Coulomb Scattering in a Strong Magnetic Field

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Abstract

The presence of a strong magnetic field intrinsically changes the orbit of electrons in their Coulomb interactions with ions for a range of parameters. The characteristic scale for the orbit modification is \( r_0 = (Ze^2/4\pi\epsilon_0 B^2)^{1/3} \) where \( Ze \) is the ion charge, \( m_e \) the electron mass and \( B \) the magnetic field strength. The scale length is comparable to the interparticle spacing when \( n_e^{1/3} r_0 = (\omega_{pe}/\omega_{ce})^{2/3}(Z/4\pi)^{1/3} \sim 1 \) where \( \omega_{pe}/\omega_{ce} \) are the electron plasma and cyclotron frequencies, respectively. For large angle scattering events we show complex, chaotic scattering interaction events for low-energy electrons. The scattering angle has a fractal dependence on the impact parameter in the chaotic scattering intervals. The process is thought to be important in strongly magnetized, low-temperature plasmas, but the overall macroscopic effects remain to be determined. Test particle simulations are presented that show probability distributions are required to describe the outgoing states of the events with fixed impact parameter and energy which specify a unique Rutherford scattering angle.
I. INTRODUCTION

The particle orbits in the attractive Coulomb interaction between an electron and an ion are isomorphic with the Kepler orbits between a planet and the central star. While the problem of the onset of irregular and chaotic motions due to perturbations in the Kepler problem was the stimulus for the research leading to the development of the Kolmogorov-Arnold-Moser \([1,2]\) stability theory and other tools of nonlinear dynamics, the corresponding problem for the electron-ion orbits has received relatively little attention. Here we show that the presence of a strong magnetic field introduces chaos into the electron-ion orbits and produces a chaotic scattering system. From the beginning the potential for chaos is clear since we know that the Kepler orbit problem contains a homoclinic parabolic orbit that separates the bounded elliptic orbits from the hyperbolic unbounded orbits. Adding to this system the additional degree of freedom of the cyclotron orbital motion clearly produces a chaotic Hamiltonian system with 2-degrees of freedom.

The symmetries and associated invariants are as follows. After removing the center of mass motion the orbits of the effective-mass particle \(\mu \equiv m_e m_i/(m_i + m_e)\) the unmagnetized system has rotational symmetry with all three components \(L\) of the angular momentum vector constant. The total energy \(E = T + V\) is conserved so the phase space is \(6 - 4 = 2\) dimensional. The \((r, p_r)\) relative motion has the separatrix at \(E = 0\) and the characteristic Kepler orbital frequency \(\Omega_K = (Ze^2/4\pi\epsilon_0 r^3)^{1/2}\). The hyperbolic orbits \(E > 0\) have straight-line motion \((x = vt)\) before and after the interaction that differ in direction by the angle \(\theta\). The Rutherford cross-section \(d\sigma_R = (Ze^2/8\pi\epsilon_0 \mu v^2)^2 \sin^{-4}(\theta/2) d\Omega\) defines the ring of area \(d\sigma_R = 2\pi bd\) within which the incident electron is scattered into a symmetric cone of angle \(\theta\) with solid angle \(d\Omega = 2\pi \sin \theta d\theta\) in the asymmetric state after the collision. Thus, \(d\sigma_R\) and \(d\Omega\) are measures in the phase space of the initial and final states for scattering events characterized by fixed energy \(E = \frac{1}{2} m_e v^2\) and scattering angle \(\theta\) between the incident and final velocity (and momentum) vectors.

The presence of the magnetic field \(B\) breaks the rotational symmetry leaving the system
with one canonical angular momentum components $P_\phi$ as a constant of the motion. The canonical momentum $P_\phi$ reduces to $L_z$ when $B \to 0$. The invariants $E, P_\phi$ reduce the phase space of the relative motion to $6 - 2 = 4$ dimensions. The second characteristic frequency (we note that the Kepler frequency is not a fixed frequency, but the characteristic frequency of trapped particle orbits, which depends on the length scale of the particle orbits in the trapping direction $z$) is the electron cyclotron frequency $\omega_{ce} = eB/m_e$ and strong chaos is expected when the Kepler frequency $\Omega_K(r) = eB/m_e$ equals the cyclotron frequency. This relation defines the scale length $r_0$ in the combined system which makes the Hamiltonian and equations of motion free of the parameters $Ze^2/m_e v^2$ and $eB/m_e$. Different regimes of motion are characterized by different initial conditions only.

In Sec. II, we define the dimensionless dynamical equations and the two invariant of the motion. Then we describe the typical chaotic scattering orbits and show their extreme sensitivity to the initial conditions. In Sec. III we discuss the plasma regime where the chaotic orbits occur. We briefly describe conditions where a laboratory experiment may look for these orbits or effects associated with the orbits. In Sec. IV we summarize the results and suggest directions for future numerical and laboratory experiments.

II. DYNAMICAL EQUATIONS AND SCALING

Time scales for the problem are the cyclotron period from $B$ and the Kepler period from Coulomb interaction. Thus, we define dimensionless variables

$$\omega_{ce} t \to t \quad \frac{r}{r_0} \to r \quad \frac{v}{\omega_{ce} r_0} \to v$$

where

$$\frac{Ze^2}{4\pi\varepsilon_0 r_0^2} = m_e \omega_{ce}^2 r_0 \quad \omega_{ce} = \frac{eB}{m_e} \quad (1)$$

The orbital equation for the effective mass, which we approximate as the electron mass, transforms from

$$m_e \frac{d^2\mathbf{r}}{dt^2} = eB \mathbf{r} \times \mathbf{\dot{z}} - \frac{Ze^2}{4\pi\varepsilon_0} \frac{\mathbf{r}}{r^3} \quad (2)$$
\[ \ddot{r} = \dot{r} \times \hat{z} - \frac{\dot{\hat{r}}}{r^2}, \]  

(3)

In the Euclidean components used in the numerical integrations the system is

\[ \ddot{x} = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}} - \dot{y}, \] 

\[ \ddot{y} = -\frac{y}{(x^2 + y^2 + z^2)^{3/2}} + \dot{x}, \] 

\[ \ddot{z} = -\frac{z}{(x^2 + y^2 + z^2)^{3/2}}, \]  

(4)

with constants of motion

\[ E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{(x^2 + y^2 + z^2)^{1/2}} \] 

\[ P_\phi = x\dot{y} - y\dot{x} - \frac{1}{2}(x^2 + y^2). \]  

(5)

The degree of constancy of \( E, P_\phi \) in the numerical solutions is used to monitor the errors in the integration rather than to further reduce the system of equations to four ordinary differential equations (odes). We have used the initial value Runge-Kutta adaptive integrator, IVPRK, on the IBM SP parallel machine with 71 nodes and 256NB/nodes at ACCES. The integrator compares the new \((\mathbf{x}, \dot{\mathbf{x}})\) vector from the 5th and 6th order Runge-Kutta formulas and adjusts the timestep to maintain the difference between the 5th and 6th order results to be less than a specified tolerance. Typically, \( E \) and \( P_\phi \) can be maintained constant to within 11 digits if desired. The change in \( E \) is larger than that in \( P_\phi \) and occurs at the time of nearest approach.

