Penning Traps

Contents

B and E fields used to increase time that an electron remains within a discharge: Penning, 1936. Can now trap a particle 'indefinitely in a combined homogeneous B and electrostatic quadrupole (now known as a Penning trap). A small cloud of such trapped particles is like a many electron atom, with the nucleus replaced by a trapping field.

Electrons introduced by applying HV to the field emmission point. The beam of energetic e's collide with sparse neutral, to produce slow electrons, which are captured in the trap. Electrodes are hyperboloids of revolution which produce a quadrupole field. Superimpose a uniform B. Resultant motion is fast circular cyclotron motion with a small radius carried along by a slow circular magnetron drift motion in a large orbit. i.e. an epicyclic motion in the x-y plane. Radius of cyclotron motion shrinks as synchrotron radiation is emitted. The axial oscillation is coupled to an external detector at low temperature. The large magnetron motion is a circle about an effective potential hill.

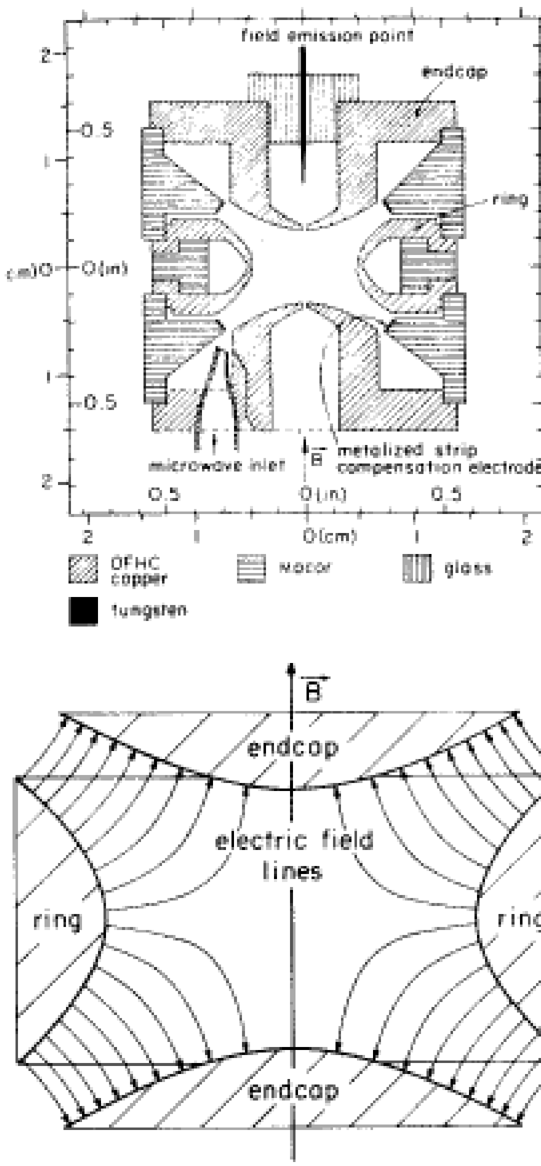


FIG. 2. Electric and magnetic field configurations of the Penning trap.

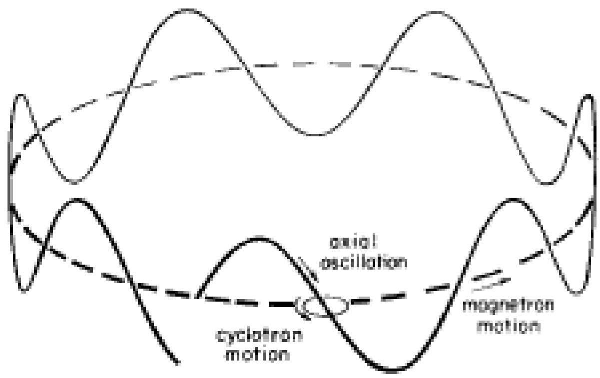


FIG. 3. Orbit of a charged particle in a Penning trap. The dashed line is the large and slow magnetron circle component of the motion. This, added to the axial oscillation, produces the guiding-center motion shown by the solid line. The total motion is given by adding the fast but small cyclotron circular motion about this moving guiding center. (Adapted from Ekstrom and Wineland, 1980.)

