_n(M))$ into statistical equivalence classes is the same whether we use $\Sigma_n(M)$ or $\Sigma'_n(M)$. Second, it is now easy to see why if we use the symmetric product M^n/S_n as our n particle configuration space we only obtain Bose statistics. The reason is that the "local" interchanges σ_i are all homotopically trivial if

Δ is not removed, and hence the analog of Σ_n is trivial.[41]

4. STRUCTURE OF THE BRAID GROUPS AND EXAMPLES

Since the S_n action on $M^n - \Delta$ is free, we have the following fibration[40]

$$\begin{array}{ccc} S_n & \hookrightarrow & M^n - \Delta \\ & & \downarrow \\ & & Q_n(M). \end{array} \tag{3}$$

The long exact homotopy sequence [42] of this fibration yields the following short exact sequence for $B_n(M)$

$$\{e\} \longrightarrow \pi_1(M^n - \Delta) \xrightarrow{\alpha} B_n(M) \xrightarrow{\beta} S_n \longrightarrow \{e\}. \tag{4}$$

The generators $\mathcal{L}$ of $B_n(M)$ are in the kernel of the epimorphism β, while the generators σ_i in $\mathcal{P}$ map onto the corresponding transpositions in S_n. Thus $\beta \downarrow \Sigma_n$ is an epimorphism from $\Sigma_n(M)$ onto S_n. Given any IUR ρ of S_n, one can "lift" it to an IUR $\tilde{\rho}$ of $B_n(M)$, i.e., $\tilde{\rho}(b) \equiv \rho(\beta(b))$ for all $b \in B_n(M)$. Clearly $\tilde{\rho} \downarrow \Sigma_n$ is the lift of ρ to $\Sigma_n(M)$. So there are at least as many distinct choices of statistics for the n particles as there are IUR's of S_n. The number of IUR's of S_n is equal to the number of partitions of the integer n, denoted by $p(n)$. Some values of $p(n)$ are given below.[43]

n	$p(n)$	n	$p(n)$
2	2	10	42
3	3	20	627
4	5	50	204,266
5	7	100	190,569,292

$p(n)$ grows rapidly as n increases. For large n, one can use the Hardy-Ramanujan asymptotic formula[43]

$$p(n) \simeq \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{2n/3}}. \tag{5}$$

For any $n \geq 2$, there are only two IUR's of degree 1. Namely the trivial IUR and the IUR sending each permutation to its sign. The corresponding statistics are Bose and Fermi respectively. For $n \neq 4$ all other IUR's have dimension at least $n-1$ (S_4 has a 2-dimensional IUR). The statistics obtained as above by lifting IUR's of S_n correspond to the parastatistics mentioned earlier. (Here we consider Bose and Fermi statistics as special cases of parastatistics.) In general, there will be many other possibilities. We first analyze the case with $dim\ M \geq 3$.

The codimension of Δ in M^n is d. Therefore if $d \geq 3$, standard general position arguments give[44] $\pi_1(M^n - \Delta) \cong \pi_1(M)^n$. Hence by Eq. (4), $B_n(M) \cong S_n$ if $\pi_1(M) = \{e\}$, and only parastatistics are possible. In particular this is true for $M = \mathbb{R}^d$, $d \geq 3$. Now let $f : D \to M$ be the inclusion of the d-disk D of the previous section into M. Of course D is homeomorphic to $\mathbb{R}^d$. By the naturality[42] of the long exact homotopy sequence of Eq. (3) we obtain the following commutative diagram ($d \geq 3$)

$$\begin{array}{ccccccccc} \{e\} & \longrightarrow & \pi_1(D^n) & \longrightarrow & B_n(D) & \xrightarrow{\gamma} & S_n & \longrightarrow & \{e\} \\ & & \downarrow f_*^n & & \downarrow g & & \downarrow h & & \\ \{e\} & \longrightarrow & \pi_1(M)^n & \xrightarrow{\alpha} & B_n(M) & \xrightarrow{\beta} & S_n & \longrightarrow & \{e\} \end{array} \tag{6}$$

where h is the identity map and γ is an isomorphism since $\pi_1(D^n) = \{e\}$. The map $\tau \equiv g \circ \gamma^{-1} \circ h^{-1}$ is a splitting homomorphism for the short exact sequence for $B_n(M)$, i.e., $\beta \circ \tau$ is the identity map on S_n. Hence $B_n(M)$ is a semidirect product of $\pi_1(M)^n$ with S_n

$$B_n(M) = \pi_1(M)^n \rtimes_\mu S_n. \tag{7}$$

The defining map $\mu : S_n \to Aut(\pi_1(M)^n)$ is given by ($\pi \in S_n$, $\ell_i \in \pi_1(M)$, $1 \leq i \leq n$)

$$\mu(\pi)(\ell_1, ..., \ell_n) = (\ell_{\pi(1)}, ..., \ell_{\pi(n)}). \tag{8}$$

This semidirect product is called the *wreath product*[45,46] of $\pi_1(M)$ with S_n and denoted by $\pi_1(M) \wr S_n$. So $B_n(M) = \pi_1(M) \wr S_n$ for $d \geq 3$. The image of τ in $B_n(M)$ is $\Sigma_n(M)$,

that is, $\Sigma_n(M) \cong S_n$. However this does not imply that we only obtain parastatistics, for given an IUR ρ of $B_n(M)$, $\rho \downarrow \Sigma_n$ may contain inequivalent irreducible components. Also, the quantization corresponding to an IUR ρ is not just a "direct sum" of many parastatistical quantizations, one for each irreducible component of $\rho \downarrow \Sigma_n$, since ρ is irreducible. However since $\Sigma_n(M)$ is just S_n, we call the associated statistics *generalized parastatistics*. An important consequence of the above is that the only types of statistics associated with scalar quantizations (which we call *scalar statistics*) of n identical particles on M, dim $M \geq 3$, are Bose and Fermi.