**A. Typical Chaotic Scattering Orbits**

The orbits are bounded for \( E < 0 \) and unbounded for \( E > 0 \). The chaos in the bounded orbits is easy to understand and has been explored by Schmidt et al. [4]. Here the motion along the magnetic field from Eq. (4) has the frequency \( \omega_z = \langle \rho^2 \rangle^{-3/4} \) for small vertical motions \(|z| \ll \langle \rho^2 \rangle^{1/2} = (x^2 + y^2)^{1/2}\). The Kepler frequency also describes in order of magnitude of the angular frequency of the large vertical motions, \( \omega_z \sim |z|^{-3/2} \). There are
island chains at the lower resonances where \( n\Omega_K - m\omega_z = 0 \). The \( z = 0 \) surface of section (SOS) plot then shows the usual chains of chaotic points between the island chains. The vertical and radial oscillations about the central Kepler orbit would be called the betatron orbits in the corresponding particle accelerator/storage ring system. The closely-related case where the perturbation is due to the fixed rotation of a large planet such as Jupiter has been extensively studied. We have reproduced Schmidt’s case in his Fig. 2 where \( E = -1.6 \). The surface of section in the \( z = 0 \) SOS for the \( r, \dot{r} \) plane shows bands of chaos separated by island chains (periods 1, 2 and 3 islands are large).

In the case of unbounded scattering orbits the chaos is more difficult to characterize. Three quantities are needed: distance from the ion to the electron guiding center line, pitch angle, and gyrophase. The asymptotic states are helical cyclotron orbits. We may characterize these states with the \( 3 \times 3 \) matrix \( T = S^1 \times D \) where \( S^1 \) is the circular motion \( v_\perp \cos(\zeta - \Omega t), v_\perp \sin(\zeta - \Omega t) \) and \( x(t) = x(t_0) + (v_\perp / \Omega) [\sin(\zeta - \sin(\zeta - \Omega t)] \) and \( y(t) = y(t_0) + (v_\perp / \Omega) [\cos(\zeta - \Omega t) - \cos \zeta] \). \( D \) is the translation \( z(t_0) = z_0 + v_\parallel (t - t_0) \). The key variable in the magnetized Coulomb scattering event is the particle’s pitch angle \( \alpha \) defined by the angle between \( \mathbf{v}(t) \) and \( \mathbf{B} \) in velocity space. Thus \( v_\parallel = v \cos \alpha \) and \( v = (2E/m_e)^{1/2} = \text{constant} \).

In the scattering process the pitch angle changes continuously in a complex, basically unpredictable, manner within the meaning of deterministic chaos. The method of Lyapunov exponents is not applicable due to the limited time of interaction. Thus, the concepts and definitions proposed by Henon [6] for chaotic scattering are the relevant tools. Thus, the outcome of the scattering event is measured by probability distribution functions of the change in the electron’s pitch angle. In the limit of a high-velocity short time \( \tau_v = b/v_\parallel \ll \omega_c^{-1} \), the scattering event pdf for the change in pitch angle \( \Delta \alpha = \alpha^f - \alpha^i \) is expected to collapse to a delta function on the Rutherford scattering angle \( \theta_R \) defined uniquely by \( bv^2 \). In the dimensionless units

\[
\tan \left( \frac{\theta_R}{2} \right) = \frac{1}{bv^2}.
\]  

(6)

Here \( \alpha^i \) and \( \alpha^f \) are the asymptotic pitch angles before and after the scattering event. The
collapse of the pdf $P(\Delta \alpha)$ to a delta function $\delta(\Delta \alpha - \theta_R(bv^2))$ is never complete, however. The reason for this is easily seen by examining several scattering events. One sees that there is not a unique definition of the parameter $b$ for the input state. The problem is that the unique initial pitch angle $\alpha^i$, kinetic energy $E = \frac{1}{2}v^2$, and position $b$, do not uniquely determine the particle’s approach to the ion or nucleus due to the cyclotron phase angle $\zeta$. The difficulty is due to the different types of asymptotic states. The closest corresponding definition of the impact parameter for the helical orbits is the distance from the geometric center of the circular orbit (defined by Alfvén as the guiding center) and the nucleus. We define this corresponding parameter as $b_X$ where the capital $X, Y$ coordinates are those fixed values $X = x(t) + v_y(t)/\Omega$ and $Y = y(t) - v_x(t)/\Omega$ of the geometric center of the helical orbit. The particle never resides at $(X, Y)$ and thus, as the 3D perspective figures will show the Coulomb interaction sets in with a varying value of $b$ as the influence of magnetic force losses control over the electron’s motion with respect to ever-increasing strength of the Coulomb attraction to the nucleus. At low, positive energies $E$ there is, however, a much more important source of spread in the outgoing pitch angle which we consider next.

B. Chaotic Motion of Quasi-trapped Electrons

To see how low-energy scattering events can have unpredictable outgoing pitch angles, consider the case shown in Fig. 1. Here we take an initial zero pitch angle ($\alpha^i = 0$) electron $v_\| = v$ for which the classical impact parameter $b$ is the same as the $b_X$ since there is no helical motion in the incoming state. Now, for $v = 1.0$ and $b = 0.5$ the Rutherford scattering angle is $\theta_R = 2\tan^{-1}(2) = 126.78^\circ$. From the orbit in Fig. 1 this rotation of the incoming velocity can be roughly seen in the first outgoing loop. Now the electron has a gyro-orbit size of roughly $\rho = v \sin \theta_R = 0.8$ and the parallel velocity is $\dot{z} = v_\| \cong -0.6$. With this speed the electron cannot escape the attraction of the ion and makes a second encounter without a well-defined impact parameter. The result is a slight increase of $v_\|$ but still insufficient to escape. Finally, on the third encounter with ions, the velocity vector
rotates to be sufficiently anti-parallel to $B$ for the electron to escape with, in fact, a very weak helical motion. Thus, the effective overall pitch angle rotation for this complex event is through $180^\circ$, not $\theta_R \cong 127^\circ$. Adding an initial $v_x, v_y$ component to the input velocity vector complicates the interaction still further, by introducing a new preferred direction of $v_0 \times B$ in the event.

Evidently, the Coulomb scattering at sufficiently low energies is dramatically altered by the magnetic field. For the next two examples we keep $v = 1$ and $\alpha_0 = 0$, but double the impact parameter to near one, $b \approx 1.0$. We now show that by very small changes in $b$ that the outcome can shift from a back-scattering to a forward-scattering event. That is, the scattering angle charge $\Delta \alpha = \alpha^f - \alpha^i$ is extremely sensitive to the precise value of the impact parameter. This is the hallmark (definition) of chaotic scattering.