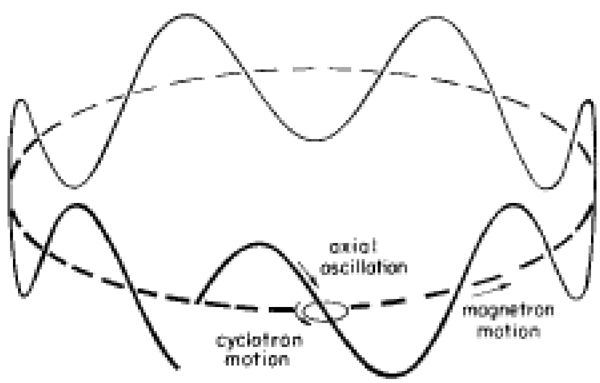


FIG. 3. Orbit of a charged particle in a Penning trap. The dashed line is the large and slow magnetron circle component of the motion. This, added to the axial oscillation, produces the guiding-center motion shown by the solid line. The total motion is given by adding the fast but small cyclotron circular motion about this moving guiding center. (Adapted from Ekstrom and Wineland, 1980.)

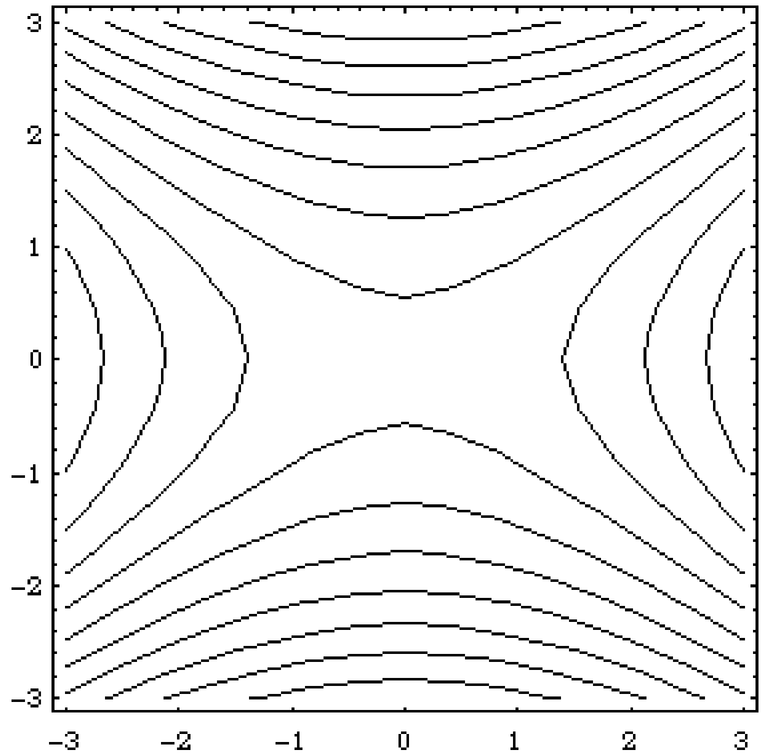
Classical picture

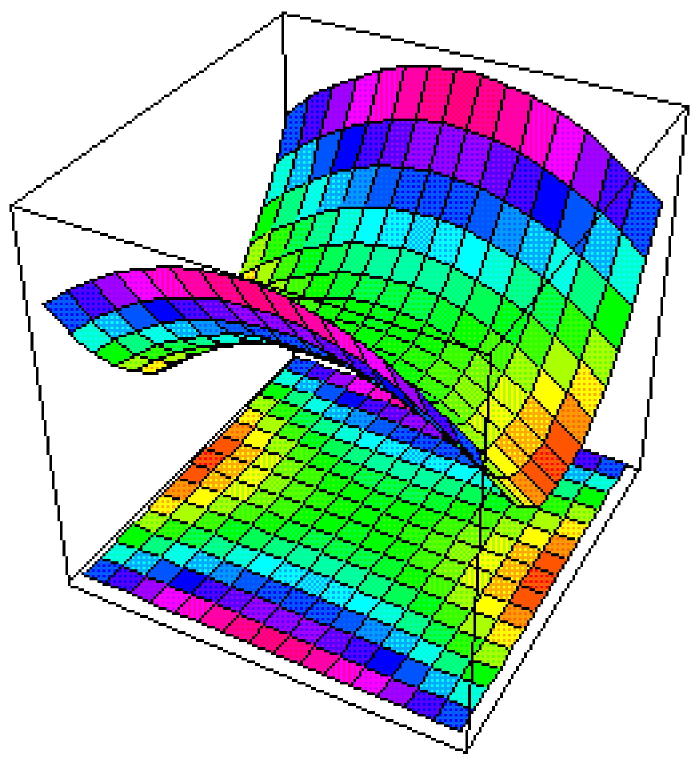
charge e, mass m. z axis parallel to **B**, positive so that cyclotron motion is right handed rule. i.e. z axis is indirection -eB. Cyclotron frequency is

$$\omega_c = \frac{|eB|}{mc}\hat{z} = \omega_c \hat{z}$$

e.g. 6T ω_c = 164 GHz, wavelength is 2 mm. Proton would oscillate at 89 MHz (radio). Charged particle is bound radially, but not axially. In an ideal Penning trap, superimpose a restoring force to any small perturbation by a quadruple field Write potential as