We can easily write down a presentation of $\pi_1(M) \wr S_n$ in terms of the presentation of $\pi_1(M)$. We have as generators

$$\ell_{i\alpha},\ \sigma_k,\ \ 1 \leq i \leq n,\ \ 1 \leq k \leq n-1,\quad \alpha \in A\ . \tag{9}$$

For fixed i, the $\ell_{i\alpha}$'s generate a subgroup isomorphic to $\pi_1(M)$ (whose generators are indexed by the set A). We will take the relations in $\pi_1(M)$ as given and not display them explicitly. The σ_k's represent the transpositions which generate $\Sigma_n(M) \cong S_n$. The other relations we need are

$$\ell_{i\alpha}\ell_{j\beta} = \ell_{j\beta}\ell_{i\alpha} \quad i \neq j;\ \ 1 \leq i,j \leq n;\ \ \alpha,\beta \in A \tag{10}$$

$$\sigma_i^{-1}\ell_{i\alpha}\sigma_i = \ell_{i+1,\alpha} \quad 1 \leq i \leq n-1;\ \ \alpha \in A \tag{11}$$

$$\ell_{i\alpha}\sigma_k = \sigma_k\ell_{i\alpha} \quad i \neq k, k+1;\ \ \alpha \in A \tag{12}$$

$$\sigma_k\,\sigma_{k+1}\,\sigma_k = \sigma_{k+1}\,\sigma_k\,\sigma_{k+1} \quad 1 \leq k \leq n-2 \tag{13}$$

$$\sigma_k\,\sigma_\ell = \sigma_\ell\,\sigma_k \quad |\,k-\ell\,| \geq 2 \tag{14}$$

$$\sigma_k^2 = e \quad 1 \leq k \leq n-1\ . \tag{15}$$

Our generators $\mathcal{L}$ in section 3 correspond to the $\ell_{1\alpha}$'s. As noted earlier these and the σ_k's alone generate $\pi_1(M) \wr S_n$ since we may use Eq. (11) to eliminate each $\ell_{i\alpha}$, $i \geq 2$, from our presentation. (For many purposes the above presentation with "too many"

generators is very convenient.) As an example consider 2 identical particles moving on real projective 3-space $\mathbb{R}P^3 = S^3/\mathbb{Z}_2$. Since $\pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$ we have $B_2(\mathbb{R}P^3) \cong \mathbb{Z}_2 \wr \mathbb{Z}_2$ which is the dihedral group D_8 of order 8. This group is generated by two elements ℓ and σ subject to the relations $\ell^2 = \sigma^2 = e$ and $(\ell\sigma)^2 = (\sigma\ell)^2$. We have $\Sigma_2(\mathbb{R}P^3) \cong \mathbb{Z}_2$. There are 5 IUR's of D_8, 4 have dimension one and one has dimension two. They are given by

$$\begin{aligned} &\rho_1(\ell) = 1,\ \rho_1(\sigma) = 1 && \rho_3(\ell) = -1,\ \rho_3(\sigma) = 1 \\ &\rho_2(\ell) = 1,\ \rho_2(\sigma) = -1 && \rho_4(\ell) = -1,\ \rho_4(\sigma) = -1 \\ &\rho_5(\ell) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, && \rho_5(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{16}$$

ρ_1 and ρ_3 give Bose statistics for the 2 particles, ρ_2 and ρ_4 yield Fermi statistics and ρ_5 provides a new type of "half Bose-half Fermi" exotic statistics. Note that

$$\rho_5(\ell)\ \rho_5(\sigma)\ \rho_5(\ell) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{17}$$

which shows the coupling of $\mathcal{L}$ and $\mathcal{P}$. Although there are 5 inequivalent quantizations of the system, there are only 3 inequivalent statistics. We call the statistics associated with ρ_5 *ambistatistics*.

For $d = 2$ the situation is much more complex, leading to a richer spectrum of statistics. One complication is that the codimension of Δ in M^n is 2 and hence $\pi_1(M^n - \Delta) \ncong \pi_1(M)^n$ in general. As a consequence, even though the generators of $B_n(M)$ can be written as in Eq.(9) where A indexes the generators of $\pi_1(M)$, the $\ell_{i\alpha}$'s for fixed i no longer (in general) generate a subgroup isomorphic to $\pi_1(M)$. Also, only the relations in Eqs.(11-14) hold in general for $d = 2$. As an example consider $M = \mathbb{R}^2$. $B_n(\mathbb{R}^2) = \Sigma_n(\mathbb{R}^2)$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$, and the only defining relations are Eqs. (13-14).[47] $B_n(\mathbb{R}^2)$ is infinite and nonabelian for $n \geq 3$ ($B_2(\mathbb{R}^2) \cong \mathbb{Z}$). For $n \geq 2$, $H_1(Q_n(\mathbb{R}^2)) \cong B_n(\mathbb{R}^2)_{ab} \cong \mathbb{Z}$ and since $Hom(\mathbb{Z}, U(1)) \cong U(1)$ the scalar quantizations

and statistics of the system are labelled by an angle θ. These are the fractional statistics discussed previously. We already see that in two dimensions, unlike the situation in higher dimensions, the representation of $\Sigma_n(M)$ defining the statistics need not be equivalent to a representation over $\mathbb{R}$. For $n \geq 3$, the system of n identical prticles on $\mathbb{R}^2$ also possesses nonscalar quantizations and statistics.[25,48] We will concentrate on the case $n = 3$. A slightly more useful presentation of $B_3(\mathbb{R}^2)$ is[47]

$$B_3(\mathbb{R}^2) = < a, b \mid a^3 = b^2 > \tag{18}$$

where $a = \sigma_1\sigma_2$ and $b = \sigma_1\sigma_2\sigma_1$. The degrees of the finite-dimensional IUR's of $B_3(\mathbb{R}^2)$ are unbounded. This can be seen as follows.[48] It is known that[49]

$$\pi_1((\mathbb{R}^2)^3 - \Delta) \cong F_2 \times \mathbb{Z} = < x, y, z \mid xz = zx,\ yz = zy > \tag{19}$$

where F_2 is the free group on two generators (x and y). It is easy to see that $F_2 \times \mathbb{Z}$ has an IUR in every positive dimension m. Just send x to any diagonal $m \times m$ unitary matrix all of whose eigenvalues are distinct, and send y to any $m \times m$ unitary matrix all of whose off diagonal elements are nonzero. Finally, send z to any unitary scalar matrix. These three matrices will generate a unitary representation ρ of $F_2 \times \mathbb{Z}$, and since the only matrices commuting with all of them are the scalar matrices, ρ is irreducible. Now by Eq.(5) we have that $F_2 \times \mathbb{Z}$ is a normal subgroup of *finite index* in $B_3(\mathbb{R}^2)$. It is a well known result in the theory of induced representations[50] that if a group G has a normal subgroup of finite index with an IUR of degree m, then G *must* have an IUR of finite degree $\geq m$. We therefore have that the degrees of the finite-dimensional IUR's of $B_3(\mathbb{R}^2)$ are unbounded. So we see that there are an infinite number of "types" of nonscalar statistics for 3 particles on $\mathbb{R}^2$.