In Fig. 2 on the left panel we have $b = 1.02$ and in the right panel $b = 1.03$. We follow the electron until it escapes the pull of the nucleus. In both cases the Rutherford angle is

$$\theta_R \cong 2\tan^{-1}(1) = \pi/2$$

so that the first encounter leaves the electron with small $|v_\parallel| \ll v$. The electron then makes a series of oscillations in and along the magnetic field line in the helical cyclotron orbit. After an unpredictable number of passages close to the nucleus the electron scatters to a pitch angle small enough to allow it to escape the pull of the nucleus. In the left panel the electron has two encounters and after the second encounter the final pitch angle is $\alpha^f \approx 135^\circ$. In the right panel the escape occurs after the third encounter and the final pitch angle is $\alpha^f \approx 30^\circ$. Thus, two nearby initial parameters, differing by one percent, have qualitatively different final orbits. Thus, we must give up the description of phenomena in terms of single orbits and introduce probability measures for the outgoing states.

C. Ensemble of Zero Pitch Angle, Low-Energy Scattering Events

To have a global view of this sensitivity and a measure of the final states we integrated the orbits for ensemble of $N_p = 2000$ particles with initial speed $v = 1.0$, zero initial pitch
angle $\alpha = 0$ and obtained the final pitch angle $\alpha_f = \cos^{-1}(v_z/v)$ as a function of the impact parameter for the interval $b \in [0.3, 1.1]$. For this setup there is axial symmetry around $B$ and the only remaining conditions are $z_0 = -50.0$ and $b$. The final pitch angle $\alpha_f$ is the same as the change in the pitch angle $\Delta \alpha$ and in Rutherford theory of the scattering events the result is $\alpha_f = \Delta \alpha = \theta_R = 2 \tan^{-1}(1/bv^2)$. Thus, we plot in the upper panel of Fig. 3 $\cos \alpha_f$ versus the $b$ values with points joined by to form the curve or scattering function. The result shows strong (discontinuous) variation from neighboring values of the scattering angle. In the upper panel the $n_z$ is plotted as a function of $b$, where $n_z$ is the number of times $v_z$ changes sign. Thus $n_z$ describes how many oscillations the electron goes through.

There are domains in which $\cos \alpha_f$ does not look chaotic, because in each of those domains, every electron has $v_z$ changed for the same number of times before getting kicked out, its final state is a continuous (slowly-varying) function of the incoming state. In each of this kind of non-chaotic domain, all the electrons get forward or backward-scattered, hence $\cos \alpha_f$ is either positive or negative. Each non-chaotic backward-scattering/forward-scattering domain has two non-chaotic forward-scattering/backward-scattering domain as neighbors. In the chaotic domain, the electrons have quite different $n_z$ values.

Moreover, in each chaotic domain, we can still find chaotic and non-chaotic subdomains. This was verified computationally by looking at a small section in $b$, namely $b \in [0.2775, 0.2776]$, and the results were shown in the lower panel of Fig. 3. In the chaotic domain we find that the scattering angle has a fractal dependence on the impact parameter at least down to the scale $\delta b = 10^{-7}$.

Notice that the final pitch angle and the bouncing time have a infinite number of discontinuities, forming a self-similar Cantor set, which was described by Henon [6] in another chaotic scattering model.

Figure 4a shows the probability distribution $P(\cos \alpha)$ of the number of particles from the ensemble that scatter into evenly-spaced bins of $\cos \alpha$. Figure 4b shows the number of electrons scattered into the bins versus $\cos \alpha_f$ for Rutherford scattering without magnetic field. We see large difference between the two of distributions. With magnetic field, the
distribution has a large portion of backward-scattering.

To show the approach to the Rutherford regime, we increase the incoming velocity.

Figure 5 shows results of an ensemble of initial velocity \( v = 3 \), and \( b \in [0.09, 0.13] \). Figure 6 shows results of an ensemble of higher initial velocity \( v = 10 \), and smaller \( b \) values, \( b \in [0.0096, 0.0104] \). We see large difference between the two of distributions. Compared to the Rutherford scattering, the distribution with magnetic field is not very peaked. The magnetic field introduces an anomalous number of back-scattering.

In a typical laboratory plasma confinement experiment, the frequency of small-angle scattering events completely overwhelms the number of side-scatter and back-scatter events. In Sec. III we will determine the plasma parameter regimes where the chaotic scattering shown in Fig. 4 is expected to be important. Here we continue with the analysis and numerical experiments to determine the characterization of the new non-Rutherford scattering regimes.

Several other ensembles have been constructed and investigated and the results will be reported in another work since they require more space to display and explain. One ensemble is that of the product of a ring of radius \( b_X \) at \( z = -15r_0 \) and for each \( \mathbf{a}(t = 0) \) a cone in velocity space with a small initial angle \( \alpha_i \) with respect to the \( \mathbf{B} \) vector. The results show a wide range of orbits. We construct the transverse momentum transfer \( v_i^2 \) as a statistic of the orbits and study its probability distribution. We have constructed histograms of the maximum kinetic energy \( T_{\text{max}} \) equivalent to the distance of closest approach, which is not the in the presence of \( \mathbf{B} \) for the ensembles. For Rutherford scattering \( v_i^2 \) is uniquely determined by \( v^2b \) which is not in the presence of \( \mathbf{B} \).

Finally, we note that we see very clearly the well-known adiabatic cut-off of the scattering that occurs for \( b_X \gg v_\parallel /\omega_{ce} \) [8,9]. For \( b \gg v_\parallel /\omega_{ce} = \rho \) there are several rotations of the particle in the collision time \( \tau_c = v_\parallel /b \). Thus, the magnetic momentum \( \mu = m v_\perp^2 /2B \) becomes an adiabatic invariant and there can only be exponentially small changes in the pitch angle. The formulas of Geller and Weisheit [9] involving the \( K_1(b\omega_{ce}/v_\parallel) \)-Bessel functions arising from perturbation theory describe well this cut-off of the interaction for \( b > b_{\text{max}} = \)
For $\omega_{ce} > \omega_{pe}$ this cut-off replaces the Debye length $\lambda_{De}$ from collective effects in the Coulomb logarithm.

