Single Particle Motion

Contents

Uniform E and B
- E = 0 - guiding centers
- Definition of guiding center
- E ≠ 0
- gravitation

Non Uniform B
- 'grad B' drift, \( \nabla B \perp \vec{B} \)
- Curvature drift
- Grad -B drift, \( \nabla B \parallel \vec{B} \)
- invariance of \( \mu \).
- Magnetic mirrors, loss cone
- Bounce Period in a dipole
- Drift period in a dipole
- The Ring Current around the earth

Non Uniform E (finite Larmor radius)

Time varying E (polarization drift)

Time Varying B (magnetic moment)

Adiabatic Invariants

Appendix (Geometry)
Plasmas somewhere between fluids and single particles. Here consider single particles.

**Uniform E and B**

E = 0 - guiding centers

\[
m \frac{d\mathbf{v}}{dt} = q \mathbf{v} \times \mathbf{B}
\]

Take \( \mathbf{B} = B \hat{z} \). The

\[
m \dot{v}_x = qBv_y; \quad m \dot{v}_y = -qBv_x; \quad m \dot{v}_z = 0
\]

\[
\dot{v}_x = \frac{qB}{m} \dot{v}_y = -\left( \frac{qB}{m} \right)^2 v_x
\]

\[
\dot{v}_y = -\frac{qB}{m} \dot{v}_x = -\left( \frac{qB}{m} \right)^2 v_y
\]

Harmonic oscillator at cyclotron frequency

\[
\omega_c = \frac{|q|B}{m}
\]

i.e.

\[
v_{x,y} = v_\perp e^{\pm i\omega_c t + i\delta_{x,y}}
\]

\( \pm \) denotes sign of \( q \). Choose phase \( \delta \) so that

\[
v_x = v_\perp e^{i\omega_c t} = \dot{x}
\]

where \( v_\perp \) is a positive constant, speed in plane perpendicular to vector \( B \). Then

\[
v_y = \frac{m}{qB} \dot{v}_x = \pm \frac{1}{\omega_c} \dot{v}_x = \pm v_\perp e^{i\omega_c t} = \dot{y}
\]

Integrate

\[
x - x_0 = \pm \frac{v_\perp}{\omega_c} e^{i\omega_c t} \quad ; \quad y - y_0 = \pm \frac{v_\perp}{\omega_c} e^{i\omega_c t}
\]

Define Larmor radius
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\[ r_L = \frac{v_L}{\omega_c} = \frac{m v_L}{|q| B} \]

Taking real parts \((fn + \text{complex conj})/2\) gives

\[ x - x_0 = r_L \sin(\omega_c t); \quad y - y_0 = \pm r_L \cos(\omega_c t) \]

i.e. a circular orbit around B. Ions and electrons circulate in opposite directions. The sense is such that the B field generated by the particle always tends to reduce the external B, i.e. plasmas are diamagnetic. Electrons have smaller Larmor radii than ions.

**Definition of guiding center**

Defined as

\[ \vec{r}_{gc} = \vec{r} - \vec{R}_c \]

\( \vec{r} \) is position vector of particle and \( \vec{R}_c \) is radius of curvature, which is a vector from the position of the particle to the center of gyration. In the plane perpendicular to B vector write in terms of momentum vector p:

\[ m \omega_c^2 \vec{R}_c = q \vec{v} \times \vec{B} \]

Now \( \omega_c^2 = \frac{q^2 B^2}{m^2} \), so

\[ \vec{R}_c = \frac{\vec{p} \times \vec{B}}{q B^2} \]

and

\[ \vec{r}_{gc} = \vec{r} - \frac{\vec{p} \times \vec{B}}{q B^2} \]

Now consider a collision in which a force \( f \) is applied to the particle in a direction perpendicular to the vector B, and this force is \( f \gg \) the Lorentz force. Let the field be homogeneous. Let the impact time be very short, \( \ll \) the Larmor frequency. At the collision the momentum p changes a lot but the particle position vector \( r \) does not. The momentum p changes from p to \( p' = p + \Delta p \), where

\[ \Delta \vec{p} = \int_{t}^{t+\Delta t} \vec{f} dt \]
Now the guiding center must move by an amount
\[ \Delta r_{gc} \approx \frac{\Delta p \times B}{qB^2} \]