All the two-dimensional IUR's of $B_3(\mathbb{R}^2)$ are easily found. They are labelled by two angles ϕ and θ, $0 \leq \phi < 2\pi$, $0 \leq \theta < \pi/2$, and are generated by (see Eq.(18))

$$a = e^{2i\phi}\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \quad b = e^{3i\phi}\begin{pmatrix} sin\theta & cos\theta \\ cos\theta & -sin\theta \end{pmatrix} \tag{20}$$

where $\omega = e^{2\pi i/3}$. (For $\theta = 0$, one must restrict the range of ϕ to $0 \leq \phi < \pi$, in order not to overcount). The case $\phi = \pi/3$, $\theta = 0$ corresponds to the "lift" of the two-dimensional IUR of S_3. A 3-parameter family of three-dimensional IUR's of $B_3(\mathbb{R}^2)$ is generated by[48] $(0 \leq \phi < 2\pi)$

$$a = e^{2i\phi}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad (b)_{jk} = e^{3i\phi}(-\delta_{jk} + 2n_j n_k), \quad 1 \leq j,k \leq 3, \tag{21}$$

where δ_{jk} is the Kronecker delta symbol and $\overline{n} = (n_1, n_2, n_3)$ is a real positive unit vector all of whose components are nonzero. (If $n_1 = n_2 = n_3 = 1/\sqrt{3}$, then one must restrict the range of ϕ to $0 \leq \phi < 2\pi/3$ so as not to overcount). These exhaust the three dimensional IUR's of $B_3(\mathbb{R}^2)$. The higher dimensional IUR's of $B_3(\mathbb{R}^2)$ can also be explicitly constructed although the procedure is much more tedious. Representations of $B_n(\mathbb{R}^2), n \geq 4$, are more difficult to construct. The representations of $B_n(\mathbb{R}^2)$ are also of use in constructing invariant polynomials in knot theory,[51] as well as in the classification of conformal field theories.[52]

Next we consider $M = S^2$. $B_n(S^2)$ has a presentation identical to $B_n(\mathbb{R}^2)$ except for the additional relation[53]

$$\sigma_1\sigma_2 \ldots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \ldots \sigma_2\sigma_1 = e \quad . \tag{22}$$

$B_2(S^2) \cong \mathbb{Z}_2$ and there are only the Bose and Fermi quantizations of the 2 particle system. For $n \geq 2$ $B_n(S^2)_{ab} \cong \mathbb{Z}_{2n-2}$ and the number of scalar quantizations and

statistics grows with n $(Hom(\mathbb{Z}_{2n-2}, U(1)) \cong \mathbb{Z}_{2n-2})$.[22] For a given n, they are labelled by the angles $\theta = j\pi/(n-1)$, $0 \leq j \leq 2n-3$. $B_3(S^2)$ can be presented as

$$B_3(S^2) = < a, b \mid a^3 = b^2, \ \ b = aba > \tag{23}$$

where a and b are as above for $M = \mathbb{R}^2$. $B_3(S^2)$ has order 12 since $\pi_1((S^2)^3 - \Delta) \cong \mathbb{Z}_2$ (see Eq.(4)).[53] It is called the metacyclic group of order 12. There are six IUR's of $B_3(S^2)$; four of them have degree 1 and the remaining two have degree 2, yielding six distinct statistics. They are given by[48]

$$a = \lambda^2 \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad b = \lambda^3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{24}$$

with $\lambda = 1$ and $\lambda = \mathrm{i}$. The groups $B_n(S^2), n \geq 4$, are infinite, nonabelian and more difficult to deal with. The above examples show that for $d = 2$, $B_n(M)$ depends on more than just $\pi_1(M)$, unlike the case for $d \geq 3$.

We now know that for 2-manifolds M, the extension in Eq.(4) will not be split in general. However, $\pi_1(M) \wr S_n$ is always a homomorphic image of $B_n(M)$ as we will now demonstrate. Let M be a 2-manifold and $f : M \to M \times \mathbb{R}$ the inclusion map, $f(m) = (m, 0)$. The naturality of the long exact sequence of Eq. (3) gives

$$\begin{array}{ccccccc} \{e\} \longrightarrow & \pi_1(M^n - \Delta) & \longrightarrow & B_n(M) & \xrightarrow{\gamma} & S_n & \longrightarrow \{e\} \\ & \downarrow f^n_* & & \downarrow g & & \downarrow h & \\ \{e\} \longrightarrow & \pi_1(M)^n & \xrightarrow{\alpha} & B_n(M \times \mathbb{R}) & \xrightarrow{\beta} & S_n & \longrightarrow \{e\} \end{array} \tag{25}$$

where h is the identity map. In Ref.54 it is shown that the induced map f^n_* is an epimorphism. Since h is also an epimorphism the five lemma[42] implies that g is one as well. Also, because $dim\ (M \times \mathbb{R}) = 3$, $B_n(M \times \mathbb{R}) \cong \pi_1(M) \wr S_n$. Hence $\pi_1(M) \wr S_n$ is a homomorphic image of $B_n(M)$. Moreover, $g(\Sigma_n(M)) = S_n$. As a consequence, the distinct quantizations and statistics of n identical particles on $M \times \mathbb{R}$ (or any manifold N of dimension three or more with

$\pi_1(N) \cong \pi_1(M)$) will also appear for the n particle system on M, along with possible others. For example, consider $B_2(\mathbb{R}P^2)$ which is generated by ℓ and σ with relations $\ell^2 = \sigma^2 = (\sigma\ell^{-1})^4$; $\Sigma_2(\mathbb{R}P^2) \cong \mathbb{Z}_4$. $B_2(\mathbb{R}P^2)$ is the dicyclic group of order 16 and has 7 IUR's, 4 of dimension one and 3 of dimension two. They are the "same" as those for $B_2(\mathbb{R}P^3)$ (see Eq. (16)) except there are two additional IUR's given by

$$\rho_6(\ell) = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \rho_6(\sigma) = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho_7(\ell) = \frac{-i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \rho_7(\sigma) = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{26}$$

ρ_6 and ρ_7 are statistically equivalent, yielding a type of "fractional" ambistatistics. Thus the scalar statistics are only Bose and Fermi, but there are 2 types of exotic nonscalar statistics.