In Fig. 7 we show the agreement between the test particle scatterings experiments and the adiabatic reduction factor effect $\text{RF}(x)$. The details of the reduction factor $\text{RF}(b\omega_{ce}/v_{\|})$ are given in the Appendix using the Fermi approximation. Here we describe the comparisons. Figure 7 shows the Rutherford scattering angle $\tan(\theta/2)$ as a function of $v^2b$ as the thin solid line with the largest value of $\tan(\theta/2)$. Next, the simulations are shown for a series of decreasing initial injection velocities for the zero-pitch angle particles $v_{\|} = v = [4, 2, 1]$ in units of $v_0 = \omega_{ce}r_0$. As the interaction time becomes longer, gyro-phase angle increases, the scattering angle decreases strongly. For comparison we show lines for the analytic model of Geller-Weisheit derived with the Fermi approximation of fixed unperturbed helical particle orbits. We see the good agreement between the Fermi approximation [13] and the numerical scattering experiments for $\kappa = b\omega_{ce}/v_{\|} = [1, 5]$. Figure 8 gives the same information for $\tan(\theta/2)$ down to $10^{-7}$ on a logarithmic scale. Above $\kappa = 5$ the reduction is so strong, there is a monotonic trend for the Fermi approximation to overestimate the reduction factor at lower normalized velocities. Since the reduction is large in the low energy when the effect is integrated over a Maxwellian one obtains the factor of $\exp\left(-C_{\text{Oh}}\kappa_{ih}^{2/5}\right)$ with $C_{\text{Oh}} = 2.04$ given in O’Neil and Hjorth [8]. For our case with the Fermi approximation, the coefficient is $C_{\text{FA}} \approx 1.71$. The $\kappa_{ih}$ derivation and definition of the thermal $\kappa$ and $C_{\text{FA}}$ are given in Eqs. (A24)-(A29) in the Appendix.

The Fermi approximation for the magnetic Coulomb scattering problem are described in the Appendix. The result for the rate of increase of the perpendicular kinetic energy $K_{\perp}$ from the ensemble of incident zero pitch-angle electrons is given by

$$\frac{dK_{\perp}}{dt} \approx \frac{4\pi n_i Z^2 e^4}{m|v_{\|}|} \int_{b_{\min}}^{D} \frac{db}{b} \left[ \frac{\omega_{ce} b}{v_{\|}} K_1 \left( \frac{\omega_{ce} b}{v_{\|}} \right) \right]$$  \hspace{1cm} (7)

where $b_{\min} = Z e^2/m v_{\|}^2$, $D$ is the Debye length $\nu_{c}/\omega_{pe} = (T_e e^2/n e^2)^{1/2}$ and $\omega_{ce} = \epsilon B/m_e$ [7,9]. The impact parameter integral is analytic in terms of $K_0(x)$ and $K_1(x)$ with $x = \omega_{ce}T_{\min}$ and $x_D = \omega_{ce}T_D$ where $T_{\min} = b_{\min}/v_{\|}$ and $T_D = D/v_{\|}$. The result of the $db/b$-integral is
determined by

\[ F(x) = \frac{x^2}{2} \left( K_1^2(x) - K_0^2(x) \right) - xK_0(x)K_1(x) \]

where

\[ \ell n \Lambda_B = \int \frac{db}{b} \left[ \frac{\omega_{ce} b}{v_\parallel K_1} \left( \frac{\omega_{ce} b}{v_\parallel} \right) \right]^2 = F(x_m) - F(x_D). \]

The function of \( F(x) \) which drops from \( F(x) \rightarrow \ell n(x) - 0.11 \) at small \( x \) to \( F(x) \rightarrow \pi \exp(-2x) \). A wild range of impact parameters contribute to \( dK_\perp \) for typical laboratory confinement plasmas the chaotic orbits occur in the transition of \( F(x) \) to the exponentially small regime \( x \gtrsim 1 \). For large \( D/b_{\min} \) values the chaotic scattering orbits are not expected to change the macroscopic transport quantities such as \( dK_\perp /dt, \) resistivity and viscosity.

In Fig. 9 we show scaling property of the squared transverse momentum transfer as a function of the initial pitch angle \( \alpha_i \), normalized to the “guiding-center scattering angle” derived from two sets of guiding-center ensembles. The first set (closed circles) has the guiding center at \( x_0 = \sqrt{2} \) from the ion, the initial velocity \( v_\parallel = 2 \) and the gyroradius varies from 0.001 to 0.6. The second set (open circles) has \( x_0 = 1, \ v_\parallel = 4, \) and the gyroradius from 0.001 to 0.50. A direct numerical integral shows that there is a continuous approach to the zero pitch-angle limit with a deviation quadratic in \( \alpha_i \). The curve is the quadratic behavior predicted by Geller and Weisheit [9] with the coefficient of the quadratic term being \( 1/(vx_0^2) = 1/4 \). The predicted small-\( \alpha \) quadratic behavior is in a reasonable agreement with the numerical points.

III. SCATTERING REGIMES OF PLASMAS

Here we ask in what plasma regimes may the chaotic scattering effects be important. It is clear from the outside that in the usual high temperature \( (T_e > 100\text{eV}) \)-moderate \( B \)-field \( (B \leq 1\text{T}) \) the effects are small since the weak small-angle scattering events completely dominate the finite scattering-angle events.

Recall that the scale length \( r_0 = (Zm_e/4\pi\epsilon_0B^2)^{1/3} = 20l(Z/B^2)^{1/3}\text{nm} \) for \( B \) in Tesla. For \( B < 8.6 \times 10^3T_e^{3/2}(\text{eV})[\text{T}] \) we have \( b_{\min} < r_0 < \rho_e \) with \( r_0 \approx b_{\min}^3\rho_e^{2/3} \).
To be quantitative we find for $T_e$ in ev, $B$ in Tesla ($T$) and electron-density $n_{20}$ in units of $10^{20} \text{m}^{-3}$ what are the characteristic scales of the magnetized Coulomb scattering problem.

The scale length $r_0$ may be expressed in terms of the electron Debye length and gyroradius alternatively as

$$r_0 = 0.27 \lambda_{\text{De}} \, Z^{1/3} \, n_{20}^{1/2} \, B^{2/3} \, T_e^{1/2}, \quad (8)$$

$$= 0.049 \rho_e \, Z^{1/3} \, B^{1/3} \, T_e^{1/2}. \quad (9)$$

Thus, $r_0$ is small compared to both $\rho_e$ unless the field exceeds $B^{1/3} > 20.5 T_e^{1/2}$

$$B > 8.6 \times 10^3 \, T_e^{3/2} \text{(ev)} \, [T]. \quad (10)$$

There are astrophysical plasmas, with strong magnetic fields and relatively low temperatures where the thermal electrons at average interval particle spacing are in the chaotic scattering regimes. We give these parameters in Table I for the atmospheres of the white dwarf and the neutron star the parameters $b_{\text{min}}$, $r_0$, $\rho_e$ and $\lambda_{\text{De}}$. For the neutron star the scaled thermal velocity $v/\omega_e r_0 = 0.5$ as in the numerical experiments. Recently, these types of plasmas have been roughly simulated in the laboratory by using lasers to cool to very low temperatures small groups of Xenon atoms [11]. The electron temperature is reduced to 100 mK for which the normalized $v \sim 1$ when the magnetic field is $B = 40 \mu \text{T}$. The electron density of $n \sim 2 \times 10^{15} \text{m}^{-3}$ is such that the average inner particle space $n^{-1/3} \approx 7.9 \times 10^{-6} \text{m}$ which is comparable to the scale radius $r_0 \approx 3.7 \times 10^{-5} \text{m}$ required for chaos.

