Consider a cold (zero-temperature) plasma waves traveling along the x-direction in the presence of a magnetic field pointing along the z-direction. The speed of the plasma waves in front of the disturbance is $V_0$, and the magnetic field is $B_0$. The plasma equations are given by

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} nV = 0 \quad (1)$$

$$\left( \frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) V = \frac{q}{m} (E + V \times B) \quad (2)$$

We assume the plasma consists of electrons and ions. Since in each of the above equations, only one species involved, for brevity the species indices have been suppressed. For the masses and charges we will use the notations: $q_e = -e$, $q_i = e$, $m_e = m$ and $m_i = M$. In front of the disturbance $n_e = n_i = n_0$. We are looking for solutions which correspond to time-independent structure. We assume that the fields are a function of x only. From Faraday’s law, i.e. $\nabla \times E = 0$, it follows that $\partial E_y / \partial x = 0$, so $E_y = \text{const}$, i.e. it is independent of x. Without lost of generality, we may set $E_z = 0$ and it can be shown that there are only three field-components involved, which are $E_x$, $E_y = \text{const}$, and $B_z \equiv B$, which as mentioned, are assumed to be a function of x only.

This system of fields and plasma fluid may be described by following set of time-independent equations

$$\frac{d}{dx} (nV_x) = 0, \quad (3)$$

$$\frac{d}{dx} E_x = \sum_k \frac{q_k n_k}{\varepsilon_0}, \quad (4)$$

$$\frac{dE_y}{dx} = 0, \quad (5)$$
\[
\begin{pmatrix}
0 \\
\frac{dB}{dx}
\end{pmatrix} = \mu_0 \sum_k q_k n_k \begin{pmatrix}
V_x \\
V_y
\end{pmatrix},
\]
(6)
\[
V_x m \frac{d}{dx} \begin{pmatrix}
V_x \\
V_y
\end{pmatrix} = q \begin{pmatrix}
E_x \\
E_y
\end{pmatrix} + q \begin{pmatrix}
V_y B \\
- V_x B
\end{pmatrix}.
\]
(7)

We assume quasi-neutrality throughout the waves, i.e.
\[n_e \approx n_i \equiv n, V_{ex} \approx V_{ix} = V_x \equiv V.\]
(8)

We will postpone the discussion of the condition for this approximation.

From the y-component of eq.(7), by adding the electron-equation to the ion equation one finds
\[V_{iy} = \frac{m}{M} V_{ey}.\]
(9)

Thus in the y-direction, compared to the motion of electrons, the motion of ions is negligible. From the x-component of eq.(7), subtracting the electron-equation from the ion-equation leads to
\[E_x = -BV_{ye} + O(m/M).\]
(10)

Adding the electron-equation and the ion-equation in the x-component of eq.(7), gives
\[nMV_x \frac{dV_x}{dx} \approx B(-enV_{ey}) = \frac{B dB}{\mu_0 dx},\]
(11)
where in the last step, Ampere’s law of eq.(6) was used. Continuity implies \(nV_x = n_0V_0\) (see eq.(3)), which may be substitute on the left hand side of eq.(11). Upon integration, it leads to
\[\frac{V_x}{V_0} = \left(1 - \frac{b^2 - 1}{2\alpha^2}\right),\]
(12)
where the normalized z-component of the magnetic field is defined by \(b \equiv B/B_0\), the March number \(\alpha = V_0/V_A\) where \(V_A\) is the Alfvèn speed satisfied the relation: \(n_0MV_A^2 = B_0^2/(2\mu_0)\). In front of the disturbance, \(b = 1\) and
$V_x = V_0$. As $b$ increases, the normalized speed in the $x$ direction decreases. Our quasi-neutrality assumption for the motions of electrons and ions is valid, strictly speaking only [2]

$$\frac{|N_i - N_e|}{N} << 1, \text{ or } \frac{B_i^2}{\mu_0}(b - 1) << n_0mc^2.$$ (13)

Within the present approximation there are only to velocity functions involved, which are

$$u \equiv \frac{V_{ix}}{V_0} \approx \frac{V_{ex}}{V_0}, \text{ and } v \equiv \frac{V_{ey}}{V_0}.$$ (14)

