Simple harmonic motion (Section 15.1)
1. SHM: From initial $x$ and $v$ to $A$ and $\phi$
2. SHM: Projection of uniform circular motion
3. Mass–spring: The initial phase angle

Kinetic energy and potential energy in SHM (Section 15.3)
5. Total energy of a simple harmonic motion

Simple pendulum and physical pendulum (Section 15.4)
6. Physical pendulum
7. Simple harmonic oscillation of a loop
8. Torsional pendulum
Consider a simple harmonic motion (SHM) along the x axis centered about the origin. The displacement x and velocity v are given by the following (see Eq. (15.4) with $\delta$ replaced by $\phi$):

$$x = A \cos (\omega t + \phi) \quad \text{and} \quad v = \frac{dx}{dt} = -A\omega \sin(\omega t + \phi).$$

At $t = 0$,

$$x = x_0 \quad \text{and} \quad v = v_0.$$

Determine the amplitude $A$:

<table>
<thead>
<tr>
<th>A</th>
<th>$x_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>$\omega$</td>
</tr>
<tr>
<td>C</td>
<td>$\sqrt{x_0^2 + \left(\frac{\omega}{v_0}\right)^2}$</td>
</tr>
</tbody>
</table>

**Hint:** At $t = 0$ there are two equations:

$$x_0 = A \cos \phi. \quad (1)$$

$$v_0 = -\omega A \sin \phi. \quad (2)$$

**Extra:** Express $\phi$ in terms of $x_0$ and $v_0$.

**Explanation:** This is the situation of two equations (1) and (2) with two unknowns: $A$ and $\phi$. Using $\cos^2 \phi + \sin^2 \phi = 1$, we may eliminate $\phi$. More specifically, $(A \cos \phi)^2 + (A \sin \phi)^2 = x_0^2 + (v_0/\omega)^2 = A^2$. Solving for $A$ leads to Answer = C.

**Explanation—extra:** Taking the ratio of Eq. (2) and (1) leads to the elimination of $A$: $\tan \phi = \sin \phi/\cos \phi = -(v_0/\omega)/x_0 = -v_0/(\omega x_0)$.  

1. SHM: From Initial $x$ and $v$ to $A$ and $\phi$  
   PhysiQuiz  
   127
Consider a simple harmonic motion (SHM) $x = A \cos \theta$ as a projection of a uniform circular motion with $\theta = \omega t + \phi$. If at $t = 0$, $x = 0$ and $v = v_0 > 0$, determine $\phi$:

<table>
<thead>
<tr>
<th>Point on the circle</th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>90°</td>
</tr>
<tr>
<td>C</td>
<td>180°</td>
</tr>
<tr>
<td>D</td>
<td>270°</td>
</tr>
</tbody>
</table>

**Hint:** Consider both of the initial conditions $x = 0$ and $v_0 > 0$.

**Explanation:** Because $x = 0$ when $t = 0$, we may choose either $B$ or $D$. Notice that at $B$ the velocity is along the negative $x$ direction. But at $D$ the velocity is along the positive $x$ direction. So $D$ is the correct choice—that is, at $t = 0$, $\theta = \phi = 270^\circ$. Answer = D.
Consider a mass–spring system in which the mass oscillates according to simple harmonic motion: \( x = A \cos(\omega t + \phi) \). At \( t = 0 \) the mass is at the equilibrium position moving to the left. Determine the phase angle \( \phi \):

<table>
<thead>
<tr>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A 0</td>
</tr>
<tr>
<td>B ( \pi/2 )</td>
</tr>
<tr>
<td>C ( \pi )</td>
</tr>
<tr>
<td>D ( 3\pi/2 )</td>
</tr>
</tbody>
</table>

**Hint:** \( v = dx/dt = -\omega A \sin(\omega t + \phi) \).

**Extra:** Find an expression for the amplitude \( A \) of the periodic motion in terms of the maximum speed \( v_0 \) and the angular velocity \( \omega \).

---

**Explanation:** From the given information at \( t = 0 \), \( x = 0 = A \cos \phi \), and \( v = -|v_0| = -\omega A \sin \phi \). Because \( \cos \phi = 0 \), we have \( \phi = \pi/2 \) or \( 3\pi/2 \). From the velocity equation, \( |v_0| = \omega A \sin \phi \). This implies that \( \sin \phi \) must be positive. In other words, \( \phi = \pi/2 \) is the correct choice. Answer = B.

**Explanation—extra:** The velocity equation gives \( A = |v_0|/(\omega \sin \phi) = |v_0|/\omega \).
Consider the mass–spring system shown here at two different time $t_1$ and $t_2$, where the mass oscillates according to simple harmonic motion: $x = A \cos(\omega t + \phi)$. The motion is modeled as the projection of the circular motion shown at the bottom. Locate the point on the circle that corresponds to the time $t_1$:

$$\begin{array}{c|c}
\text{At } t_1 \\
A & A_1 \\
B & A_2 \\
\end{array}$$

**Hint:** Consider both the position and the velocity.

**Extra:** Which point on the circle corresponds to moment $B$?

**Explanation:** At the time $t_1$, as far the location of the mass is concerned, both $A_1$ and $A_2$ are possible. Because the velocity at moment $A$ is positive, it limits the point $A_2$ to be the correct choice. Answer = B.