Explicit presentations of the braid groups of all closed 2-manifolds are known.[43,55,56] Except for $B_2(S^2)$, $B_3(S^2)$ and $B_2(\mathbb{R}P^2)$ these groups are all infinite and nonabelian.[57] For all closed 2-manifolds $M \neq S^2$ and any $n \geq 2$, the only scalar statistics are Bose and Fermi since $B_n(M)_{ab} \cong H_1(M) \oplus \mathbb{Z}_2$ and the image of $\Sigma_n(M)$ under the abelianization map is the $\mathbb{Z}_2$ direct factor. This can be demonstrated by explicitly abelianizing the presentations in Ref. 56. Thus these manifolds allow the same scalar statistics as all manifolds of higher dimension. Also, if M is a submanifold of $\mathbb{R}^2$ (which is necessarily open, e.g., M can be the plane with any number of punctures) then it can be shown that $B_n(M)_{ab} \cong H_1(M) \oplus \mathbb{Z}$ and one obtains the full range of fractional statistics among the scalar quantizations. Note the peculiarity of the case $M = S^2$ where $B_n(S^2)_{ab}$ depends on n. We do not know if this is the only manifold which displays this behavior. However, it can be shown[58] that for any open 2-manifold M, $B_n(M)_{ab} \cong B_{n+1}(M)_{ab}$ for $n \geq 4$. Also, the scalar statistics for n identical particles on *any* 2-manifold are always labelled by a *single* angle taking on certain values between 0 and 2π. We collect the results of this section on scalar quantizations and statistics below.

M	$B_n(M)_{ab}$	Statistics angle
dim $M \geq 3$	$H_1(M) \oplus \mathbb{Z}_2$	$\theta = 0, \pi$
closed 2-manifold $\neq S^2$	$H_1(M) \oplus \mathbb{Z}_2$	$\theta = 0, \pi$
S^2	$\mathbb{Z}_{2n-2}$	$\theta = \frac{j\pi}{(n-1)}, j = 0, 1, \ldots, 2n-3$
$M \subseteq \mathbb{R}^2$	$H_1(M) \oplus \mathbb{Z}$	$0 \leq \theta < 2\pi$

5. REPRESENTATION THEORY OF WREATH PRODUCTS

In this section we will prove some general results concerning statistics, mostly for $d \geq 3$ using the fact that $B_n(M) \cong \pi_1(M) \wr S_n$. We begin with the question: When does the system of n identical particles on M have only parastatistical quantizations? Equivalently, when is every IUR of $B_n(M)$ statistically equivalent to the lift of an IUR of S_n (as in section 4)? Such a $B_n(M)$ will be said to have property P.

Theorem 4: If *dim* $M \geq 3$, then $B_n(M)$ has property P if and only if $\pi_1(M)$ is U-inert (see section 2).

Before proceeding to the proof, we will need the following.

Definition: Let ρ be an m-dimensional representation of a group G and $\{v_1, v_2, \ldots, v_m\}$ a basis for the representation space V with $\rho(g)v_i = \sum_{j=1}^{m} [\rho(g)]_{ji}\ v_j$ for all $g \in G$. The *n-fold modified outer direct sum* of ρ with itself, $\overset{n}{\tilde{\oplus}}\ \rho$, (of dimension nm) is the representation of $G \wr S_n$ given by ($g_i \in G, \pi \in S_n$)

$$(\overset{n}{\tilde{\oplus}}\ \rho)(g_1, g_2, \ldots, g_n, \pi)(v_{ij}) = \sum_{\ell=1}^{m} [\rho(g_j)]_{\ell i}\ v_{\ell\pi(j)} \tag{27}$$

where $\{v_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a basis for $\overset{n}{\underset{k=1}{\oplus}} V$.

Lemma 1: ρ is a nontrivial IUR of G if and only if $\overset{n}{\tilde{\oplus}}\ \rho$ is an IUR of $G \wr S_n$ ($n \geq 1$).

Proof: It is obvious that ρ is a unitary representation of G if an only if $\overset{n}{\tilde{\oplus}}\rho$ is a unitary representation of $G \wr S_n$. If ρ is trivial, then clearly $\overset{n}{\tilde{\oplus}}\rho$ is reducible. To show that the irreducibility of ρ implies the irreducibility of $\overset{n}{\tilde{\oplus}}\rho$, we use the converse of Schur's lemma. Assume L commutes with the image of $\overset{n}{\tilde{\oplus}}\rho$. Write L as an $n \times n$ matrix with components L_{ij} each of which is itself an $m \times m$ matrix. The fact that L commutes with all matrices of the form $(\overset{n}{\tilde{\oplus}}\rho)(g, e, \ldots, e, e)$ implies that if ρ is a nontrivial IUR, then $L_{11} = \lambda_1 \mathbb{1}_m$ and $L_{1j} = L_{j1} = O_m$, $j > 1$. (Here $\mathbb{1}_m$ is the identity matrix of dimension m and O_m is the $m \times m$ matrix all of whose entries are zero.) Similarly we find $L_{ij} = O_m$ for all $i \neq j$ and $L_{ii} = \lambda_i \mathbb{1}_m$. Finally, the fact that L commutes with $(\overset{n}{\tilde{\oplus}}\rho)(e, \ldots, e; \pi)$, where π is the cyclic permutation $(1\ 2 \ldots n)$, implies $\lambda_i = \lambda_j$ for all i, j. Thus L is a scalar matrix and $\overset{n}{\tilde{\oplus}}\rho$ is irreducible. Finally if ρ is reducible then the image of ρ commutes with some nonscalar matrix B, and the image of $\overset{n}{\tilde{\oplus}}\rho$ commutes with $\mathbb{1}_n \otimes B$, i.e., $\overset{n}{\tilde{\oplus}}\rho$ is reducible.

■

Proof of Theorem 4:

(a) $\pi_1(M)$ U-inert $\Rightarrow$ $B_n(M)$ has property P: Let ρ be an IUR of $B_n(M)$. Then $\rho \downarrow \pi_1(M)^n$ is a unitary representation of $\pi_1(M)^n$. Now $\pi_1(M)$ U-inert implies that $\pi_1(M)^n$ is U-inert[35] and hence $\rho \downarrow \pi_1(M)^n$ is trivial. However $B_n(M)/\pi_1(M)^n \cong S_n$ and hence ρ must be a lift of an IUR of S_n.