From eq.(11),

$$v = \frac{V_{ey}}{V_0} = \frac{-1}{\mu_0neV_0} \frac{dB}{dx} = -v_A r_A \cdot \left( \frac{db}{dx} \right),$$ (15)

where

$$v_A = \frac{V_A}{V_0} \text{ and } r_A = \frac{MV_A}{eB_0}.$$ (16)

From Faraday's law, eq.(5) and the Lorenz force law, more specifically the $y$-component of the electron-equation in eq.(7),

$$E_y = E_{y0} = V_0B_0 = V_x B.$$ (17)

For the last step the mass of electrons is ignored. Continuity equation implies that

$$nV_x = n_0v_0, \text{ or } \frac{n}{B} = \frac{n_0}{B_0}.$$ (18)

Again, mass of electrons is neglected for the last equality here. The soliton problem was discussed by a number of authors in late 50’s and 60’s. See for examples refs.[1, 2, 3]. One finds that the problem can be reduced to following second order nonlinear equation:

$$\frac{d^2B}{d\tau^2} = -\frac{\partial \Phi}{\partial B} = \frac{\omega_{pe}^2}{c^2} B_0(b - 1) \left(1 - \frac{b(b + 1)}{2\alpha^2} \right),$$ (19)
where \( u \, d\tau = dx \) with \( u = V_x / V_0 \) and
\[
\Phi \equiv \frac{\omega_{pe}^2}{2c^2} B_0^2 (b - 1)^2 \left[ \frac{(b + 1)^2}{4\alpha^2} - 1 \right].
\] (20)

We will set \( \omega_{pe}^2 / c^2 = 1 \). This amounts to expressing \( \tau \) in the length units of \( c / \omega_{pe} = \lambda_p / (2\pi) \). The \( \Phi \) function is essentially the Sagdeev potential function[3]. The \( x \) variable is also in units of \( c / \omega_{pe} \).

Proceed to solve the second order differential equation, eq.(19), one obtains
\[
\left( \frac{db}{dx} \right) = \pm \frac{\alpha(b - 1)g(b)}{2\alpha^2 + 1 - b^2}, \text{ where } g(b) = \sqrt{(2\alpha)^2 - (1 + b)^2}.
\] (21)

Upon integration it leads to
\[
x = \frac{g(b)}{\alpha} + \frac{\alpha}{\sqrt{\alpha^2 - 1}} \ln \left[ \frac{(2\alpha^2 - 1 - b) - g(b)}{\alpha(b - 1)} \right].
\] (22)

Using eq.(22), one may determine the functional dependence of \( b \) as a function of \( x \). This is the soliton solution which is shown as the redline in Fig. 1a, for a typical March number of \( \alpha = 1.8 \). We have also solved \( b \) numerically by integrating the second order differential equation eq.(19). The result reproduces the \( b(x) \) curve. This serves as an independent check on our numerical integration code. The corresponding \( u(x) \) is shown in Fig.1b. Notice that as expected that for \( x << 0 \), \( b \) approaches unity, or \( V_x \approx V_0 \). It is only near \( x = 0 \), within a narrow width, there is a significant slow down of electron velocity. Beyond this structure, as \( x \) further increases, \( v(x) \) approaches to 1. The \( V_{ey} \) velocity is shown in Fig 1c. Notice that this velocity is proportional to \( dB / dx \). There is a peak prior to the peak of the \( b \)-function shown in Fig1a, and a dip after it.

The corresponding kinetic energy gain may be determined through the relation
\[
eE = -\frac{\partial e\phi}{\partial x} = -eBV_y = -B \left( \frac{1}{\mu_0 n} \frac{dB}{dx} \right) = -\frac{B_0^2}{\mu_0 n_0} \frac{db}{dx}.
\] (23)
where in the last step $B/n = B_0/n_0$ of eq.(18) was used. Upon integration over $x$ gives

\[ \Delta KE = -\Delta e\phi = \frac{B_0^2}{\mu_0 n_0} \left( \frac{B}{B_0} - 1 \right) = 2MV_A^2 \left( \frac{B}{B_0} - 1 \right). \] (24)

This is shown in Fig. 1d. Consider the situation of earth magnetosphere. With typical numbers: $B_0 = 20\text{nT}$, $n_0 = 1./\text{cm}^3$, we get $B^2/(\mu_0 n_0) \sim 2\text{keV}$, $v_A \sim 200\text{km/s}$, $V_0 \sim 550\text{km/s}$ and $r_A = 200\text{km}$.