Generalizing to a continuous force
\[
\frac{d}{dt} r_{gc} = \frac{d}{dr} r_{gc} + \frac{d}{dt} \frac{p \times B}{qB^2}
\]

now use \( \frac{dp}{dt} = q \nabla \times B + F \)

where \( F \) is a non magnetic force. Then
\[
\frac{dr_{gc}}{dt} = v + \frac{(F + q \nabla \times B) \times B}{qB^2}
\]
\[
= v + \frac{(F \times B + q(\nabla \times B) \times B)}{qB^2}
\]
\[
= v_{\parallel} + \frac{F \times B}{qB^2}
\]

(exand triple vector product and use \( \vec{v} = v_{\perp} + v_{\parallel} \))

i.e. if the force is continuous, the guiding center motion can be viewed as a continuous series of small impacts.
\( E \neq 0 \)

Will find motion is sum of usual circular Larmor orbit plus a drift of the 'guiding center'. Choose \( E \) in the \( x-z \) plane so that \( E_y = 0 \). As before, \( z \) component of velocity is unrelated to the transverse components and can be treated separately.

\[
m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})
\]

\( z \) component

\[
m \frac{dv_z}{dt} = q(E_z); \quad v_z = \frac{qE_z}{m} t + v_{z0}
\]

i.e. acceleration along the vector \( B \). Transverse components give

\[
\dot{v}_x = \frac{q}{m} E_x \pm \omega_c v_y; \quad \dot{v}_y = 0 \mp \omega_c v_x
\]

Differentiate

\[
\ddot{v}_x = -\omega^2 v_x
\]

\[
\ddot{v}_y = \mp \omega_c \left( \frac{q}{m} E_x \pm \omega_c v_y \right) = -\omega^2 \left( \frac{E_x}{B} + v_y \right)
\]

i.e.

\[
\frac{d}{dt}^2 v_x = -\omega^2 v_x
\]

\[
\frac{d}{dt}^2 \left( v_y + \frac{E_x}{B} \right) = -\omega^2 \left( v_y + \frac{E_x}{B} \right)
\]

i.e. just like \( E = 0 \) case except replace \( v_y \) by \( v_y + E_x/B \). Therefore solution is
$$v_x = v_\perp e^{i\omega_c t}$$

$$v_y = \pm i v_\perp e^{i\omega_c t} - \frac{E_x}{B}$$

i.e. there is a drift in the -y direction. Electrons and ions drift in the same direction.

More generally, can obtain an equation for the 'guiding center' drift $v_{gc}$. Omit $dv/dt$ terms, as we know this just gives the Larmor orbit and frequency. Then

$$(E + \vec{v} \times \vec{B}) = 0$$

$$E \times \vec{B} = -(\nabla \times \vec{B}) \times \vec{B} = \vec{B} \times (\nabla \times \vec{B})$$

$$= \nu B^2 - \vec{B}(\nabla \cdot \vec{B})$$

(Use

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

$$a = B; b = \nu; c = B$$

$$\vec{B} \times (\nabla \times \vec{B}) = \nabla(\vec{B} \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{B})$$

transverse component gives the electric field drift of the guiding center"

$$\vec{v}_{gc} = \frac{E \times \vec{B}}{B^2}, \text{ independent of } q, m, \omega_c.$$ 

This is because on first half of orbit a particle gains energy from E field, so velocity and Larmor radius increase. But on second half of orbit the particle loses energy and Larmor radius decreases. The difference in Larmor radius causes the drift.

**gravitation**

Generalize by replacing electrostatic by a general force. Then

$$\vec{v}_{gc} = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$$

e.g. for gravity

$$\vec{v}_{gc} = \frac{m \vec{g} \times \vec{B}}{q B^2}$$
Note this is charge dependent, but very small. Therefore under gravity the charges drift and produce a net current:

\[ j = \sum n_i q_i v_i = n(m + M) \frac{g \times B}{B^2} \]

**Non Uniform B**

Need to expand in a small parameter; orbit theory

'grad B' drift \( \nabla B = \nabla B \perp B \)

Take straight field lines in y direction, but with a gradient in their density, . Anticipate result: gradient in \( |B| \) causes the Larmor radius to be different, If gradient in y direction then smaller radius at top than bottom, , which will give a drift opposite for e and i, perpendicular to both the field and its gradient .