For laboratory magnetic fields of 10 T or less, only very low-energy electrons near the ionization threshold will have large scattering angles. For example, the characteristic speed $v = 1$ used in the numerical examples corresponds to an electron kinetic $T$ of only a few times $10^{-3} \text{ev}$ or $10^5 \text{K}$. Clearly, the study of such low-energy, free-electron states would require a cryogenic plasmas to avoid recombination from removing all the low-energy electrons. We observe in Table I that it is always the combination of $ZB$ that enters the problem so that using a high $Z$ ion enhances the effective strength of the $B$-field.

The complex quasi-trapped orbits near the ionization threshold suggests that the ionization and radiation levels maybe the observable effect of the chaos.
Note that the radiated power formulas may be modified by a strong magnetized field. We propose in a future work to calculate the combined synchrotron and Bremsstrahlung from these orbits.

IV. CONCLUSIONS

The magnetic field plays the role of taking away relative kinetic energy from the Coulomb interaction. Thus, low-energy electrons make transitions into orbits that remain in the neighborhood of ions for long, unpredictable time periods. When the cyclotron rotation and the Coulomb electric field develop the correct phase relation, the electron is ejected with sufficient parallel velocity to escape the pull of the ion.

Two immediate consequences of these long quasi-trapped oscillations of the electron about the ion is that the scattering angle $\theta$ becomes a fractal function of the impact parameter $b$. Thus, the differential cross-section requiring the derivation $d\theta/db$ does not exist. The second effect is a prolonged shield of the ion with numerous violet accelerations of the electron. We are investigating pulses of synchrotron-Bremstrahlung radiation produced during these periods of trapping. Summers and McWhirter [12] point out that radiation and atomic ionization-recombination rates are likely to be enhanced by strong magnetic fields.

This suggests that one search for the macroscopic evidence for this microscopic effect in low-temperature transport coefficients. The rate of ionization and recombination of electrons to atomic and molecular ions may also be an area of application of this anomalous Rutherford scattering interaction.

There are two time scales associated with the electron motion, one is the trapping time, the other is the time for the electron travel from one ion to another.

Experiments at low temperature and high magnetic fields may show anomalously high levels of back-scattered electrons. Ultracold neutral plasma maybe the plasma to search for these chaotic electron-ion interaction effects. Laser cooled Xenon atoms at 100 mK electron temperature and 10 $\mu$K ion temperature have been used by Killian et. al. [11] to show
classical Debye scattering and plasma oscillation. Their systems as plasma for studying the effect has the dimensionless speed \( v = 20.5 \frac{T_{eV}^{1/2}}{(ZB)^{1/3}} \). In order for the normalized velocity \( v \sim 1 \), with 100 mK electron temperature and \( Z = 1 \), \( B \sim 0.4 \) mT is needed. The density is \( n \sim 2 \times 10^{15} \) m\(^{-3}\), \( n^{-1/3} \sim 7.9 \mu m \) and \( r_0 = 37 \mu m \). Thus, we see that these ultra-low plasma traps may be the most fruitful place to look for the effects, such as anomalous back-scattering levels, of the chaotic orbits in the magneto-Coulomb scattering system.

Electron particle-in-cell (PIC) simulations may be a useful method of clarifying the role of these low-energy anomalous Coulomb interactions. Clearly, much more work is required to find practical applications of this interesting aspect of the magnetic field in the Coulomb interaction problem.

Acknowledgments

The authors acknowledge useful discussions with Todd Ditmire and Anthony Chan on the problem.

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Appendix: Fermi Approximation

The analysis of electron-ion collision in the presence of a constant magnetic field has been carried out in detail for incident electrons at general impact parameters and pitch angles [9]. Even just for the zero pitch angle case, the problem for general $b$ is already relatively complex. This is especially so when $b$ is comparable to the cyclotron radius $v_\perp/\Omega$, where the electron can be close to the ion as it passes the ion. In this Appendix we will mainly consider the zero pitch angle case in the impact parameter region where the angle of deflection is small, i.e. $b/v_\perp \gg b/v_\parallel \geq 1/\Omega$. In this region, the Fermi’s Approximation [13] is applicable. Here the problem may be setup in the following way.

**Equation of Motion.** Go to the rest frame of the guiding center of the cyclotron motion of the electron. In this frame the electron satisfies the approximate equation of motion

$$\ddot{y} + \Omega^2 y = \frac{eE_y}{m},$$

and the ion is traveling parallel to the direction of the magnetic field with an initial speed $v_\parallel$. The transverse kinetic energy ($K_\perp = \frac{1}{2}m v_\perp^2$) transferred to the electron is given by

$$\frac{dK_\perp}{dt} = v_\perp \cdot \frac{md\mathbf{v}}{dt} = -e\mathbf{v}_\perp(t) \cdot \mathbf{E}(x(t), t).$$

**$K_\perp$ transfer for fixed $v_\parallel$ and $b$.** The change in the transverse kinetic energy for an incident electron with an initial speed $v_\parallel$ and the initial impact parameter $b$ is given by

$$K_\perp(+\infty) - K_\perp(-\infty) = -e \int_{-\infty}^{+\infty} dt \mathbf{v}_\perp(t) \cdot \mathbf{E}(x(t), t)$$

$$\approx -e \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \mathbf{v}_\perp(\omega) \cdot \mathbf{E}(b,-\omega) d\omega$$

$$\approx \frac{e^2}{m} \int_{-\infty}^{+\infty} \frac{d\omega |E_y(b,\omega)|^2(i\omega)}{\Omega^2 - \omega^2 + i\epsilon \omega}.$$  

In the second step, a straight line trajectory approximation was used, so the impact parameter $b$ remains fixed throughout the scattering process. We will adopt this fix $b$ approximation
as was done in the original discussion by Fermi. In the last step, the Fourier transform of Eq. (A1) was used which leads to

\[ v_y(\omega) = -i\omega y(\omega) \approx \frac{i\omega E_y(b, \omega)}{\Omega^2 - \omega^2 - i\epsilon \omega}, \quad (A6) \]

where

\[ E_y(b, \omega) \approx \int_{-\infty}^{\infty} dt \frac{Zebe^{i\omega t}}{(b^2 + v_{||}^2 t^2)^{3/2}} = \frac{2Ze}{bv_{||}} \left[ \frac{\omega b}{v_{||}} K_1 \left( \frac{\omega b}{v_{||}} \right) \right]. \quad (A7) \]

By evaluating the pole contribution in Eq. (A5) one finds that the change in the transverse kinetic energy is given by

\[ \Delta K_\perp = \left[ \frac{2}{m} \left( \frac{Ze^2}{bv_{||}} \right)^2 \right] \left[ \frac{\Omega b}{v_{||}} K_1 \left( \frac{\Omega b}{v_{||}} \right) \right]^2. \quad (A8) \]