$$V = V_0 \frac{z^2 - \rho^2/2}{2d^2}$$





This function satisfies Laplace. Can be produced by placing electrodes along equipotential contours. Three are required. Two endcaps and a ring.

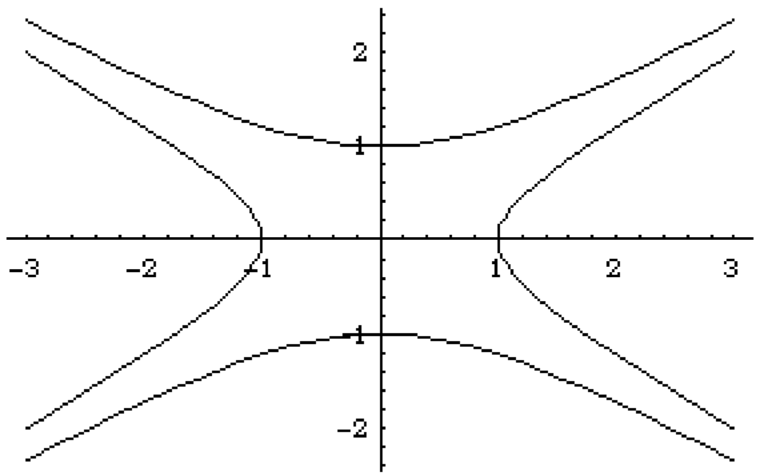
Endcaps: $z^2 = z_0^2 + \rho^2 / 2$

ring electrode: $z^2 = \frac{1}{2}\left(\rho^2 - \rho_0^2\right)$

Constants z_0 and r_0 are the minimum axial and radial distances to the electrodes. Choose characteristic trap dimension d as

$$d^2 = \frac{1}{2}\left(z_0^2 + \rho_0^2 / 2\right)$$

Then V_0 is the potential difference between endcap and electrodes



Axial motion

is de coupled from B field, and is a simple harmonic motion.

$$\ddot{z} + \omega_z z = 0; \quad \omega_z^2 = \frac{eV_0}{md^2}$$

Usual to have $\omega_z << \omega_c$. Typically might have V_0 = 10 V, and d = 0.3 cm, so that ν_z = 62 MHz. But can easily get 10 keV, so ν_z = 62 GHz.