Average the Lorentz force over a gyration. Note \( \langle F_i \rangle = 0 \) as equal time moving up and down. To calculate average \( \langle F_i \rangle \) use undisturbed orbit. Expand the field vector around the point \( x_0, y_0 \) so that

\[ B = B_0 + (r \cdot \nabla)B + .. \]

\[ B_z = B_0 + y \frac{\partial B}{\partial y} + .. \]
\[
F_y = -qv_x B_z(y) = -qv_x \cos(\omega_c t) \left[ B_0 + y \frac{\partial B}{\partial y} + ... \right]
\]
\[
-qv_x \cos(\omega_c t) \left[ B_0 \pm r_L \cos(\omega_c t) \frac{\partial B}{\partial y} + ... \right]
\]
This assumes \( r_L/L \ll 1 \), where \( L \) is scale of \( \partial B/\partial y \). Then
\[
\left\langle F_y \right\rangle = \pm \frac{1}{2} q v_{\perp} r_L \left( \frac{\partial B}{\partial y} \right)
\]
and the guiding center drift is
\[
v_{\text{vb}} = \pm \frac{1}{2} v_{\perp} r_L \frac{\vec{B} \times \nabla B}{B^2}
\]
with \( \pm \) standing for the sign of the charge.

**Curvature drift**

Assume lines of force have constant radius of curvature \( R_c \). Take \( |B| \) constant. This does not satisfy Maxwell's equations, so we will always add the grad B drift to the answer we are about to get. Let \( v_{\parallel}^2 \) be the average of the square of the (random) velocity along the vector \( B \). then the average centrifugal force is
\[
\vec{F}_c = \frac{m v_{\parallel}^2}{R_c} \hat{r} = \frac{m v_{\parallel}^2 R}{R_c^2}
\]
Therefore
\[
\bar{v}_r = \frac{1}{q} \frac{\vec{F}_c \times \vec{B}}{B^2} = \frac{m v_{\parallel}^2 R}{q B^2 R_c^2}
\]
Now compute the associated grad-B drift. Use cylindrical coordinates. \( \nabla \times \vec{B} = 0 \) in vacuum.. \( \vec{B} \) has only a \( \theta \) component, and \( \nabla B \) only a radial component , so
\[
(\nabla \times \vec{B})_z = \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) = 0; \quad B_\theta \propto \frac{1}{r}
\]
and
\[
|B| \propto \frac{1}{R_c}; \quad \frac{\nabla |B|}{|B|} = -\frac{R}{R_c^2}
\]
so that
\[ v_{vB} = \pm \frac{1}{2} v_{\perp} r_L \frac{B \times |B| \bar{R}}{B^2 R_c^2} = \pm \frac{1}{2} \frac{v_{\perp}^2 \bar{R} \times \bar{B}}{B R_c^2} = \frac{1}{2} q \frac{v_{\perp}}{B} \]

Adding the curvature term to this gives
\[ v_r + v_{vB} = \frac{m}{q} \frac{\bar{R} \times \bar{B}}{B^2 R_c^2} \left( \frac{v_{\perp}^2}{2} + v_{\parallel}^2 \right) \]

Now for a Maxwellian plasma \( v_{\parallel}^2 = \frac{1}{2} v_{\perp}^2 = k_B T \) (because the perpendicular component has 2 degrees of freedom), so that
\[ v_{r+vB} = \pm \frac{v_{\perp}^2}{\bar{R} \omega_c} \hat{y} = \pm \frac{\bar{R}_m v_{\perp}}{\bar{R}_c} \hat{y} \]

where \( \hat{y} = \frac{\bar{R} \times \bar{B}}{\bar{R}_c \times \bar{B}} \)

i.e. dependent on \( q \) but not \( m \).

**Grad - B drift, \( \nabla B \parallel \bar{B} \)**

Now consider B field along z, with magnitude dependent on \( z \). Consider symmetric case; \( \partial / \partial \theta = 0 \). Then lines of force diverge and converge, and \( B_r \neq 0 \). From \( \nabla \cdot \bar{B} = 0 \)
\[ \frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{\partial B_z}{\partial z} = 0 \]

Suppose \( \partial B_z / \partial z \) is given at \( r = 0 \), and is not strongly dependent on \( r \), then
\[ r B_z = - \int_0^r \frac{\partial B}{\partial r} dr \approx - \frac{1}{2} r^2 \left[ \frac{\partial B_z}{\partial r} \right]_{r=0} \]
\[ B_r \approx - \frac{1}{2} r \left[ \frac{\partial B_z}{\partial r} \right]_{r=0} \]

Variation of \( |B| \) with \( r \) causes a grad-B drift of the guiding centers about the axis of symmetry. However there is no radial grad - B drift because \( \partial B / \partial \theta = 0 \).