**Digression on tan \( \frac{\theta}{2} \) dependence.** It is convenient to introduce the “reduction factor function” RF defined as follows:

\[ (\Delta K_\perp)_B = (\Delta K_\perp)_{B=0} \cdot RF, \quad \text{with } RF(x) = [x K_1(x)]^2, \quad \text{and } x = \frac{\Omega b}{v_{||}}. \quad (A9) \]

Here \((\Delta K_\perp)_{B=0}\) is the multiplicative factor given by the first square bracket in Eq. (A8). Equation (A9) implies that

\[ \sin \theta = \sin \theta_{B=0} \sqrt{RF(x)}. \quad (A10) \]

where \(\theta\) is the scattering angle in the presence of \(B\), and \(\theta_{B=0}\) the scattering angle where \(B = 0\). The Rutherford scattering formula for the incident electron moving along the \(z\) direction, where the incident velocity \(v = v_{||}\), we have

\[ \tan \frac{\theta_{B=0}}{2} = \frac{Ze^2}{v_{||}^2 b}. \quad (A11) \]

For our numerical calculation, we work in the units where \(\frac{Ze^2}{m} = 1\). For \(\Omega = 1\), \(x = \frac{\Omega b}{v_{||}} = \frac{v_{\perp}^2}{v_{||}^2}\).

So once \(v\) and \(b\) are given, from Eqs. (A10) and (A11) tan \(\frac{\theta}{2}\) may be obtained. The situation of tan \(\frac{\theta}{2}\) vs. \(v^2 b\) is shown in Fig. 7, where 3 cases of \(\Omega = 1\): \(v = 1\) (short dash), \(v = 2\) (dash) and \(v = 4\) (long-dash) and the case of \(B = 0\) (or \(\Omega = 0\)) (solid curve) are included. Notice
that for $B = 0$ case, the scattering angle depends only on one independent variable $v^2 b$. The points are based on numerical calculations. The figure shows that in the presence of the magnetic field the scattering angle decreases monotonically with the decrease of the incident velocity. The overall reduction pattern predicted by the theory in Fermi approximation is confirmed by the numerical calculation. On the other hand as shown in Fig. 7 and more clearly in Fig. 8, when $v$ is small, e.g. $v = 1$, Fermi’s approximation over estimates the reduction factor. This is reasonable, since for small $v$, the straight-line approximation is no longer valid. Figure 8 suggests that the extra excursion of a trajectory which is particularly pronounce when $v$ is small leads to a slightly larger scattering angle compared to that given by a straight-line approximation.

**Guiding Center Ensemble:** Although in this Appendix we are mainly concerned with cases where the the initial pitch angle is 0, we digress here to consider the the effect of nonzero pitch angle, when the pitch angle is relatively small. As alluded to in Sec. II.C, here we replace a single electron by a “guiding center ensemble.” The ensemble consists of $N$ electrons uniformly distributed along a “gyro-ring” which is perpendicular to the direction of the magnetic field. All electrons are performing cyclotron motion with a common tangential speed $v_{\perp} = r \Omega$, where $r$ is the radius of the ring. The ensemble has a common parallel component of the initial velocity $v_{\parallel i}$, along the direction of the magnetic field. The center of the ring is located at $x = x_0$. From the geometry there is a common initial pitch angle given by

$$\tan \alpha_i = \frac{v_{\perp i}}{v_{\parallel i}}.$$

We will only consider the case where $r$ is small compared to $x_0$. The transverse kinetic energy transfer averaged over the ensemble is defined by

$$\Delta K_{\perp} \equiv \frac{1}{N} \sum_{i=1}^{N} \Delta K_{\perp i}, \text{ where } [\Delta K_{\perp}]_k = \frac{m}{2} \cdot \left[ (v_{f_x} - v_{i_x})^2 + (v_{f_y} - v_{i_y})^2 \right]_k.$$

Since we will be interested in the region where $v_{\perp i} \ll v_{\parallel i}$. We will use speed of the electron $v$ which is essentially a constant, to denote $v_{\parallel i}$.
Based on Geller and Weisheit’s model [see their Eq. (52)] in the small angle approximation, the square of the transverse velocity transfer associated with the scattering of one electron may be written as follows:

$$ [\Delta v_{\perp}]^2 \approx (v \theta_f (r = 0))^2 + \frac{v_{\perp}^2}{2 v^2} \left( \frac{2}{v b^2} \right)^2 + (\ldots) \cos \theta_0 $$

(A14)

The first term on the right-hand side is due to the so-called “off-cone” scattering. The scattering angle for the case where $x_0 = b$ and $r = 0$ in a small angle approximation is given by $\theta_f (r = 0) \propto \frac{1}{2 (v^2 b)} [x K_1(x)]$, with $x = \Omega b / v$ [see Eqs. (A8)-(A11)]. The second term is due to “on-cone” scattering which is proportional to the square of $v_{\perp}$. The third term is linear in the cosine of the gyro-phase at $t = 0$. After averaging over the ensemble, this term vanishes. Finally one arrives at

$$ \frac{\Delta K_\perp (\alpha_i)}{\Delta K_\perp (0^\circ)} = \left[ \frac{\Delta v_{\perp} (\alpha_i)}{\Delta v_{\perp} (0^\circ)} \right]^2 \approx 1 + \left( \frac{2}{v b^2} \right)^2 \bar{\alpha}_i^2 $$

(A15)

where the normalized initial pitch angle $\bar{\alpha}_i = \alpha_i / \theta_f (r = 0)$ and $\alpha_i \approx v_{\perp} / v$. In Fig. 9, this ratio is plotted versus $\bar{\alpha}_i$ for two cases: solid points are for $v = 2$, $x_0 = \sqrt{2}$ and circles are for $v = 4$ and $x_0 = 1$. They both have a common value of $v x_0^2 = 4$. The points are to be compared with the curve $1 + \bar{\alpha}_i^2 / 4$. Within the region shown the curve agrees well with the solid points within. For the circles, the deviation begins to be noticeable at $\alpha / \theta_f \sim 0.5$. We conclude that the model works reasonably well at least in the small pitch angle region.