Radial motion:

$$m\ddot{\rho} = e\left[\mathrm{E} + \frac{\rho}{c} \times \boldsymbol{B}\right]$$

$$\boldsymbol{E} = \frac{V_0}{2d^2}\rho$$

In terms of axial and cyclotron frequency

$$\ddot{\rho} - \omega_c \times \dot{\rho} - \frac{1}{2}\omega_z^2 \rho = 0$$

See repulsive last term from electrostatic potential. Find two consequences. First cyclotron frequency reduced, as repulsive radial potential reduces centrifugal force. Second, fast cyclotron orbit is superimposed upon a slower circular magnetron orbit, angular frequency ω_m. Resultant is as shown in figure b below, but now add axial harmonic motion.

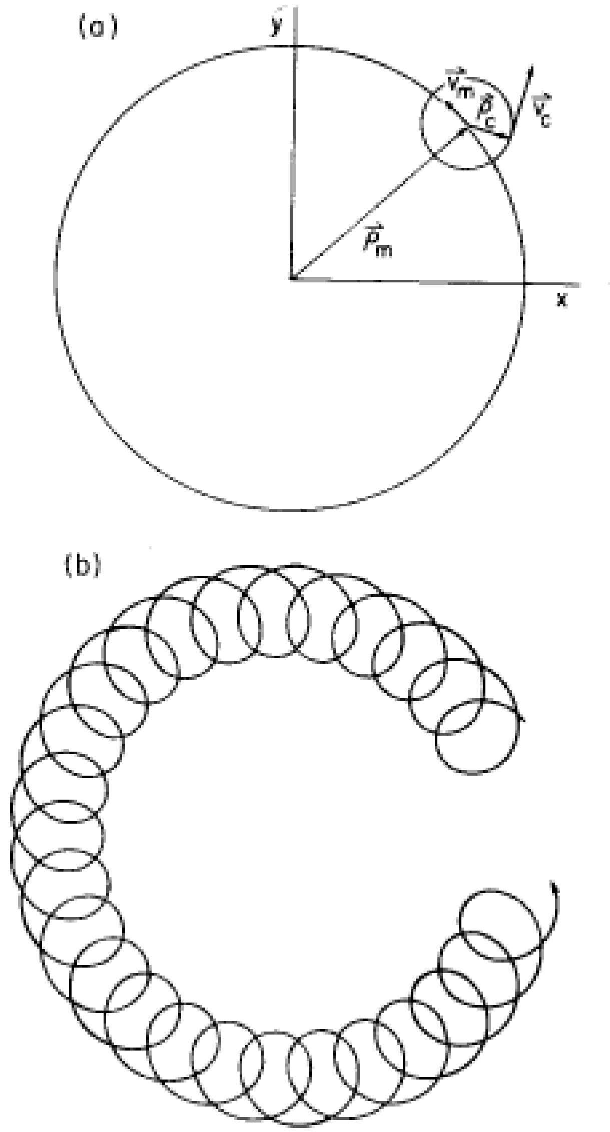


FIG. 7. Projection of the motion of a particle in a Penning trap upon the xy plane. The motion is the superposition of (a) circular magnetron and cyclotron motions producing (b) epicycles. The orbits are not to scale.

To understand magnetron motion, note that in perpendicular E and B fields, a charged particle with a drift velocity

$$\boldsymbol{u} = c\mathbf{E} \times \boldsymbol{B} / B^2$$

will move unimpeded, because in the basic radial motion equation **u**/cx**B** will cancel **E** (a velocity filter). Strictly true only for constant fields, i.e. a constant drift velocity **u.** But approximately true anyway. The filter velocity does not depend on m or e. Subst for E

$\boldsymbol{E} = V_0\rho/\left(2d^2\right)$ to get $\omega_m = u/\rho = \boldsymbol{E}/B/\rho = V_0/\left(2Bd^2\right) = \omega_z^2/\omega_c$ shows that the drift or magnetron motion is a circular motion with the same sense of rotation as the cyclotron orbit, independent of e and m, but $\omega_m << \omega_z << \omega_c$.