The component of the Lorentz force are
\[ F_r = q(v_o B_z - v_z B_o) = q(v_o B_z) \]
\[ F_\theta = q(-v_z B_z + v_o B_r) \]
\[ F_z = q(v_o B_z - v_o B_o) = q(-v_o B_r) \]

The terms \( F_r = q(v_o B_z) \); \( F_\theta \) part a \( = q(-v_o B_z) \) give the Larmor gyration. The term \( F_\theta \) part b \( = q(v_o B_r) \) vanishes on axis, and off axis it causes a drift in the radial direction. This makes the guiding centers follow the 'lines of force'. The final term, \( F_z = q(-v_o B_r) \) is written as

\[ F_z = \frac{1}{2} qv_0 r \left( \frac{\partial B}{\partial z} \right) \]

Average over a gyration period. Consider a particle with guiding center on axis. The \( v_\theta = v_\perp; \quad r = r_L \)

\[ \langle F_z \rangle = \frac{1}{2} q v_\perp r_L \left( \frac{\partial B}{\partial z} \right) = \frac{1}{2} q \frac{v_\perp^2}{\omega_c} \left( \frac{\partial B}{\partial z} \right) = -\frac{1}{2} \frac{m v_\perp^2}{B} \left( \frac{\partial B}{\partial z} \right) \]

Define magnetic moment

\[ \mu = \frac{1}{2} \frac{m v_\perp^2}{B} \]

Then in general the parallel force on a particle is given in terms of the element ds along the vector B:

\[ \vec{F}_\parallel = -\mu \frac{\partial B}{\partial s} = -\mu \nabla_\parallel B \]

Note for a current loop area A current I then

\[ \mu = IA = \frac{e \omega_c}{2 \pi} \pi r_L^2 = \frac{e \omega_c}{2 \pi} \frac{\pi v_\perp^2}{\omega_c} = \frac{1}{2} \frac{e v_\perp^2}{\omega_c} = \frac{1}{2} \frac{mv_\perp^2}{2B} \]

**Invariance of \( \mu \).**

Consider parallel equation of motion

\[ m \frac{d v_\parallel}{dt} = -\mu \frac{\partial B}{\partial s} \]
\[ m v_\parallel \frac{d v_\parallel}{dt} = \frac{d}{dt} \left( \frac{1}{2} m v_\parallel^2 \right) = -\mu \frac{\partial B}{\partial s} \frac{ds}{dt} = -\mu \frac{dB}{dt} \]
where $dB/dt$ is the variation of $B$ seen by the particle. ($B$ itself is or can be constant). Now conserve energy

$$
\frac{d}{dt}\left(\frac{1}{2}m\nu_{\|}^2 + \frac{1}{2}m\nu_{\perp}^2\right) = \frac{d}{dt}\left(\frac{1}{2}m\nu_{\|}^2 + \mu B\right) = 0
$$

Then

$$
\frac{d}{dt}(\mu B) - \mu \frac{dB}{dt} = 0; \quad \frac{d\mu}{dt} = 0
$$

**Magnetic mirrors, loss cone**

Finally we have the magnetic mirror. A particle moves from weak to strong field. $B$ increases, so $\nu_{\perp}$ must increase to conserve $\mu$. Then $\nu_{\|}$ must decrease to conserve energy. If $B$ is high enough at some place along the trajectory, then $\nu_{\|} = 0$, and the particle is reflected. A particle with small $\nu_{\perp}/\nu_{\|}$ at the mid plane where $B = B_0$ is a minimum can escape if the maximum field $B_m$ is not sufficiently large. Let $B = B_0$, $\nu_{\|} = \nu_{\|0}$ and $\nu_{\perp} = \nu_{\perp0}$ at the mid plane. At the turning point $B = B'$, $\nu_{\|} = 0$, and $\nu_{\perp} = \nu_{\perp}'$. Conserve $\mu$

$$
\frac{1}{2} m\nu_{\perp0}^2 B_0 = \frac{1}{2} m\nu_{\perp}'^2 B'
$$

Conserve energy

$$
\nu_{\perp}'^2 = \nu_{\perp0}^2 + \nu_{\|0}^2 \equiv v_0^2
$$

$$
\frac{B_0}{B} = \frac{\nu_{\perp0}^2}{\nu_{\perp}^2} = \frac{\nu_{\|0}^2}{\nu_{\perp0}^2} = \sin^2(\theta)
$$