**Explanation—extra:** At the time $t_2$, the velocity of the mass is negative. By inspection, $B_1$ is the correct choice.
Consider a mass–spring system in which the oscillation is described by $x = A \cos \omega t$. The kinetic energy is $K = m(\frac{dx}{dt})^2/2$. The potential energy is $U = kx^2/2$. The maxima are $K_{\text{max}} = m(\omega A)^2/2$ and $U_{\text{max}} = kA^2/2$. Which choice here gives the total energy of oscillation?

<table>
<thead>
<tr>
<th>Choice</th>
<th>Total Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$E = K_{\text{max}} = U_{\text{max}} = m(\omega A)^2/2$</td>
</tr>
<tr>
<td>B</td>
<td>$E = K_{\text{max}} + U_{\text{max}} = m(\omega A)^2$</td>
</tr>
<tr>
<td>C</td>
<td>$E = K_{\text{max}} + U_{\text{max}} = kA^2$</td>
</tr>
</tbody>
</table>

**Extra:** At which point during the oscillation do you expect the mass–spring system to have the most energy?

- When the spring is relaxed (at $x = 0$).
- When the spring is fully stretched (at $|x| = x_{\text{max}}$).

**Explanation:** The total energy $E = K + U$ of the mass–spring system is a conserved quantity. $E$ remains at the same value throughout the oscillations. When the mass passes the point $x = 0$, its potential energy is 0 and its kinetic energy is at its maximum. At the maximum stretch, its potential energy is at its maximum and its kinetic energy is 0. Answer = A.

**Explanation—extra:** Because $E$ stays the same throughout the oscillations, the total energy of the mass–spring system at $x = 0$ is the same as the total energy at $|x| = x_{\text{max}}$. 

5. Total Energy of a Simple Harmonic Motion PhysiQuiz 131
Consider a physical pendulum of mass $m$ where $P$ is the pivot point and $b$ is the distance between $P$ and the center of gravity. In a small $\theta$ approximation, $b_{\perp} = b \sin \theta \approx b\theta$, and the equation of motion is given by

$$\tau = I\alpha = I\ddot{s}/dt^2 = -mgb \sin \theta \approx -(mgb)\dot{\theta} \quad (1)$$

Determine the period $T$ of this physical pendulum:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$\sqrt{\frac{l}{\kappa}}$</td>
<td>$2\pi \sqrt{\frac{l}{mgb}}$</td>
<td>$\sqrt{\frac{\kappa}{l}}$</td>
<td>$2\pi \sqrt{\frac{mgb}{l}}$</td>
</tr>
</tbody>
</table>

**Hint:** For simple harmonic motion,

$$d^2s/dt^2 = -\omega^2 s \quad (2)$$

with $\omega = 2\pi/T$.

**Extra:** Find $T$ of a simple pendulum that has a mass $m$ and a length $L$.

**Explanation:** Comparing the equations of motion (1) and (2) with $\theta = s$ implies that $\omega^2 = mgb/l$. This leads to

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{mgb}}. \text{ Answer = B.}$$

**Explanation—extra:** For a simple pendulum, $l = ml^2$ and $b = L$. So

$$T = 2\pi \sqrt{\frac{l}{g}}.$$
The period of a physical pendulum is $T = 2\pi \sqrt{\frac{l}{mb}}$, where $m$ is the mass, $l$ is the moment of inertia about the pivot point, and $b$ is the distance between the pivot point and the center of gravity. Consider a circular loop oriented vertically where the pivot point $P$ is at the top of the loop (see the sketch). Find $b$ and $l$ for a loop with radius $r$ and mass $m$:

$$
\begin{array}{cccc}
A & B & C & D \\
\hline
b & r & r & 2r \\
l & mr^2 & 2mr^2 & mr^2 & 2mr^2
\end{array}
$$

**Hint:** Use the parallel axis theorem: $l = l_{cm} + MD^2$.

**Extra:** Find the period $T$ of the oscillation of the loop.

**Explanation:** $b$ is the distance between $P$ and the center, so $b = r$.

$l = l_{cm} + MD^2 = mr^2 + mr^2 = 2mr^2$. Answer = B.

**Explanation—extra:** $T = 2\pi \sqrt{\frac{l}{mb}} = 2\pi \sqrt{\frac{2mr^2}{mgr}} = 2\pi \sqrt{\frac{2r}{g}}$. 

---

7. Simple Harmonic Oscillation of a Loop  

PhysiQuiz 133
A circular disk is suspended by a wire attached at the top of some fixed support. When the disk is twisted through some small angle $\theta$, the twisted wire exerts a restoring torque on the body that satisfies $\tau = l\alpha = l\frac{d^2\theta}{dt^2} = -\kappa \theta$, where $\kappa$ is referred to as the torsion constant of the wire. Find the period of the oscillation:

$$T = 2\pi \sqrt{\frac{l}{\kappa}}$$

**Hint:** For simple harmonic motion, $\frac{d^2s}{dt^2} = -\omega^2 s$, with $\omega = 2\pi / T$.

**Explanation:** The present equation of motion implies that $\omega^2 = \kappa / l$, and in turn, $2\pi \sqrt{\frac{l}{\kappa}}$. Answer = B.