Note: cyclotron motion is almost exclusively kinetic. Axial motion alternates between kinetic and potential. Reducing energy in either reduces amplitude: they are stable. magnetron motion is almost exclusively potential. i.e. it is an orbit about the top of a potential hill. exciting the magnetron motion makes the particle roll down the hill.. The motion is unbounded: any dissipative process which removes energy from the magnetron motion increases the magnetron radius until the particle hits the ring and is lost But typical damping time is of order years!

Radiation damping

Accelerated charge radiates em waves. Therefore motion is damped. Transition probability for such an electric dipole is proportional to high power of transition frequency; appreciable radiative decay occurs only for high transition frequencies/ For protons or heavier the frequencies of motion are in radio frequency range, and radiative decay is ignorable. Also true for axial an magnetron motion of electrons, but not cyclotron motion.

Energy in a cyclotron orbit is decreased by power radiated, according to

$$-\frac{d}{dt}E = \frac{2e^2}{3c^3}\ddot{\boldsymbol{r}}^2$$

$$\ddot{\boldsymbol{r}} = \boldsymbol{w}_c \times \dot{\boldsymbol{r}}$$

$$E = \frac{1}{2}m\dot{\boldsymbol{r}}^2$$

$$\frac{dE}{dt} = -\gamma_c E$$

$$\gamma_c = \frac{4e^2\omega_c^2}{3mc^3}$$

$$E(t) = E_0 e^{-\gamma_c t}$$

Convenient to write in terms of classical radius r_0 of charged particle:

$$r_0 = \frac{e^2}{mc^2}$$

$$\gamma_c = \left[\frac{4r_0\omega_c}{3c}\right]\omega_c$$

for electron $r_0 = 2.8\times10^{-13}$ cm. So if $\omega_c/(2\pi) = 160$ GHz, $\gamma_c = 80$ ms. Note $1/m^3$ dependence.

Number density

Canonical Angular momentum.

Look at old paper by Brillouin (Phys Rev 1945) Define momentum p by a standard Lagrangian. For a charge e, e/m vector potential A, electrostatic scalar potential Φ, with independent variable x, v and t:

$$\boldsymbol{p}_k = m\dot{x}_k + e\boldsymbol{A}_k = \frac{\partial L}{\partial \dot{x}_k}$$

$$L(\dot{x}_1\dot{x}_2\dot{x}_3x_1x_2x_3t) = \frac{m}{2}\mathrm{v}^2 - e\Phi + e(\mathbf{v}\cdot\boldsymbol{A})$$

$$\mathbf{v}\cdot\boldsymbol{A} = \dot{x}_1A_1 + \dot{x}_2A_2 + \dot{x}_3A_3$$

Note L(**v**,**x**,t). Note A_1, A_2, A_3 are components in a rectangular coord system x_1 x_2 x_3. The E and B components are given by

$$E_k = -\frac{\partial\Phi}{\partial x_k} - \frac{\partial A_k}{\partial t} \quad k = 1,2,3$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} \quad B_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$$

Then Lagrange's eqn of motion is

$$\frac{dp}{dt} = \dot{p} = \frac{\partial L}{\partial x}$$

$$\frac{dp_k}{dt} = \dot{p}_k = \frac{\partial L}{\partial x_k} = -e\frac{\partial\Phi}{\partial x_k} + e\left(\dot{x}_1\frac{\partial A_1}{\partial x_k} + \dot{x}_2\frac{\partial A_2}{\partial x_k} + \dot{x}_3\frac{\partial A_3}{\partial x_k}\right)$$

Remember that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \left(\dot{x}_1\frac{\partial}{\partial x_1} + \dot{x}_2\frac{\partial}{\partial x_2} + \dot{x}_3\frac{\partial}{\partial x_3}\right)$$