$\theta$ is pitch angle of particle in weak field. Particles with smaller pitch $\theta$ will mirror in regions of higher $B$. If $\theta$ is too small $B' > B_m$ and the particle escapes. The smallest $\theta$ of a confined particle is given by the mirror ratio $R_m$:

$$
\sin^2(\theta) = \frac{B_0}{B_m} = \frac{1}{R_m}
$$

Particles which escape are said to be in the loss cone, independent of $q$ and $m$. Collisions can scatter particles into loss cones. Electrons are lost easily because collision frequency is higher.
Consider the particles at the low field point where \( B = B_0 \). Let there be a uniform distribution of pitch angles \( \alpha \). The probability of getting lost is

\[
P = \Phi / (2\pi) = \int_{0}^{\alpha_0} \sin(\alpha) d\alpha = 1 - \cos(\alpha_0)
\]

\[
= 1 - \left(1 - \sin^2(\alpha_0)\right)^{1/2} = 1 - \left(1 - \frac{B_0}{B_M}\right)^{1/2} = 1 - \left(1 - \frac{1}{R_M}\right)^{1/2}
\]

Planetary Loss Cones exist. Take account of atmosphere. e.g. define equatorial loss cone for Earth as that pitch angle \( \alpha_0 \) such that its mirror point is at 100 km from the surface. This is arbitrary, but used because at 100 km the density is high enough for scattering of electrons to occur. Therefore at 100 km and below a mirroring electron will probably get absorbed by the atmosphere and lost from the radiation belt. The equatorial loss cone for a dipole line of force crossing the equator at 6 \( R_E \) is about 30°. All electrons within 30° equatorial pitch angle cone are precipitated because they are mirroring below 100 km. Electrons outside of the loss cone mirror at heights above 100 km are trapped radiation belt electrons.
**Bounce Period in a dipole**

Consider trapped particles in the Earth's dipole field. Period of north south motion of guiding center is

$$T_b = \int \frac{ds}{v_\parallel}$$

ds along path, $v_\parallel$ along path vector B, integral over complete period. Note

$$v_\parallel^2 = v^2 - v_\perp^2 = v^2 \left(1 - \sin^2(\alpha)\right) = \left(1 - \frac{B}{B_0} \sin^2(\alpha_0)\right)$$

where conservation of the first invariant $\mu$ has been applied, $v^2 = v_0^2$ and subscript 0 means at the equator.

We need an expression for $ds$. Work in spherical coordinates

(r,θ) is polar coordinate of point P. PM perpendicular to OQ.

$$\sin(OQP) = r \frac{\sin(\delta \theta)}{PQ} = r \frac{\sin(\delta \theta)}{\delta \theta} \frac{\delta s}{\delta s}$$

Let Q approach P, then angle OQP becomes the angle between the tangent and the radius vector, denoted as $\phi$. Also $\sin(\delta \theta) / \delta \theta \rightarrow 1$; $\delta s / PQ \rightarrow 1$; $\delta \theta / \delta s \rightarrow d \theta / ds$ and

$$\sin(OQP) \rightarrow \sin(\phi) = r \frac{d \theta}{ds}$$

Similarly
\[
\cos(OQP) = \frac{MQ}{PQ} = \frac{OQ - OM}{PQ}
\]
\[
= \frac{r + \delta r - r \cos(\delta \theta) \delta s}{PQ}
\]
\[
= \left( \frac{r(1 - \cos(\delta \theta))}{\delta s} + \delta r \right) \frac{\delta s}{PQ}
\]

i.e.
\[
\cos(\phi) = \left(0 + \frac{dr}{ds}\right) = \frac{dr}{ds}
\]

Then
\[
\tan(\phi) = r \frac{d\theta}{dr}
\]

and since
\[
\sec^2(\phi) = 1 + \tan^2(\phi); \quad \cos e c^2(\phi) = 1 + \cot^2(\phi)
\]

then
\[
\left(\frac{ds}{dr}\right)^2 = 1 + r^2 \left(\frac{d\theta}{dr}\right)^2
\]
\[
\frac{1}{r^2} \left(\frac{ds}{d\theta}\right)^2 = 1 + \frac{1}{r^2} \left(\frac{dr}{d\theta}\right)^2
\]

i.e.
\[
\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2
\]