**Coulomb logarithm.**

If the electrons travel through ions of density $n_i$, then the number of nucleus encountered in a distance $dz = v || dt$, is $dN = 2\pi b db dz n_i$. Integrating over the impact parameter, the rate of transverse kinetic energy imparted to the electrons is given by

$$ \left( \frac{dK_{\perp}}{dt} \right)_B = \int 2\pi b db n_i v || \Delta K_{\perp} (b, v ||) $$

(A16)

$$ \approx \frac{4 \pi n_i z^2 e^4}{m |v ||} \int_{b_m}^{b} \frac{db}{b} \left( \frac{\Omega b}{v ||} \right)^2 K_1 \left( \frac{\Omega b}{v ||} \right) $$

(A17)

where the limits are $b_m = \frac{Z e}{m v ||}$ and $D$ the Debye length.
To define the effective Coulomb logarithm in the presence of the magnetic field we write

\[
\left( \frac{dK_\perp}{dt} \right)_B = \left( \frac{dK_\perp}{dt} \right)_{B=0} \left( \frac{ln\Lambda_B}{ln\Lambda_{B=0}} \right), \quad \text{with} \quad \left( \frac{dK_\perp}{dt} \right)_{B=0} = 8\pi n v^3 b_m^2 ln\Lambda_{B=0}.
\]  

(A18)

From Eqs. (A17) and (A18), it follows that the coulomb logarithm in the presence of \(B\) is given by:

\[
ln\Lambda_B = \int_{b_m}^D db \left[ \frac{\Omega b}{v} K_1 \left( \frac{\Omega b}{v} \right) \right]^2 \equiv F(x_m) - F(x_D). \tag{A19}
\]

where the \(F\) function is defined by

\[
F(x) = -xK_0(x)K_1(x) + \frac{x^2}{2} \left( K_1^2(x) - K_0(x)^2 \right). \tag{A20}
\]

The arguments of two \(F(x)\)'s on the right-hand side of Eq. (A19) are

\[
x_m = \frac{\Omega b_m}{v} \equiv \Omega T_m, \quad \text{and} \quad x_D = \frac{\Omega b_D}{v} \equiv \Omega T_D,
\]  

(A21)

where \(T_m = b_m/v\), which is the minimum collision time, so \(x_m = \Omega T_m\) may be identified as the gyro-phase, i.e. the phase angle which the electron has acquired during a minimal collision time interval. \(T_D = D/v\). It is the maximum effective collision time interval beyond which the electric field due to the nucleus is negligible. Correspondingly, \(x_D\) is the gyro-phase acquired during the maximal effective collision time interval. Equations (A18)-(A21) agree with the result of [9] [See their Eq. (61), and also their Fig. 7. Notice that their \(v\) is the present \(v\). They work with the variable \(R^* = v/\Omega\). For our discussion below we find it more natural to work with the original variable \(\Omega\) and associate the \(v\)-dependence with the collision time \(T \sim b/v\) and refer the quantity \(x = \Omega T\) as the gyro-phase, which the electron has acquired during the collision.] The \(x\)-dependence of \(F(x)\) of Eq. (A20) is as follows.

- For small \(x\), with \(xK_1(x) \to 1 \) and \(K_0 \to -ln \frac{x}{2} - 0.557 + \ldots\), so

\[
F(x) \to -ln x - 0.11. \tag{A22}
\]
• For large \( x \), i.e. \( \ln x > \ln x_c \sim 1 \), with \( K_0(x) \), \( K_1(x) \rightarrow \sqrt{\frac{2}{\pi x}} \exp(-x) \). Taking into account the full expression of Eq. (A20) we arrive at

\[
F(x) \rightarrow \pi \exp(-2x). \tag{A23}
\]

The Coulomb logarithm as defined by Eq. (A19) for the case where \( D/b_m = 100 \), or the corresponding \( B = 0 \) Coulomb logarithm \( \ln \Lambda_B=0 \equiv ln \frac{D}{b_m} = 4.6 \) is shown in Fig. 10. The first term \( F(x_m) \) is represented by the short-dash curve. Notice that when \( B = 0 \), \( \Omega = 0 \) and \( \ln x_m = \ln(\Omega T_m) = -\infty \). As \( B \) varies from 0 to \(+\infty\), \( \ln x_m \) varies from \(-\infty\) to \(+\infty\).

Using Fig. 10 as a guide, one sees following general behavior of the Coulomb logarithm as a function of the logarithm of the gyro-phase. There are two landmarks:

(1) One is at \( \Omega = \Omega_0 \), where \( \ln x_0 \equiv \ln \Omega_0 T_m = b_m/D = -\ln \Lambda_B=0 \) and (2) \( \Omega = \Omega_c \), where \( \ln x_c \equiv \ln \Omega_c T_m \approx 1 \).

These two landmarks divide \( \Omega \) into three regions.

• For \( \Omega < \Omega_0 \), it is the plateau region where \( \ln \Lambda_B = \ln \Lambda_B=0 \).

• For \( \Omega > \Omega_c \), it is the exponential fall off region where \( \ln \Lambda_B \sim \pi \exp(-2\Omega T_m) \).

• The intermediate interval \( \Omega_0 < \Omega < \Omega_c \) is the transition region which smoothly connects the plateau to the left and the exponential fall to the right.

**Average \((dK_\perp/dt)_B\) over a velocity distribution.** So far we have investigated situations where for each case the initial velocity \( v_\parallel \) is fixed. We conclude with a discussion where the longitudinal velocity of incident electrons has a distribution. We take the example of a gaussian distribution, with the average velocity denoted by \( v_{th} \). For this velocity, the lower impact parameter cutoff is given by

\[
l_{th} = \left( \frac{Ze^2}{m v_{th}^2} \right) = \left( \frac{Ze^2}{m v^2} \right) \cdot \frac{v^2}{v_{th}^2} = b_m \cdot \frac{v^2}{v_{th}^2}. \tag{A24}\]

So the gyro-phase variable in the velocity integration is given by

\[
\Phi = \frac{\Omega T_m}{\ln \frac{D}{b_m}} = \frac{V_{th}^2}{\sqrt{2 e A}}.
\]
\[ x_m = \frac{b_m \Omega}{v} = \left( \frac{b^{th}_m}{v_{th}} \right) \cdot \left( \frac{b^{th}_m}{v} \right) \cdot \left( \frac{\nu^{th}}{v} \right) \equiv \frac{\kappa_{th}}{\sigma^3} \]  

where \( \kappa_{th} = \frac{b^{th}_m}{v_{th}} \) and \( \sigma = \frac{v}{v_{th}} \). We use angular-brackets to denote an average over the velocity distribution. We are interested in the asymptotic behavior of the reduction factor for large \( \kappa_{th} \).

The mathematics of this problem was considered in detail in [8] and further refined in [7], where these authors studied the effect of a strong constant magnetic field on electron-electron collisions in a plasma. In the next few steps we will continue to work within the Fermi approximation where a straight-line approximation for the denominator function of the integrand of Eq. (A7) was assumed. Based on Eqs. (A19), (A20), and (A23), for the present case the appropriate reduction factor is given by

\[ RF(\kappa_{th}) \equiv \frac{\langle \frac{dK_s}{dt} \rangle_B}{\langle dK_s \rangle_{B=0}} \propto \int \frac{d\sigma}{\sigma} \exp \left( -\frac{\sigma^2}{2} \right) \exp \left( -\frac{2b_m \Omega}{v} \right), \]  

which leads to

\[ RF(\kappa_{th}) \propto \int \frac{d\sigma}{\sigma} \exp(-h(\sigma, \kappa_{th})), \quad \text{with} \quad h(\sigma, \kappa_{th}) = \frac{\sigma^2}{2} + \frac{2\kappa_{th}}{\sigma^3}. \]  

Denote \( h(\sigma) = C \cdot \sigma^{2/5} \). This leads to the asymptotic behavior

\[ RF(\kappa_{th}) \propto \exp(-C \cdot \kappa_{th}^{2/5}) \]  

For the present case with Fermi’s straight-line approximation, \( C = C_{FA} = 5(6)^{2/5}/6 \approx 1.71 \).