It is instructive to look at the $b$-dependence of the $\Phi$ function defined in eq.(20), which is shown in Fig 1e. The soliton solution begins from $b = 1$ where $\Phi = 0$. It evolves following the curve $\Phi$ curve until it reaches the maximum point $b \approx 2.6$. After passing through this peak value it follows the $\Phi$-curve in a reversed manner until it reaches the starting point $b = 1$ where $\Phi = 0$. The corresponding negative derivative with respect to $b$ is shown in Fig 1f.

**Shock waves – damped soliton solution**

Following ref.[2], correcting for the effect due to collision along the y-component, the Lorenz force equation becomes

\[ V_x \approx \frac{q}{m}(E_x - V_x B) - \frac{1}{N} \int CV_y dV_y. \] (25)

The collision term is replaced by a simple form $-\nu V_y$, and lead to the damped soliton equation

\[ \frac{d^2 B}{d\tau^2} = -\frac{\partial \Phi}{\partial B} - \frac{\nu}{V_0} \frac{dB}{d\tau}. \] (26)

Blue oscillatory curves in Fig. 1a-d are the curves corresponding to damped soliton solution. Notice in particular Fig1a shows the damped solution approaches asymptotically the limit which corresponds to, as expected, the minimum of the $\Phi$-function of Fig1e, at

\[ b = -\frac{1}{2} + \left(2\alpha^2 + \frac{1}{4}\right)^{1/2} \] (27)
For the present case, $\alpha = 1.8$, the $b$ value where $\Phi$ is minimum occurs at $b \sim 2.1$, which is confirmed by Fig. 1e and Fig 1f. Fig 1b. shows that the outgoing speed beyond the narrow structure is finite. This implies that the damped soliton corresponds to a shock wave case. The corresponding KE gain implied in Fig 1d shows that the maximum energy gain is reduced from the soliton case although the gain here is stretched out over many times of the length-units of $c/\omega_p$.

**Acceleration of reflected electrons.** Consider the interaction between shock waves and magnetic field at oblique angles. This problem has been investigated extensive (see for example refs [5, 4]. In the relativistic domain, within a certain parameter range, one finds that the reflected electrons tend to linger near the point of reflection within some characteristic spatial interval $\Delta x$, which leads to a significant energy gain in the $y$-direction, since $E_0 \Delta y = E_0(v_y/v_x)\Delta x$, where $v_x$ is small and $\Delta x$ is finite. So the acceleration in the $y$-direction can be large.

Below we apply a similar consideration following ref.([4]), albeit nonrelativistic, to the interaction between the solar wind and the magnetosphere of the earth. We make use of the guiding-center approximation. The velocity of the guiding center of an electron is

$$\vec{v}_g = \vec{v}_d + \frac{\vec{B}}{B^2} v_d, \quad \text{with} \quad v_d \approx \frac{\vec{E} \times \vec{B}}{B^2}. \quad (28)$$

The setup is as follows. In front of the shock the plasma is traveling along the $x$-direction with a speed $v_{sh}$ and the magnetic field is in the $x$-$z$ plane. Assume the Cartesian components of the fields depend on $x$-only. Without the lost of generality, one may set $E_z = 0$. Then, it can be shown, the components of the magnetic and electric field vectors are given by $\vec{B} = (B_{x0}, B_y, B_z)$ and $\vec{E} = (E_x, E_{y0}, 0)$, where quantities with subscript ”0” are their values at the upstream, i.e. before the shock. From the setup, by definitions $B_{y0} = 0$, $v_{x0} = -v_{sh}$ and $v_{y0} = v_{z0} = 0$. The guiding center
velocity components are given by
\[
\begin{align*}
v_{gx} &= \frac{E_y B_z + v_{\parallel} B_{z0}}{B^2} \\
v_{gy} &= -\frac{E_x B_z + v_{\parallel} B_{y0}}{B^2} \\
v_{gz} &= \frac{E_z B_y - E_{y0} B_{z0}}{B^2} + v_{\parallel} \frac{B_z}{B}.
\end{align*}
\] (29)

