A linearly polarized traveling Gaussian pulse

We follow the method described in Appendix B of Bernhard Rau's Dissertation. Our E_x expression agrees with Rau's. Our B_x expression has corrected some minor typos present in Rau's.

The pulse: The pulse travels in the z direction, and it is liearly polarized in x, i.e. $E_y = 0$. At any fixed time, the pulse satisfies

$$(\nabla^2 + k^2)E_x(r,t) = 0. \tag{1}$$

The wave packet is made out of the superposition of traveling harmonic plane waves with frequency $\omega = ck$, i.e.

$$E_x(r,t) = \int d^3k F(k;\sigma,w) exp[i(kr - \omega t)]. \tag{2}$$

Here F-function is the form factor, which ensures that the pulse peaks at $k^2 = k_0^2$, or $k = \pm k_0$, and has a logitudinal gaussian width σ and the spot size w. More explicitly

$$F(k;\sigma,w) = N f(k,\sigma) g(k_T, w), \tag{3}$$

with

$$f(k,\sigma) = exp\left[-\frac{(k-k_0)^2\sigma^2}{2}\right] + exp\left[-\frac{(k+k_0)^2\sigma^2}{2}\right],$$

and

$$g(k_T, w) = exp\left[-\frac{k_T^2 w^2}{4}\right]$$

where N is the overall normalization factor. In eq.(2), we write the traverse itegration elements in polar coordinate, i.e.

$$d^3k = dk k_T dk_T d\phi$$

In the configuration space, denote $\rho^2 = x^2 + y^2$. Integrating over ϕ one gets

$$E_x = N \int_{-\infty}^{\infty} dk \int_0^k k_T dk_T f(k) g(k_T) e^{i(k_z z - ckt)} J_0(k_T \rho). \tag{4}$$

We will show due to the constraint of Maxwell's equation, the loggitudinal component of the B field takes the form

$$B_z = N\int_{-\infty}^{\infty} dk \int_0^k k_T dk_T f(k) g(k_T) e^{i(k_z z - ckt)} \frac{y}{\rho} \cdot \frac{ik_T}{k} J_1(k_T\rho). \tag{5}$$

Consistency check: Begin from the Maxwell equation in free space

$$\frac{1}{c}\frac{\partial}{\partial t}B = \nabla \times E.$$

For the present case, the longitudinal component of the equation

$$\frac{1}{c}\frac{\partial}{\partial t}B_z = \partial_x E_y - \partial_y E_x = -\partial_y E_x. \tag{6}$$

We proceed to show that the LHS of eq.(6) based eq.(5) agrees with the RHS of eq.(6), based on eq.(4). The left hand side,

$$\begin{aligned} LHS &= \int\int \cdots \frac{1}{c}\frac{\partial}{\partial t}\left[e^{i(k_z z - ckt)}\right] \frac{y}{\rho} \cdot \frac{ik_T}{k} J_1(k_T\rho) \\ &= \int\int \cdots \left[e^{i(k_z z - ckt)}\right] \cdot \frac{yk_T}{\rho} \cdot J_1(k_T\rho). \end{aligned} \tag{7}$$

The right hand side,

$$\begin{aligned} RHS &= \int\int \cdots \left[e^{i(k_z z - ckt)}\right] \cdot \left[-\partial_y J_0(k_T\rho)\right] \\ &= \int\int \cdots \left[e^{i(k_z z - ckt)}\right] \cdot \frac{yk_T}{\rho} \cdot J_1(k_T\rho) = LHS. \end{aligned} \tag{8}$$

In the second step, the Bessel function identity: $J_0'(\zeta) = -J_1(\zeta)$ was used. We proceed with the evaluation of eq.(4). Since the integrand peaks at $k^2 = k_0^2$, the transverse part has an approximate weight factor

$$g(k_T) \approx exp\left[-\frac{1}{4}k_0^2\left(1 - \frac{k_z^2}{k_0^2}\right)w^2\right].$$

The dominant region of integration is in the region,

$$\begin{aligned} \frac{1}{4}k_T^2 w^2 \sim k_0^2\left(1 - \frac{k_z^2}{k_0^2}\right)\frac{w^2}{4} \leq 1 \\ \text{or } \left(1 - \frac{k_z^2}{k_0^2}\right) \leq \left(\frac{2}{k_0 w}\right)^2 = \left(\frac{\lambda}{\pi w}\right)^2. \end{aligned} \tag{9}$$

This allows the extension of k_T integration to ∞. Furthermore we assume that the denominator πw is sufficiently large compared to λ, so the last quantity in eq(9) is small and the plane wave phase factor in the tegrand may be approximated by

$$(k_z z - ckt) \approx k(z - ct). \tag{10}$$

This approixmation together with extending k_T integral to ∞ gives a factorizable form. At $t = 0$,

$$E_x \approx N F_1(z, \sigma) G_1(\rho, w), \tag{11}$$

where

$$F_1(z, \sigma) = \int_{-\infty}^{\infty} dk f(k, \sigma) exp[ik(z - ct)], \tag{12}$$

and

$$G_1(\rho, w) = \int_0^{\infty} k_T dk_T g(k_T, w) J_0(k_T \rho). \tag{13}$$

Similarly for the longitudinal component of B,

$$B_z \approx N F_2(z, \sigma) G_2(\rho, w) \tag{14}$$

where

$$F_2(z, \sigma) = \int_{-\infty}^{\infty} \frac{dk}{k} f(k, \sigma) exp[ik(z - ct)], \tag{15}$$

$$G_2(\rho, w) = \int_0^{\infty} k_T dk_T g(k_T, w) J_1(k_T \rho). \tag{16}$$

Substituting $f(k, \sigma)$ given in eq.(3) into the integrand of eq.(12) gives

$$F_1(z, \sigma) = 4 \int_0^{\infty} dk exp\left(-\frac{(k\sigma)^2}{2}\right) coskz \cdot cosk_0 z. \tag{17}$$

Using the math-identity G3.896.4 (here after math identities beginning with a letter G are given in the Appendix),

$$F_1 = \frac{2\sqrt{2\pi}}{\sigma} cosk_0 z exp\left(-\frac{z^2}{2\sigma^2}\right). \tag{18}$$

Using the math identity G6.631, eqs. (13) and (16) lead to respectively

$$G_1 = \frac{2}{w^2} {}_1F_1(1, 1, -(\rho/w)^2), \text{ and} \tag{19}$$

$$G_2 = \frac{4\rho}{w^4} {}_1F_1(2,2,-(\rho/w)^2). \tag{20}$$

Lastly we evaluate the F_2 fuction. From eq.(15) and the fact that $f(k,\sigma)$ is an even function in k,

$$F_2(z,\sigma) = i\int_{-\infty}^{\infty}\frac{dk}{k}f(k,\sigma)sinkz$$

$$= 2iexp\left(-\frac{(k_0\sigma)^2}{2}\right)\int_0^{\infty}\frac{dk}{k}sinkz\left[exp\left(-k_0k\sigma^2 - \frac{(k\sigma)^2}{2}\right) + (k_0 \to -k_0)\right]. \tag{21}$$

As shown in the Appendix, this eventually leads to the asymptotic expression:

$$F_2(z,\sigma) \approx 2\sqrt{\pi}i\frac{exp\left[-\frac{1}{2}\left[\frac{z}{\sigma}\right)^2\right]}{\left[\frac{z^2}{2\sigma^2} + \frac{(k_0\sigma)^2}{2}\right]}\left[\frac{z}{\sqrt{2}\sigma}\cdot cosk_0z + \frac{k_0\sigma}{\sqrt{2}sink_0z}\right]. \tag{22}$$