For a dipole field \( r = r_0 \cos^2(\lambda) \), where \( \lambda = \pi/2 - \theta \), and finally
\[
ds = r_0 \cos(\lambda) \left(1 + 3\sin^2(\lambda)\right)^{1/2} d\lambda
\]

Also for a dipole \( \frac{B}{B_0} = \left(\frac{1 + 3\sin^2(\lambda)}{\cos^6(\lambda)}\right)^{1/2} \), so that
\[
T_b = \int_{0}^{\lambda_{\text{max}}} \frac{r_0 \cos(\lambda) \left(1 + 3\sin^2(\lambda)\right)^{1/2}}{\sqrt{1 - \sin^2(\alpha_0) \left(\frac{1 + 3\sin^2(\lambda)}{\cos^6(\lambda)}\right)^{1/2}}} d\lambda = \frac{4r_0 I_1}{v}
\]
\[ I_1 = \int_0^{\lambda_{\text{max}}} \frac{\cos(\lambda) (1 + 3 \sin^2(\lambda))^{1/2}}{1 - \sin^2(\alpha_0) \frac{(1 + 3 \sin^2(\lambda))^{1/2}}{\cos^5(\lambda)}} \, d\lambda \]

\( \lambda_1 \) is the mirror point of the particle in the northern hemisphere. Here \( v_\parallel = 0 \) and \( \lambda_m \) is given by the solution of

\[ \cos^6(\lambda_m) - \sin^2(\alpha_0) (1 + 3 \sin^2(\lambda_m))^{1/2} = 0 \]

Now \( I_1 \) is dimensionless, and solved numerically to be

\[ I_1 \approx 1.3 - 0.56 \sin(\alpha_0) \]

Note \( 4r_0/v \) depends on the particle energy.

Now particles are precipitated (lost) if the distance of closest approach

\[ \frac{R_{\text{min}}}{r_0} < \frac{1}{L} \]

i.e. particles are trapped if the distance of closest approach is larger than \( R_{\text{min}} \). Now the field line equation is \( r = r_0 \cos^2(\lambda) \), so that for particles mirroring at the plane surface we have \( \lambda_{\text{min}} \) given by

\[ \cos^2(\lambda_m) = \frac{1}{L} \]

For Earth, \( r_0 = L R_E \) (L is a parameter introduced by McIlwain). Figure shows bounce period as a function of electron energy for the marginally trapped electrons, for \( L = 1 \) to 8. For Auroral lines of force (\( L = 6 \)), and typical electron energies of 10 keV to 50 keV, the bounce period is a few seconds.

**Drift period in a dipole field**

To calculate the guiding center drift period in a dipole field, remember that there are two components, the gradient drift and the curvature drift. They can be combined together as

\[ v_R + v_{VB} = \frac{m}{q} \frac{\vec{R} \times \vec{B}}{B R^2_c} \left( \frac{v^2_\perp}{2} + v^2_\parallel \right) = \frac{1}{\omega_c R_c} \frac{\vec{R} \times \vec{B}}{B R^2_c} \left( \frac{v^2_\perp}{2} + v^2_\parallel \right) \]

p 3.15
where $\omega_c$ is the cyclotron frequency, and $R_c$ is the radius of curvature. This drift is perpendicular to vector $B$ and vector $\rho$, that is around the word (equivalent to along the equator). The expression for the radius of curvature of a line in polar coordinates is

$$\rho = \frac{\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - 2\frac{d^2r}{d\theta^2}}$$

For the dipole field

$$r = r_0 \cos^2(\lambda); \quad \lambda = 90 - \theta$$

so that the radius of curvature vector is

$$\rho = r_0 \cos(\lambda) \left(1 + 3\sin^2(\lambda)\right)^{\frac{3}{2}} \frac{3(1 + \sin^2(\lambda))}{3}$$

The cyclotron frequency for the dipole is

$$\omega_c = \omega_0 \frac{\left(1 + 3\sin^2(\lambda)\right)^{\frac{1}{2}}}{\cos^6(\lambda)}$$

where $\omega_0$ is the value at the equator. Then the drift velocity at a latitude $\lambda$ is written as

$$v_D = \frac{3v^2}{2} \frac{\left(1 + \sin^2(\lambda)\right) \cos^5(\lambda)}{\omega_0 r_0 \left(1 + 3\sin^2(\lambda)\right)^{\frac{1}{2}} \cos^6(\lambda)} \left[2 - \sin^2(\alpha_0) \frac{\left(1 + 3\sin^2(\lambda)\right)^{\frac{1}{2}}}{\cos^6(\lambda)} \sin^2(\alpha_0)\right]$$