(b) $B_n(M)$ has property $P \Rightarrow \pi_1(M)$ U-inert: Assume $\pi_1(M)$ is not U-inert. Thus $\pi_1(M)$ has a nontrivial IUR ρ and by Lemma 1, $\overset{n}{\tilde{\oplus}}\rho$ is an IUR of $\pi_1(M) \wr S_n$. Now by construction $(\overset{n}{\tilde{\oplus}}\rho) \downarrow S_n$ is a permutation representation[45] with orbits $\{v_{ij} \mid i \text{ fixed}, 1 \leq j \leq n\}$. Make a change of basis to $\{w_k \mid 1 \leq k \leq mn\}$ such that $w_1 = (v_{11} + v_{12} \ldots + v_{1n})/\sqrt{n}$, and $\{w_k \mid k \neq 1\}$ spans the orthogonal complement to W_1 in $\overset{n}{\underset{k=1}{\oplus}} V$, call it W. Since $\overset{n}{\tilde{\oplus}}\rho$ is unitary and the span of w_1 is an invariant subspace, so is W and hence $(\overset{n}{\tilde{\oplus}}\rho) \downarrow S_n$ breaks up into a direct sum of 2 representations of S_n, one on w_1, and the other on W. Clearly

$(\overset{n}{\tilde{\oplus}}\ \rho)(e,\ldots,e;\pi)(w_1) = w_1$ for all $\pi \in S_n$, so one of the representations is the trivial one, say T. Now for $\overset{n}{\tilde{\oplus}}\ \rho$ to be statistically equivalent to a lift of an IUR of S_n we need $(\overset{n}{\tilde{\oplus}}\ \rho) \downarrow S_n \simeq \mathbb{1}_{mn} \otimes T$. But since $n \geq 2$ we see that $(\overset{n}{\tilde{\oplus}}\ \rho) \downarrow S_n$ has a nonidentity image for some $\pi \in S_n$. Thus $\overset{n}{\tilde{\oplus}}\ \rho$ is not such a lift and $B_n(M)$ cannot have property P. ∎

Given the results of section 2, Theorem 4 is not a surprise. There is an interesting corollary for 2-manifolds.

Corollary 1: Let $dim\ M = 2$. Then $B_n(M), n \geq 2$, never has property P .

Proof: From section 3 we know that there is an epimorphism $\phi :\ B_n(M) \longrightarrow \pi_1(M) \wr S_n$ such that $\phi(\Sigma_n(M)) = S_n$. By Theorem 4 if $\pi_1(M)$ is not U-inert, then $\pi_1(M) \wr S_n$ does not have property P which implies (by lifting) that $B_n(M)$ does not have property P. Thus we may assume $\pi_1(M)$ is U-inert. However, from Ref. 59 and Theorem 2 it follows that $\pi_1(M)$, M a 2-manifold, cannot be nontrivial and U-inert. Hence we may assume M is simply connected. The only simply connected 2-manifolds (up to homeomorphism) are $\mathbb{R}^2$ and S^2, and from our treatment of these spaces in section 4 we know that $B_n(\mathbb{R}^2)$ and $B_n(S^2)$ do not have property P. ∎

So for n identical particles on M, $dim\ M = 2$, there are always nonparastatistical quantizations. We also have

Theorem 5: Let $dim\ M \geq 3$. If $\rho_1, \rho_2 \in \mathcal{R}(B_n(M))$ with $\rho_1 \sim \rho_2$, then $dim\ \rho_1 = 1$ implies $dim\ \rho_2 = 1$.

Proof: If $dim\ \rho_1 = 1$ and $\rho_1 \sim \rho_2$ then under ρ_2 each σ_i, $1 \leq i \leq n-1$, is represented as a scalar matrix and $\ell_{i\alpha}\ \ell_{i\beta} = \ell_{i\beta}\ \ell_{i\alpha}$ for all α, β (see Eq. (2)). The relation in Eq. (11) then implies that under ρ_2 we have $\ell_{i+1,\alpha} = \ell_{i\alpha}$ for all $1 \leq i \leq n-1$, $\alpha \in A$. Finally Eq. (10) then gives $\ell_{i\alpha}\ \ell_{j\beta} = \ell_{j\beta}\ \ell_{i\alpha}$ for *all* i, j, α, β. Thus under ρ_2 all generators commute

with each other and since ρ_2 is irreducible it must have dimension one. ∎

So for n identical particles on M, $dim\ M \geq 3$, any quantization which is statistically equivalent to a scalar quantization must also be scalar. If M is a closed 2-manifold this result still holds and can be deduced from the presentations in Ref. 56.

Corollary 2: The system of 2 identical particles moving on M, with $\pi_1(M)$ U-inert, has only scalar quantizations.

The proof follows immediately from Theorems 4 and 5.[60]

Although we have obtained some general theorems, it is unclear how to classify completely the statistics of n identical particles on an arbitrary manifold M. However, if $dim\ M \geq 3$ and $\pi_1(M)$ is a *finite* group we can come close to doing so by using the theory of representations of wreath products of finite groups.[45] Before stating the main classification theorem, we need the following.

Definition: Let ρ be a representation of a subgroup H of a finite group G. Pick a transversal $\{g_i\}$ to H in G and define $(g \in G)$

$$\dot{\rho}(g) = \begin{cases} \rho(g) & g \in H \\ O_m & g \notin H \end{cases} \tag{28}$$

where m is the dimension of ρ. The *induced representation* $\rho \uparrow G$ from H to G is given by

$$[(\rho \uparrow G)(g)]_{ij} = \dot{\rho}(g_i^{-1} g g_j)\ . \tag{29}$$

$\rho \uparrow G$ has dimension mr where r is the index of H in G.

Definition: If ρ is a representation of a group G on a vector space V, the *n-fold modified outer tensor product* of ρ with itself, $\overset{n}{\tilde{\#}}\ \rho$, is a representation of $G \wr S_n$ on $\bigotimes_{k=1}^{n} V$ given by $(g_i \in G, \pi \in S_n)$

$$(\overset{n}{\tilde{\#}}\ \rho)(g_1, \ldots, g_n; \pi)(v_1 \otimes \ldots \otimes v_n) = \rho(g_1)v_{\pi(1)} \otimes \ldots \otimes \rho(g_n)v_{\pi(n)}. \tag{30}$$

Clearly if ρ is irreducible, so is $\overset{n}{\#}\rho$.

Definition: A k-fold partition of an integer n is a k-tuple o f positive integers $(n_1, \ldots, n_k)$ with $\sum_{i=1}^{k} n_i = n$.

The following result[45] completely classifies the IUR's of $G \wr S_n$ in terms of those of G.

Theorem 7: Let $\rho_1, \ldots, \rho_k$ be a complete set of IUR's of the finite group G. The set $\mathcal{R}(G \wr S_n)$ is in one-to-one correspondence with the set of (pairwise distinct) IUR's of $G \wr S_n$ of the form

$$[(\overset{n_1}{\#}\rho_1 \otimes [\alpha_1])\# \ldots \#(\overset{n_k}{\#}\rho_k \otimes [\alpha_k])] \uparrow G \wr S_n \tag{31}$$

where $(n_1, \ldots, n_k)$ is a k-fold partition of n, α_i is an IUR of $S_{n_i}, [\alpha_i]$ is the lift of α_i to $G \wr S_{n_i}$ and $\#$ is the ordinary outer tensor product of representations.
Note that *before* inducing to $G \wr S_n$, Eq. (31) provides an IUR of $(G \wr S_{n_1}) \times \ldots \times (G \wr S_{n_k}) \cong G \wr S_{(n)}$, where $S_{(n)} \equiv S_{n_1} \times \ldots \times S_{n_k}$.

