Periodic waves (Section 16.2)
1. Traveling waves: shape and direction
2. Traveling velocity and oscillatory velocity
3. Traveling speeds along two wires

Superposition of waves (Section 16.3)
4. Beats

Standing waves (Section 16.4)
5. Superposition of two opposing waves with same \( k \) and \( w \)
6. Standing waves: Locations of nodes and antinodes
7. Standing waves: Frequencies of harmonics
Consider a traveling wave: \( y = 3 \sin(\pi x + 3t) \). The two shapes (1) and (2) are indicated in the diagram here.

Choose the correct shape at \( t = 0 \), and traveling velocity:

<table>
<thead>
<tr>
<th>Shape at ( t = 0 )</th>
<th>Wave Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (1)</td>
<td>( \sim 1 )</td>
</tr>
<tr>
<td>B (1)</td>
<td>( \sim (-1) )</td>
</tr>
<tr>
<td>C (2)</td>
<td>( \sim 1 )</td>
</tr>
<tr>
<td>D (2)</td>
<td>( \sim (-1) )</td>
</tr>
</tbody>
</table>

**Extra:** What is the velocity amplitude of oscillations at any fixed value of \( x \)?

**Explanation:** At \( t = 0 \), \( y = 3 \sin(\pi x) \). As \( x \) varies from 0 to 2, the argument of the sine function varies from 0 to \( 2\pi \). So (1) is the correct shape.

Alternative explanation for calculation of traveling velocity: We can obtain the traveling velocity by choosing \( y = \) constant and taking the time derivative of 
\( y = 3 \sin (\pi x + 3t) \). It gives 
\( 0 = 3 \cos (\pi x + 3t) \) \( \frac{d}{dt} (\pi x + 3t) \), \( \frac{d}{dt} (\pi x + 3t) \), or 
\( \frac{dx}{dt} = \frac{-3}{\pi} \approx -1 \). Answer = B.

**Explanation—extra:** \( \frac{dy}{dt} = 9 \cos(\pi x + 3t) \). So the velocity amplitude at \( y \) oscillations for any fixed \( x \) value is 9.
A traveling simple harmonic wave train is described by
\[ y = A \sin(kx + \omega t) \]
where the wave number \( k = 2\pi/\lambda \), and the angular frequency \( \omega = 2\pi f \). The traveling wave velocity \( v_{\text{wave}} \) is given by which of the following?

\[
\begin{array}{cccc}
\text{A} & \text{B} & \text{C} & \text{D} \\
v_{\text{wave}} & \lambda f & -\lambda f & \lambda/f & -\lambda/f
\end{array}
\]

**Hint:** Traveling velocity may be determined in three steps:
1. Introduce a fixed-argument equation, which is given by “argument = constant.”
2. Take the time derivative of this equation.
3. Solve for \( dx/dt \) at argument, which is \( v_{\text{travel}} \).

**Extra:** Show that the maximum velocity of the oscillations at any fixed \( x \) value is given by \( v_{\text{max}} = \omega A \).

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**Explanation:** We follow the steps outlined in the hint.
From \( d(kx + \omega t)/dt = 0 \), we get \( dx/dt = -\omega/k = -(2\pi/T)/(2\pi/\lambda) = -\lambda/T = -\lambda f \). The wave is traveling in the negative \( x \) direction. Answer = B.

**Explanation—extra:** For fixed \( x \), the oscillatory velocity is \( v_{\text{oscillations}} = dy/dt = \omega A \cos(kx + \omega t) \equiv v_{\text{max}} \cos(kx + \omega t) \). So the amplitude of the velocity oscillations is given by \( v_{\text{max}} = \omega A \).
Consider waves traveling along two different wires made from the same material. Use the following symbols:

<table>
<thead>
<tr>
<th>Wire</th>
<th>Mass/Length</th>
<th>Tension</th>
<th>Diameter</th>
<th>Length</th>
<th>Speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>( \mu_1 )</td>
<td>( T_1 )</td>
<td>( d_1 )</td>
<td>( L_1 )</td>
<td>( v_1 )</td>
</tr>
<tr>
<td>#2</td>
<td>( \mu_2 )</td>
<td>( T_2 )</td>
<td>( d_2 )</td>
<td>( L_2 )</td>
<td>( v_2 )</td>
</tr>
</tbody>
</table>

Given \( d_2 = 2d_1 \) and \( L_2 = 2L_1 \), determine the correct value for the ratio \( \mu_1/\mu_2 \):

\[
\begin{array}{cccc}
\mu_1/\mu_2 & A & B & C & D \\
1/4 & 1/2 & 1 & 2 \\
\end{array}
\]

**Hint:** Mass/Length = (Density \( \times \) Volume)/Length.

**Extra:** If \( T_2 = 2T_1 \), show that \( v_2 = \frac{v_1}{\sqrt{2}} \). Use \( v = \sqrt{\frac{F}{\mu}} \) [Eq. 16.14], where \( F \) is the tension and \( \mu = M/L \).

**Explanation:** For a wire, Volume/Length = Cross section = \( \pi(d/2)^2 \). So for two identical materials with the same densities, the ratio is given by

\[
\frac{\mu_1}{\mu_2} = \left[ \frac{\left( \frac{d_1}{2} \right)^2}{\left( \frac{d_2}{2} \right)^2} \right] = \left( \frac{d_1}{d_2} \right)^2 = \frac{1}{4}.
\]

Answer = A.

**Explanation—extra:** \( v_2 = \sqrt{\frac{T_2}{\mu_2}} = \sqrt{\frac{2T_1}{4\mu_1}} = \frac{1}{\sqrt{2}}v_1 \).
Two waves with equal amplitude but slightly different frequencies are traveling in the same direction. At a given point their displacements are described by $y_1 = A_0 \cos \omega_1 t$ and $y_2 = A_0 \cos \omega_2 t$. Evaluate $y = y_1 + y_2$. Identify the factor responsible for the beats:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\sin[(\omega_1 - \omega_2)t/2]$</td>
</tr>
<tr>
<td>B</td>
<td>$\cos[(\omega_1 - \omega_2)t/2]$</td>
</tr>
<tr>
<td>C</td>
<td>$\sin[(\omega_1 + \omega_2)t/2]$</td>
</tr>
<tr>
<td>D</td>
<td>$\cos[(\omega_1 + \omega_2)t/2]$</td>
</tr>
</tbody>
</table>

**Hint:** $\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$.

**Extra:** Show that the beat frequency is $f_{\text{beat}} = |f_1 - f_2|$.

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**Explanation:** Using the trigonometric identity in the hint, $y = y_1 + y_2 = 2A_0 \cos \left( \frac{\omega_1 - \omega_2}{2} t \right) \cos \left( \frac{\omega_1 + \omega_2}{2} t \right)$. Because the difference between $\omega_1$ and $\omega_2$ is small, the first cosine factor corresponds to a low-frequency oscillation term. Its maxima give rise to beats. Answer = B.

**Explanation—extra:** Rewrite the first cosine term as $\cos(bt)$. Beats occur when $\cos(bt) = \pm 1$. Denote the time interval between two adjacent beats by $T_{\text{beat}}$ (the beat frequency is $f_{\text{beat}}$). Then the corresponding angular interval is $\pi$—that is, $bT_{\text{beat}} = \pi$. So $b = \pi/T_{\text{beat}} = \pi f_{\text{beat}} = |\omega_1 - \omega_2|/2$. This implies that $f_{\text{beat}} = |f_1 - f_2|$.
Consider two traveling waves:
- \( y_1 = A \sin(kx - \omega t) \)
- \( y_2 = A \sin(kx + \omega t) \)

Determine the wave velocity of the superposed pattern:

<table>
<thead>
<tr>
<th>Wave Velocity of Superposed Pattern</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A Positive</td>
<td></td>
</tr>
<tr>
<td>B Zero</td>
<td></td>
</tr>
<tr>
<td>C Negative</td>
<td></td>
</tr>
</tbody>
</table>

**Hint:** Using \( \sin \alpha + \sin \beta = 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right) \), the superposed pattern is given by \( y(x,t) = 2 \sin kx \cos \omega t \).