Note we have used

$$v_\| = v \cos(\alpha); \quad v_\bot = v \sin(\alpha) \quad (\alpha \text{ is the pitch angle})$$

$$v_\|^2 + \frac{1}{2} v_\bot^2 = v^2 \cos^2(\alpha) + v^2 \sin^2(\alpha) - \frac{1}{2} v^2 \sin^2(\alpha)$$

$$= v^2 \left(1 - \frac{1}{2} \sin^2(\alpha)\right) = v^2 \left(1 - \frac{1}{2} \frac{B}{R_0} \sin^2(\alpha_0)\right)$$

$$= \frac{v^2}{2} \left(2 - \frac{(1 + 3\sin^2(\lambda))^{\frac{1}{2}}}{\cos^6(\lambda)} \sin^2(\alpha_0)\right)$$
Now the angular drift (around the earth) which occurs in one bounce period is

$$\Delta \Phi = \int \frac{v_D}{r \cos(\lambda)} \frac{ds}{v_{\parallel}} \text{ (ds along vector } B)$$

(because bounce period is $T_b = \int ds / v_{\parallel}$, so distance covered in a bounce period = $T_b = \int v_D ds / v_{\parallel}$, and angular distance is as given. The angular drift, averaged over a bounce, is then

$$\langle \omega_D \rangle = \frac{\Delta \Phi}{2\pi T_B}$$

Using expressions derived before, we can write this as

$$\langle \omega_D \rangle = \frac{3v^2}{2\omega_0 R^2} \frac{I_2}{I_1} = \frac{3E}{qB_0 R^2 L^2} \frac{I_2}{I_1}$$

with kinetic energy $E = \frac{mv^2}{2}$, and

$$I_2 = \int_0^{\lambda_{\text{max}}} \frac{\cos^4(\lambda)\left[1 + \sin^2(\lambda)\right]}{1 - \sin^2(\alpha_0)\left[1 + 3\sin^2(\lambda)\right]^{1/2}} \left[2 - 3\sin^2(\alpha_0)\left[1 + 3\sin^2(\lambda)\right]^{1/2}\right] d\lambda$$

Numerically it is found that $I_2(\alpha_0) / I_1(\alpha_0) = 0.35 + 0.15\sin(\alpha_0)$, i.e. varying from 0.35 for $\alpha_0 = 0^0$ particles to 0.5 for $\alpha_0 = 90^0$ particles. Then the bounce averaged drift period

$$\langle T_D \rangle = \frac{1}{\langle \omega_D \rangle} \approx 50/(LE)$$

with $T_D$ in minutes and $E$ in MeV. Typical values (50 keV, $L = 6$) is about an hour.

**The Ring Current around the earth**

The drifting bounced orbits just discussed are represented as below in a Mercator projection. Magnetic storms are accompanied by a decrease in the horizontal field intensity at the earth. A westward current around the earth would do this. This is expected from trapped particles. Several MA are carried.
We know the drift velocity

\[ v_R + v_{\nabla B} = \frac{m}{q} \frac{\vec{R} \times \vec{B}}{BR_c^2} \left( \frac{v_\perp^2}{2} + \frac{v_\parallel^2}{2} \right) = \frac{1}{\omega_c R_c} \frac{\vec{R} \times \vec{B}}{BR_c^2} \left( \frac{v_\perp^2}{2} + \frac{v_\parallel^2}{2} \right) \]

from which we derived the averaged angular frequency in the equatorial direction

\[ \omega_D = \frac{3v^2}{2\omega_c r_0^2} = \frac{3E}{qB_0 r_0^2} \]

i.e. the current density

\[ J = nq\omega_D = nqr_0 \omega_D = -\frac{3nE}{B_0 r_0} \hat{\phi} \]

with \( n \) the number density and \( E \) the energy of the particles. Then the total current is given by

\[ I \, d\hat{l} = \vec{J} \, dV \]

i.e.