Corollary 3: The number of IUR's of $G \wr S_n$ is

$$N_n(G) \equiv \sum_{(n)} p(n_1)p(n_2)\ldots p(n_k) \tag{32}$$

where $(n) = (n_1, \ldots, n_k)$ is a k-fold partition of n, $p(n_i)$ is the number of partitions of n_i and k is the number of conjugacy classes of G.

The proof of corollary 3 follows immediately from Theorem 7 given that S_{n_i} has $p(n_i)$ IUR's (see section 4) and G has k IUR's. So there ar e $N_n(\pi_1(M))$ inequivalent quantizations of the system of n identical particles on M, $\pi_1(M)$ finite. Combining Theorem 7 with the following well known result[62] will provide much information on particle statistics.

Theorem 8 (Mackey Subgroup Theorem): Let H and K be subgroups of a finite group G, ρ a representation of H, and S a set of representatives for the (H, K) double cosets[45] of G. For all $s \in S$, let $H_s = sHs^{-1} \cap K$ which is a subgroup of K. The map ρ^s defined by $\rho^s(x) = \rho(s^{-1}xs), K \in H_s$, is a representation of H_s, and we have

$$(\rho \uparrow G) \downarrow K \simeq \bigoplus_{s \in S} (\rho^s \uparrow K). \tag{33}$$

Corollary 4: Let $(n_1, \ldots, n_k)$ be a k-fold partition of n and $S_{(n)} = S_{n_1} \times \ldots \times S_{n_k}$. If ρ is an IUR of $G \wr S_n$ then

$$(\rho \uparrow G \wr S_n) \downarrow S_n \simeq (\rho \downarrow S_{(n)}) \uparrow S_n \ . \tag{34}$$

Proof: In Theorem 8 let G be $G \wr S_n$, $H = G \wr S_{(n)}$ and $K = S_n$. Since $[G \wr S_{(n)}] \cdot S_n = G \wr S_n$ there is just one (H, K) double coset with representative the identity element e. $\rho^e = rho \downarrow S_{(n)}$ and the result follows immediately from Eq. (33). ■

For an IUR ρ of $G \wr S_{(n)}$ as in Eq. (31) (prior to induction), notice that

$$\rho \downarrow S_{(n)} = [(\overset{n_1}{\tilde{\#}} \rho_1 \downarrow S_{n_1}) \otimes \alpha_1] \# \ldots \# [(\overset{n_k}{\tilde{\#}} \rho_k \downarrow S_{n_k}) \otimes \alpha_k] \tag{35}$$

and

$$(\overset{n_i}{\tilde{\#}} \rho_i)(e, \ldots, e; \pi)(v_1 \otimes \ldots \otimes v_n) = v_{\pi(1)} \otimes \ldots \otimes v_{\pi(n)}. \tag{36}$$

So $(\overset{n_i}{\tilde{\#}} \rho_i) \downarrow S_{n_i}$ is simply a permutati on representation and depends solely on $dim\ \rho_i$. Thus, for any finite group G we need only specify its distinct character degrees along with their multiplicities (which gives the class number) in order to classify the statistical equivalence classes of $\mathcal{R}(G \wr S_n)$ completely in terms of IUR's of S_n. In particular, if G is also abelian, the statistical classes depend only on $\mid G \mid$ (the order of G), since all IUR's of G are one-dimensional.

We have obtained a complete characterization (number and type) of the statistics available for $n = 2, 3, 4, 5$ and 6 identical particles on $M, dim\ M \geq 3$ and $\pi_1(M)$ finite.

All we need is the character of the permutation representation $\rho \downarrow S_{(n)}$ mentioned above, the character tables of S_n, $2 \leq n \leq 6$, and the formula for an induced character. That is, if α is a representation of a subgroup H of a finite group G with character χ, then the character ϕ of $\alpha \uparrow G$ is given by[62] ($g \in G$)

$$\phi(g) = \frac{1}{\mid H \mid} \sum_{x \in G} \chi(xgx^{-1}), \tag{37}$$

where we set $\chi(g) = 0$ if $g \notin H$. We will let $H = S_{(n)}$, $\alpha = \rho \downarrow S_{(n)}$ and $G = S_n$. By Theorem 7 and Eq. (34), this will suffice. (Statistical equivalence of two IUR's of $\pi_1(M) \wr S_n$ can be easily ascertained from the characters of S_n via Eq.(2).) For example take $n = 2$ and let $d_1, \ldots, d_k$ be the degrees of the k IUR's of $\pi_1(M)$. There are $N_2(G) = \frac{1}{2}(k^2 + 3k)$ IUR's of $\pi_1(M) \wr S_2$ whose characters, when restricted to $\Sigma_2 \cong S_2$, are given by

	e	σ
$\alpha_{ij} \downarrow S_2$ $1 \leq i, j \leq k, i \neq j$	$2d_i d_j$	0
$\beta_i \downarrow S_2$ $(1 \leq i \leq k)$	d_i^2	d_i
$\gamma_i \downarrow S_2$ $(1 \leq i \leq k)$	d_i^2	$-d_i$

From Eq. (2) we see that all of the α_{ij}'s are statistically equivalent, while the β_i's and γ_i's are all statistically distinct for distinct d_i. Thus, if q represents the number of *distinct* character degrees of $\pi_1(M) \neq \{e\}$ then there are $2q + 1$ types of statistics for 2 identical

particles on M. There are only 2 (Bose and Fermi) if $\pi_1(M) = \{e\}$. Each α_{ij} yields ambistatistics for the 2 particles (see section 3) while $\beta_i \downarrow S_2$ $[\gamma_i \downarrow S_2]$ is a direct sum of $d_i(d_i+1)/2$ $[d_i(d_i-1)/2]$ Bose IUR's and $d_i(d_i-1)/2$ $[d_i(d_i+1)/2]$ Fermi IUR's. We have performed a similar analysis for $3 \leq n \leq 6$. However, below we simply list the number of statistics available for the n particle system in these cases.

n	# of statistics	k
2	2	1
	$2q+1$	≥ 2
3	3	1
	5	2
	$5q+1$	≥ 3
4	5	1
	11	2
	$2q^2+8q+3\Sigma$	3
	$2q^2+8q+3\Sigma+1$	≥ 4
5	7	1
	18	2
	$8q^2+7q+9\Sigma$	3
	$8q^2+9q+9\Sigma$	4
	$8q^2+9q+9\Sigma+1$	≥ 5
6	11	1
	34	2
	$\frac{4}{3}q^3+\frac{33}{2}q^2+\frac{31}{6}q+(16+6q)\Sigma+4\kappa$	3
	$\frac{4}{3}q^3+\frac{37}{2}q^2+\frac{37}{6}q+(19+6q)\Sigma+4\kappa$	4
	$\frac{4}{3}q^3+\frac{37}{2}q+\frac{49}{6}q+(19+6q)\Sigma+4\kappa$	5
	$\frac{4}{3}q^3+\frac{37}{2}q+\frac{49}{6}q+(19+6q)\Sigma+4\kappa+1$	≥ 6

In the above q is the number of distinct character degrees of $\pi_1(M)$, Σ is the number of character degrees appearing at least twice, and κ the number of those degrees appearing at least 3 times.