Instead of making a straight-line approximation, the authors in [8] carried out a complex plane analysis which leads to a stationary point at \( \sigma_s = (3 \pi \kappa_{th})^{1/5} \), in turn the asymptotic behavior

\[ RF(\kappa_{th}) \propto \exp(-C_{OH} \kappa_{th}^{2/5}) \]  

where \( C_{OH} = 5(3 \pi)^{2/5}/6 \approx 2.04 \).
REFERENCES


FIG. 1. Perspective view of the orbit for \( v = 1[\omega r_0], \alpha_i = 0 \) and impact parameter \( b = 0.5[r_0] \). The electron first scatters back through roughly the 127° degree Rutherford angle, then reflects twice in the Coulomb potential and finally escapes in a back-scattered orbit with pitch angle almost rotated by 180° degrees.

FIG. 2. Two orbits with impact parameters \( b_1 \) and \( b_2 \) differing by less than one percent and having opposite scattering directions. In the left panel the orbit is back-scattered and is qualitatively similar to the orbit in Fig. 1. In the right panel the orbit makes one more interaction with the nucleus and exits in the forward direction with a significant pitch angle.

FIG. 3. The upper panel shows the scattering angles and \( n_z \) for a range of impact parameters \( b \in [0.2, , 0.001, 1.1] \). The lower panel shows the enlargement of \( b \)-intervals \([0.2775, 10^{-7}, 0.2776]\) in the upper panel.

FIG. 4. The probability distribution of the number of scattering events in Fig. 3 as a function of the final pitch angle \( \theta = \alpha_f \). The distribution is bi-modal with forward and backward-scattering dominating in qualitative contrast to the monotonically forward-peaked Rutherford prediction.

FIG. 5. The probability distribution of the scattering events for \( v = 3 \) and \( b \in [0.09, 10^{-4}, 0.13] \).

FIG. 6. Shows the probability distribution for the Coulomb-magnetic interaction as it tends toward the Rutherford Coulomb distribution for increased values of the electron speed. Here \( v = 10 \) and \( b \) is reduced by \( 10^{-2} \) to keep the Rutherford parameter \( bv^2 \) fixed.

FIG. 7. An illustration showing that in the presence of magnetic field, the scattering angle decreases with the velocity of the initial electron. The upper-most solid curve is
\[ \tan(\theta/2) \] for \( \Omega = 0 \), where \( \theta \) is the Rutherford scattering angle. The 3 curves below it are for \( \Omega = 1 \) (the normalized cyclotron frequency). They are predictions based on the Fermi’s approximation for \( v = 4 \) (long-dashes), \( v = 2 \) (dashes) and \( v = 1 \) (short dashes). The corresponding points are numerical results without making this approximation.

FIG. 8. The same plot as Fig. 7, except that the ordinate here is in a logarithm scale. There are 3 sets of data points with curves. From top down, the data points are for \( v = 4 \), \( v = 2 \) and \( v = 1 \). The curves are the corresponding analytic predictions based on the Fermi approximation. Notice that the short dashed curve is for \( v = 1 \). It shows that the Fermi Approximation gives a slight overestimation in the reduction of the scattering angle.

FIG. 9. The normalized transverse kinetic energy versus the normalized initial pitch angle \( \vec{\alpha} \). The two sets of points shown are as follows: the solid points are for \( v = 2 \), \( x_0 = \sqrt{2} \), \( \theta_j(r = 0) = 14.9^0 \) and \( \Delta v_1^2(0) = 0.265 \) and the circles are for \( v = 4 \), \( x_0 = 1 \), \( \theta_j(r = 0) = 6.71^0 \) and \( \Delta v_1^2(0) = 0.218 \). Both curves have \( vx_0^2 = 4 \), which leads to the curve: \( 1 + \vec{\alpha}^2/4 \), as is predicted by Eq. (A15). The first case agrees well while for the second case a deviation sets in at \( \alpha/\theta_j \sim 0.5 \), which grows relatively quickly beyond this point.

FIG. 10. Coulomb logarithm \( \ln \Lambda_B \) versus \( \ln x_m \), where the gyro-phase \( x_m = \Omega T_m \sim -4.6 \) and \( \ln \Lambda_B = F(x_m) - F(x_D) \). The short-dashed curve is for \( F(x_m) \), the dashed curve for \( F(x_D) \) and the thick solid-curve for \( \ln \Lambda_B \). Two landmarks are at \( \ln x_m = \Omega_0 T_m \sim -4.6 \) indicated by a solid vertical line, and at \( \ln x_m = \ln \Omega_e T_m \sim 1 \) indicated by a vertical short-dashed line. They divide the total range into three regions: the plateau, the transition and the exponential fall regions.
### TABLE I. Representative Astrophysical Plasmas where Chaotic Scattering is Predicted

<table>
<thead>
<tr>
<th>Plasma Environments</th>
<th>$T$ (K)</th>
<th>$B$ (T)</th>
<th>$v$</th>
<th>$n$ ($m^{-3}$)</th>
<th>$r_0$ (m)</th>
<th>$n^{-1/3}$ (m)</th>
<th>$\lambda_e$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>White dwarf atmosphere</td>
<td>$5.0 \times 10^4$</td>
<td>$1.0 \times 10^4$</td>
<td>$2.4$</td>
<td>$1.0 \times 10^{22}$</td>
<td>$4.3 \times 10^{-10}$</td>
<td>$4.6 \times 10^{-8}$</td>
<td>$3.3 \times 10^{-10}$</td>
</tr>
<tr>
<td>Neutron star atmosphere</td>
<td>$1.0 \times 10^6$</td>
<td>$1.0 \times 10^8$</td>
<td>$0.5$</td>
<td>$1.0 \times 10^{30}$</td>
<td>$9.2 \times 10^{-13}$</td>
<td>$1.0 \times 10^{-10}$</td>
<td>$7.5 \times 10^{-9}$</td>
</tr>
<tr>
<td>Ultracold neutral plasma</td>
<td>$0.1$</td>
<td>$4.0 \times 10^{-4}$</td>
<td>$1.0$</td>
<td>$2.0 \times 10^{15}$</td>
<td>$3.7 \times 10^{-5}$</td>
<td>$7.9 \times 10^{-6}$</td>
<td>$2.3 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

Here $\lambda_e$ is the electron’s thermal DeBroglie wavelength $\lambda_e = \frac{2\pi\hbar}{\sqrt{2\pi mkT}}$. 
Scattering angle $\theta$ and bouncing time $n_z$ vs. impact parameter $b$ with a magnetic field

FIG. 3.
FIG. 4.

FIG. 5.
FIG. 6.
Adiabatic Reduction Effect

FIG. 7.

FIG. 8.

FIG. 9.
FIG. 10.