So that

$$m\ddot{x}_k + e\frac{\partial A_k}{\partial t} + e\left(\dot{x}_1\frac{\partial A_k}{\partial x_1} + \dot{x}_2\frac{\partial A_k}{\partial x_2} + \dot{x}_3\frac{\partial A_k}{\partial x_3}\right) = -e\frac{\partial\Phi}{\partial x_k} + e\left(\dot{x}_1\frac{\partial A_1}{\partial x_k} + \dot{x}_2\frac{\partial A_2}{\partial x_k} + \dot{x}_3\frac{\partial A_3}{\partial x_k}\right)$$

Take k = 1, note first terms in each brackets cancels, and we get

$$m\ddot{x}_1 = -e\frac{\partial A_1}{\partial t} - e\frac{\partial \Phi}{\partial x_1} + e\dot{x}_2\left(\frac{\partial A_2}{\partial x_2} - \frac{\partial A_1}{\partial x_2}\right) + e\dot{x}_3\left(\frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3}\right)$$

which can be arranged as

$$m\ddot{x}_k = eE_k + e(\mathbf{v}\times\boldsymbol{B})$$

i.e. the equation of motion.

Now the Lagrangian appears in the principle of least action. We can now build up the Hamiltonian

$$H(p_1 p_2 p_3 x_1 x_2 x_3 t) = \sum_k p_k \dot{x}_k - L$$

$$H = \frac{1}{2}m\mathrm{v}^2 + e\Phi = \frac{1}{2m}\sum_k (p_k - eA_k)^2 + e\Phi$$

and the eqns of motion are

$$\dot{x}_k = \frac{\partial H}{\partial p_k} = \frac{1}{m}(p_k - eA_k)$$

$$\dot{p}_k = -\frac{\partial H}{\partial x_k}$$

For a single electron, in a z field, conservation of canonical momentum leads to

$$p_\theta = rmv_\theta + reA_\theta = rmv_\theta + re\frac{B_r r}{2}$$

i.e.

$$p_\theta = m(r\boldsymbol{v}_\theta - r^2\omega_c/2)$$

where $\omega_c = eB/m$, v_θ is azimuthal velocity, consisting of ExB and thermal (cyclotron) motions. If there are no external torques, **and the electrons are introduced without any canonical angular momentum,** we have

$$\dot{\theta} = \omega_c/2$$

Note: For $r\omega_c >> |v_\theta|$, then

$$p_\theta = -(m\omega_c/2) r_g^2$$

with r_g the radius of the guiding center. Note that in absence of external torques

$$P = \sum p_\theta = -(m\omega_c/2)\sum r_g^2 = const$$

i.e. if some electrons increase their radius, other must decrease theirs. Since initial average radius is less than the wall radius, , only small losses can occur. Now radial component of force balance in equilibrium is

$$eE_r + eBr\dot{\theta} + mr\dot{\theta}^2 = 0$$

(**E**+**v**x**B** balances centrifugal). Substitute $\dot{\theta} = \omega_c/2 = \omega_H$ and use conservation of energy (actually H) and let initial velocity be very small (0). Direction 3 is along B field

$$U = -\frac{2e}{m}\Phi = -\frac{2e}{m}\left(\frac{m}{2e}\mathrm{v}^2\right) = \dot{r}^2 + r^2\dot{\theta}^2 + \mathrm{v}_3^2$$

then

$$E_r = -\frac{\partial \Phi}{\partial r} = \frac{m}{2e}\frac{\partial U}{\partial r} = \frac{m}{e}\left(r\omega_H^2 + \mathrm{v}_3\frac{\partial \mathrm{v}_3}{\partial r}\right)$$

but need $\frac{\partial \mathrm{v}_3}{\partial r} = 0$, so Poisson's equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(eE_r) = \frac{\rho}{\varepsilon_0} = 2\frac{m}{e}\omega_H^2$$

Now note that ρ = ne, and get

$$n = \frac{\varepsilon_0 B^2}{2m} = \frac{B^2}{2\mu_0 mc^2}$$