We have found that if $k \geq n$ we always get an equivalence class of statistics for n identical particles on M corresponding to the regular representation of S_n. We call these statistics *complete* statistics. They are a generalization of ambistatistics, which are complete statistics for $n = 2$. Using the same combinatorial techniques as above, one can classify the statistical equivalence classes of $\mathcal{R}(\pi_1(M) \wr S_n)$ for any n.

We conclude this section with two results concerning the existence of ambistatistics.

Lemma 2: If there is an epimorphism $\phi : G \to H$, then there exists an epimorphism $\tilde{\phi} : G \wr S_n \to H \wr S_n$ such that $\tilde{\phi}$ maps $S_n \subseteq G \wr S_n$ isomorphically onto $S_n \subseteq H \wr S_n$.

Proof: Let $\tilde{\phi}(g_1, \ldots, g_n; \pi) = (\phi(g_1), \ldots, \phi(g_n); \pi)$, $g_i \in G$, $\pi \in S_n$. ∎

Theorem 9: If $\pi_1(M)$ is finitely generated and *not* U-inert, then one always obtains an ambistatistical equivalence class in $\mathcal{R}(B_2(M))$, $dim\ M \geq 2$.

Proof: By Theorem 1, $\pi_1(M)$ has a nontrivial finite homomorphic image H, and as we have shown earlier $H \wr S_2$ always has an ambistatistical IUR. By Lemma 2 this can be lifted to an ambistatistical IUR of $\pi_1(M) \wr S_2$. If $dim\ M = 2$, it can further be lifted to $B_2(M)$ (see section 4). ∎

Theorem 10: Let $dim\ M \geq 3$ and $n \geq 3$. Then for each $\rho \in \mathcal{R}(B_n(M))$ we have $\rho(\sigma_i) \neq \rho(\sigma_j)$ for some $1 \leq i, j \leq n-1$, unless $dim\ \rho = 1$.

The proof of Theorem 10 follows from the presentation of $B_n(M) = \pi_1(M) \wr S_n$ given in section 4. We leave it as an exercise for the reader.

Theorem 9 tells us that 2 identical particle systems very often have ambistatistical quantizations, while Theorem 10 tells us that for 3 or more identical particles on $M, dim M \geq 3$,

there is never a prime quantization (with corresponding IUR ρ) such that $\rho \downarrow S_n$ is a direct sum of Bose and Fermi IUR's of S_n (including ambistatistics). So the "part Bose - part Fermi" prime quantizations are a two-particle phenomenon.

In closing, we note that wreath products and their representations have also appeared in studies of nuclear magnetic resonance.[63]

6. CLUSTER PROPERTIES

In previous sections we have rigorously defined the notion of statistics for n identical particles on a manifold M, and have demonstrated the formal existence of quantizations displaying various types of exotic statistics. Even so, the reader may not be convinced of the *possible* physical utility of these quantizations. More specifically, although exotic statistics are consistent with the basic tenets of the quantum theory, it is not clear that the exotic quantizations obey the appropriate cluster decomposition properties required of a realistic model. For n identical particles on M we require the following two properties:

(a) If $m \leq n$ of the particles are localized on a submanifold N of M then we want the effective description of the m particle subsystem provided by a quantization of the full n particle system to be acceptable from the point of view of an independent m particle system on N.

(b) Each distinct quantization of an independent m particle system on N should be, in principle, obtainable via reduction (as above) from some larger system.

(a) and (b) guarantee that when describing local experiments (i.e. on N) we need only consider local degrees of freedom.

In order to study property (a), we choose a basepoint of $Q_n(M)$ with precisely m particles in N. Let $C_m(N)$ be the subgroup of $B_n(M)$ generated by interchanges of the m particles in a disk in N and loops of these particles which lie completely in N. (a) amounts to the statement that if ρ is an IUR of $B_n(M)$ then $\rho \downarrow C_m(N)$ must

"correspond" to a representation of $B_m(N)$. This is satisfied if $C_m(N)$ is a homomorphic image of $B_m(N)$. To see the epimorphism consider the commutative diagram induced by the inclusion of N in M.

$$\begin{array}{ccccccc} \{e\} \longrightarrow & \pi_1(N^m - \Delta) & \xrightarrow{\alpha} & B_m(N) & \xrightarrow{\beta} & S_m & \longrightarrow \{e\} \\ & \downarrow f & & \downarrow g & & \downarrow h & \\ \{e\} \longrightarrow & \pi_1(M^n - \Delta) & \xrightarrow{\gamma} & B_n(M) & \xrightarrow{\delta} & S_n & \longrightarrow \{e\} \ . \end{array} \tag{38}$$

Here h is the natural inclusion of S_m in S_n. It is clear that the image of g in $B_n(M)$ is $C_m(N)$, which is what we wished to demonstrate. (Note that by the five lemma $C_m(N) \cong B_m(N)$ if f is injective, i.e., if all noncontractible loops in $N^m - \Delta$ remain so when we are allowed to deform them into $M^n - \Delta$.) Of course the representation $\rho \downarrow C_m$ need not be irreducible, so that the effective quantization of the m particle subsystem provided by a prime quantization of the full system need not be prime.