\[ I = -\frac{3E_{\text{tot}}}{2\pi R_0 r_0^2} \]

where \( E_{\text{tot}} \) is the total energy associated with the particles. Then the change in field at the center of this loop is

\[ \Delta \vec{B} = \frac{\mu_0 I}{2a} \approx -\frac{3\mu_0 E_{\text{tot}}}{4\pi B_0 r_0^2} \hat{z} \]
The total perturbation must account for another contribution, namely the diamagnetic current due to the cyclotron motion. This is calculated by noticing that the diamagnetic contribution is equivalent to a ring of dipoles of radius \( r \) and total magnetic moment \( \mu \). Because \( r \gg r_c \), the cyclotron radius, we can estimate the field at the center of the dipole ring as

\[
B(r, \lambda) = \frac{\mu_0 M}{4\pi r^3} \left( 1 + 3\sin^2(\lambda)^{1/2} \right)
\]

(from previous notes,

\[
\overline{B}_{\text{diamagnetic}} = -\frac{\mu_0}{4\pi r^3} \overline{\mu} = \frac{\mu_0 E_{\text{tot}}}{4\pi B_0 r_0^3} \overline{z}
\]

and

\[
\overline{\mu} = -\frac{mv^2\overline{B}}{2\overline{B}^2} = -\frac{E_{\text{tot}}}{B} \overline{z} = -\frac{E_{\text{tot}}}{B} \overline{z}
\]

Note the individual dipoles are aligned with the magnetic field direction.

The total perturbed field is then

\[
\Delta \overline{B} \approx -\frac{3\mu_0 E_{\text{tot}}}{4\pi B_0 r_0^3} \overline{z} + \frac{\mu_0 E_{\text{tot}}}{4\pi B_0 r_0^3} \overline{z} \approx -\frac{\mu_0 E_{\text{tot}}}{2\pi B_0 r_0^3} \overline{z} = -\frac{2E_{\text{tot}}}{M} \overline{z}
\]

where we have used

\[
B = \frac{\mu_0 M}{4\pi r^3}
\]

**Non Uniform E** (finite Larmor radius)

**Time varying E** (polarization drift)

**Time Varying B** (magnetic moment)
Adiabatic Invariants

Appendix - Some Geometry

If C is a space curve defined by the function \( \mathbf{r}(u) \), then \( d\mathbf{r} / du \) is a vector in the direction of the tangent to C. If the scalar \( u \) is the arc length \( s \) measured from some field point C, then \( d\mathbf{r} / ds \) is a unit tangent vector to C, and is called \( \mathbf{T} \). The rate at which \( \mathbf{T} \) changes with respect to \( s \) is a measure of the curvature of C and is given by \( d\mathbf{T} / ds \). The direction of \( d\mathbf{T} / ds \) at any point on C is normal to the curve at that point. If \( \mathbf{N} \) is a unit vector in this normal direction, it is called the unit normal. Then \( d\mathbf{T} / ds = \kappa \mathbf{N} \), where \( \kappa \) is called the curvature, and \( \rho = 1/\kappa \) is called the radius of curvature.

The position vector at any point is

\[
\mathbf{r} = x(s) \hat{i} + y(s) \hat{j} + z(s) \hat{k}
\]

Therefore

\[
\frac{d\mathbf{T}}{ds} = \frac{d^2 x}{ds^2} \hat{i} + \frac{d^2 y}{ds^2} \hat{j} + \frac{d^2 z}{ds^2} \hat{k}
\]
But \( \frac{dT}{ds} = \kappa N \); \( \kappa = \left| \frac{dT}{ds} \right| \). \( \rho = \frac{1}{\kappa} = \left( \left( \frac{d^2x}{ds^2} \right)^2 + \left( \frac{d^2y}{ds^2} \right)^2 + \left( \frac{d^2z}{ds^2} \right)^2 \right)^{-\frac{1}{2}} \).

**Spherical Coordinates \((r, \theta, \phi)\).**

**Transformation equations:**

\( x = r \sin \theta \cos \phi, \ y = r \sin \theta \sin \phi, \ z = r \cos \theta \)

where \( r \geq 0, \ 0 \leq \theta \leq \pi, \ 0 \leq \phi < 2\pi. \)

**Scale factors:** \( h_1 = 1, \ h_2 = r, \ h_3 = r \sin \theta \)

**Element of arc length:**

\( ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \)

**Jacobian:**

\( \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta \)

**Element of volume:**

\( dV = r^2 \sin \theta \ dr \ d\theta \ d\phi \)

**Laplacian:**

\( \nabla^2 U = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2} \)