We proceed to evaluate \(E_{y0}\). In front of the shock, averaging over all electrons in a small volume element, the guiding center velocity \(< v_{gz} > = 0\). Using \(B_{y0} = 0\),
\[
< v_{\parallel} > = \frac{E_{y0} B_{z0}}{B_{0} B_{z0}}
\] (30)

With \(< v_{gx} > = -v_{sh}\), eq.(30) lead to
\[
E_{y0} = -v_{sh} B_{z0}.
\] (31)

Consider the acceleration of electron along the \(y\)-direction from point A to B during which the electron is moving along the \(x\)-direction from \(x_A\) to \(x_B\). Integrating over the conservation of energy equation obtained by taking the dot-product of \(v\) with the Lorentz force equation,
\[
m \frac{dv}{dt} = -e (E + v \times B)
\] (32)

one arrives at
\[
\Delta KE = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2 = -e (\phi_B - \phi_A) - e E_{y0} \int_A^B \frac{v_y}{v_x} \, dx,
\] (33)

In the guiding-center approximation, using eq.(29), and eq.(30) the integrand of eq.(33) may be evaluated in following manner,
\[
\frac{v_y}{v_x} \approx \frac{v_{gy}}{v_{gz}} = -\frac{E_x B_z / B^2}{-(E_{y0} B_z / B^2) + (v_{\parallel} B_{z0} / B)} = \frac{-1}{E_{y0}} \left( 1 - \frac{(v_{\parallel} B_{z0} / (v_{sh} B_{z0}))(B / B_z)}{1 - c/(v_{sh} \tan \theta)} \right).
\] (34)
Using eq. (34), one can evaluate eq. (33), it leads to
\[
|\Delta KE| \approx \left| \frac{e(\phi_B - \phi_A)}{1 - v_{sh}/(c \cos \theta)} \right|
\]  
(35)

Here we made the replacement of \( \tan \theta \) by the corresponding relativistic factor \( 1/\cos \theta \), since one finds that [5] for relativistic case \( v_{sh} \) is replaced by \( \gamma_{sh} v_{sh} \), so the relativistic pole position is given by \( \cos \theta = v_{sh}/c \), which is eq. (47) of ref. [5].

Using eq. (24), the inverse-beta function is given by
\[
\frac{1}{\beta} = \frac{|\Delta KE|}{T_e} = \left( \frac{B_0^2}{\mu_0 n_0 T_e} \right) \left( \frac{B}{B_0} - 1 \right) \cdot \frac{1}{1 - v_{sh} \cos \theta/c}.
\]  
(36)

Fig. 2 is a plot of the inverse beta-function, i.e. the gain in KE in units of the thermal energy plotted as a function of \( \cos \theta \) for nominal values in solar-wind earth magnetosphere collision at the magnetopause. They are: \( v_{sh} = 500 \text{ km/s} \), or \( v_{sh}/c = 1.7 \times 10^{-3} \), \( B_0 = 50 \text{ nT} \), and \( T_e = 500 \text{ eV} \). Notice the pronounced energy gain in the neighborhood of the critical point indicated by the dashed line, to be compared to the horizontal line where the multiplicative denominator term is not included. Thus one sees that the mechanism considered in here provides a source generating energetic electrons, with an energy many times larger than the maximum energy inferred to in the soliton solution.

References


Figure 1: Soliton and Damped solution: a) $z$-component magnetic field, b) $x$-component velocity, c) Sagdeev potential and d) KE gain.
Figure 2: Angular dependence of the inverse-$\beta$ function, i.e. $B_0^2/(\mu_0 n_0 T_e)$, near the pole at fixed incident speed of the shock waves. The critical angle where $\cos \theta = v_{sh}/c$ is indicated by the vertical dashed curve. The horizontal line indicates the inverse-beta-function when the multiplicative pole denominator is set to unity.