**Extra:** Determine the amplitude of oscillation at \( x = 0 \) and \( x = \lambda / 2 \), with the wave number \( k = 2\pi / \lambda \).

**Explanation:** From the superposed pattern \( y(x,t) \) given in the hint, the first factor depends on \( x \) only and the second depends on \( t \) only. Here the \( x \) locations of the peaks and the valleys of the wave pattern do not vary with time. In other words, the wave pattern does not “travel” in \( t \). So the traveling speed is zero. In other words, at a fixed \( x \) there is a fixed amplitude.

Answer = B.

**Explanation—extra:** We write \( y = B \cos \omega t \). From the hint, the amplitude is a function of \( x \)—that is, \( B = 2 \sin kx \). So at \( x = 0 \), \( B = 0 \). At \( x = \lambda / 4 \), \( kx = \pi / 2 \), or \( B = 2 \sin \pi / 2 = 2 \).
Consider the superposition of two traveling waves:

- \( y_1 = A_0 \sin(kx - \omega t) \)
- \( y_2 = A_0 \sin(kx + \omega t) \)

Find the values of \( kx \) for which the amplitude vanishes:

<table>
<thead>
<tr>
<th>( kx )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
</tr>
<tr>
<td>B</td>
</tr>
<tr>
<td>C</td>
</tr>
<tr>
<td>D</td>
</tr>
</tbody>
</table>

**Hint:** \( \sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \).

**Extra:** Determine the \( kx \) values at which the amplitude is \( 2A_0 \).

**Explanation:** Using the identity given in the hint,
\( y = y_1 + y_2 = 2A_0 \sin kx \cos \omega t \). The zeros of the amplitude function: \( 2A_0 \sin kx \), occur at \( kx = 0, \pi, 2\pi, \) and so on. So Answer = C.

**Explanation—extra:** For the amplitude to be \( 2A_0 \), where \( \sin kx = \pm 1 \), or \( kx = \pi/2, 3\pi/2, 5\pi/2, \ldots \).
Consider a string of length $L$ fixed at both ends. Its fundamental frequency gives the middle C note with frequency $f_c = 264$ Hz.

Now consider a second string of the same length, again fixed at both ends. Its mass per unit length is half that of the first string, and its tension is twice as large.

The frequency $f'_2$ of the second harmonic of the second string is given by which of the following?

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'_2$</td>
<td>$f_c$</td>
<td>$2f_c$</td>
<td>$3f_c$</td>
</tr>
</tbody>
</table>

**Hint:** For normal modes of vibration,

- First harmonics ($n = 1$): The fundamental or lowest frequency is $f_1 = \frac{v}{\lambda_1}$.
- Overtones ($n \geq 2$): For the $n$th harmonics, $f_n = nf_1$.
- $v = \sqrt{\frac{F}{\mu}}$ for all harmonics.

**Explanation:** $f'_2 = 2f'_1$. $f'_1 = \frac{v'}{\lambda'} = \frac{1}{\lambda'} \sqrt{\frac{F'}{\mu'}}$. From the givens, both strings are fixed at both ends and have the same strength $L$. The fundamental mode corresponds to half of a wavelength, or $\lambda_1' = 2L$. Also from the givens, $F' = 2F$ and $\mu' = 0.5\mu$. This leads to $f'_1 = \frac{1}{2L} \sqrt{\frac{2F}{0.5\mu}} = 2 \times \frac{1}{2L} \sqrt{\frac{F}{\mu}} = 2f_c$. $f'_2 = 2f'_1 = 4f_c$. Answer = D.