As an example let ρ be the IUR corresponding to the unique Fermi quantization of n identical particles on $M = \mathbb{R}^d$, $d \geq 2$, and let N be any submanifold of $\mathbb{R}^d$ with $2 \leq dim\ N \leq d$. The effective quantization of the $m \geq 2$ particle subsystem on N provided by ρ corresponds to an IUR α of $B_m(N)$ which is the lift of the alternating representation of its S_m quotient group. As a second example let $M = \mathbb{R}^3$, $n = 3$, $m = 2$ and N be a proper submanifold of $\mathbb{R}^3$ which is also homeomorphic to $\mathbb{R}^3$. Take ρ to be the 2-dimensional IUR of $B_3(\mathbb{R}^3) \cong S_3$ given by

$$\rho(\sigma_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \rho(\sigma_2) = \begin{pmatrix} -\frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} . \tag{39}$$

$C_2(N) = < \sigma_1 > \cong \mathbb{Z}_2$ and the effective quantization of the 2 particle subsystem, which is not prime, is the direct sum of the Bose and Fermi prime quantizations. (This quantization is *not* ambistatistical.) Finally, an interesting example is given by $M = S^2$, $N \approx$

$\mathbb{R}^2$, $m < n$, $n \geq 3$, and ρ an exotic scalar representation of $B_n(S^2)$ characterized by an angle $\theta = \frac{j\pi}{(n-1)}$ (recall that $B_n(S^2)_{ab} \cong \mathbb{Z}_{2n-2}$) The angle θ above is *not* an allowed statistical angle for m identical particles on S^2. However since $N \approx \mathbb{R}^2$ all angles θ are possible and the effective quantization of the m particle subsystem is acceptable. This result still holds if N is *any* proper $2D$-submanifold of S^2 (see section 4).

We now turn to part (b) of the cluster property. One cannot, in principle, rule out the appearance of a specific representation α of $B_m(N)$ in the reduction from any given "parent system". For even if none of the IUR's of the braid group of the big system restrict to α, there still may be Berry-Wilczek-Zee (BWZ) adiabatic (matrix) phases[64] arising in the dynamical reduction to the subsystem, which will modify the restricted representation. These phases depend on the specific form of the Hamiltonian of the total system. For a more quantitative account of how BWZ phases may affect the effective kinematical quantizations of subsystems, see Ref. 37.

7. STATISTICS IN ONE DIMENSION

The analysis of identical particles in one-dimension differs in various key respects from that in higher dimensions. Here M is either $\mathbb{R}$ or S^1. For $M = \mathbb{R}$, the space $\mathbb{R}^n - \Delta$ breaks up into $n!$ connected components. This reflects the fact that no continuous particle permutation can be performed without bringing at least 2 particles through each other.[65] Hence (like the situation in zero dimensions) there is no topological notion of statistics on $\mathbb{R}$. The situation for $M = S^1$ is much more interesting. The space $(S^1)^n - \Delta$ has $(n-1)!$ components, reflecting the fact that, unlike the $M = \mathbb{R}$ case, continuous *cyclic* par ticle permutations may still be performed. The components of $(S^1)^n - \Delta$ are identified under the action of S_n so that the orbit space $[(S^1)^n - \Delta]/S_n \equiv Q_n(S^1)$ is path-connected; the cyclic $\mathbb{Z}_n$ subgroup of S_n acts within a given component of $(S^1)^n - \Delta$. For each n we have $\pi_1(Q_n(S^1)) \cong \mathbb{Z}$ where the generator α corresponds to the cyclic permutation

$1 \to 2, 2 \to 3, \ldots n \to 1$ (around the circle). The element of $\pi_1(Q_n(S^1))$ corresponding to taking one particle around a loop of winding number m is just α^{mn}. Under such a loop, *all* particles must wind around the circle m times in order to avoid "collisions". Unlike the situation in higher dimensions, there is no separation of generators into "$\mathcal{L}$ and $\mathcal{P}$". If we wish to retain the concept of statistics on S^1 we must choose the analog of $\Sigma_n(M)$ to be all of $\pi_1(Q_n(S^1))$, thereby leading to something akin to θ-statistics since $Hom\ (\pi_1(Q_n(S^1)),\ U(1)) \cong U(1)$. However in contrast to θ-statistics, various particle permutations cannot be realized continuously (except in the case $n = 2$) as noted above.

8. DISCUSSION AND CONCLUSIONS

Model building proceeds most efficiently when the researcher has at his disposal the complete set of theoretical structures consistent with the broad conceptual framework within which he is working. This alone is a strong motivation for studying the inequivalent quantizations discussed here; one can obtain the complete set of theories, consistent with the basic tenets of quantum mechanics, with a given classical limit. In this paper, following Ref. 26, we have performed a comprehensive analysis of the inequivalent quantizations of the system of n identical particles moving on a manifold M. The outcome has been a rigorous and unified treatment of the statistics of identical particles, as well as examples of new types of exotic nonscalar statistics. However there are various important questions which have not been addressed. Are there "particle-like" structures occurring in realistic systems which obey, at some level of description, any of the new exotic statistics? Certainly none of which we are aware. As many of our exotic statistics rely on the nonsimply connectedness of the ambient space M, it may be some time before relevant experiments are available. But history tells us not to get discouraged too quickly, for recall that the formal demonstration of fractional statistics predated their

well known applications in condensed matter physics by several years. We therefore feel that techniques (such as those presented here) which allow one to calculate all statistical possibilities in a given situation are just as important as any specific application.

Another important question is: which, if any, of our exotic statistics survive second quantization? More precisely, do there exist relativistic quantum field theories (RQFT's) which possess particle-like excitations obeying such statistics? (Our work only addresses the existence question in *particle mechanics.*) It is well known that parastatistics[3] and (possibly) fractional statistics[66] can be obeyed by the elementary field quanta in certain RQFT's. Also, solitons in $(2+1)$ -dimensional field theory may be anyons.[16] Although there is some evidence for ambistatistical solitons in some $(1+1)$-dimensional field theories,[67] no other types of exotic statistics are known to survive second quantization. Of course, even if a specific type of statistics is ruled out in relativistic field theory, this does not mean that they cannot arise as an effective description of certain systems at the level of nonrelativistic mechanics. If a given type of statistics can be shown to exist in a RQFT, then many more interesting questions may be asked. For example, is the notion of spin well defined for such excitations, and is there a "spin-statistics" connection? These questions have been addressed for para- and θ-statistics.

Very recently, some fascinating related work on statistics in gravity and string theory has begun. More specifically, Aneziris et.al.[68] have demonstrated that three-dimensional geons[69] in quantum gravity "may be neither bosons nor fermions (nor paraparticles)." New exotic statistics for two-dimensional geons were also found. These facts, among others, lead the authors to conclude "our usual conceptions about the statistics of particle species thus do not seem to be valid in generally covariant theories". In the examples which motivate these statements the identical geons obey what we would here call (fractional) ambistatistics. Thus these strange geon statistics *do* have an analog in particle mechanics and our examples above may provide a simple theoretical laboratory

in which to study the properties of these interesting topological excitations in quantum gravity. Also it has been shown that two identical closed strings on $\mathbb{R}^3$ can be ambions![70] In the above, the "internal structure" of the strings and geons provides the necessary topological complexity for the existence of ambistatistics, while for two point particles it was the nonsimply connectedness of the ambient space M.

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