ELSEVIER

Physica D 136 (2000) 205–244

PHYSICA D

www.elsevier.com/locate/physd

# Classification and Casimir invariants of Lie–Poisson brackets

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Received 18 November 1998; received in revised form 1 June 1999; accepted 14 June 1999
Communicated by J.D. Meiss

**Abstract**

We classify Lie–Poisson brackets that are formed from Lie algebra extensions. The problem is relevant because many physical systems owe their Hamiltonian structure to such brackets. A classification involves reducing all brackets to a set of normal forms, and is achieved partially through the use of Lie algebra cohomology. For extensions of order less than five, the number of normal forms is small and they involve no free parameters. We derive a general method of finding Casimir invariants of Lie–Poisson bracket extensions. The Casimir invariants of all low-order brackets are explicitly computed. We treat in detail a four field model of compressible reduced magnetohydrodynamics. ©2000 Elsevier Science B.V. All rights reserved.

*Keywords:* Casimir invariants; Lie–Poisson brackets; Hamiltonian structure

## 1. Introduction

This paper deals with the classification of Lie–Poisson brackets obtained from extensions of Lie algebras. A large class of finite- and infinite-dimensional dynamical equations admit a Hamiltonian formulation using noncanonical brackets of the Lie–Poisson type. Finite-dimensional examples include the Euler equations for the rigid body [1] and the moment reduction of the Kida vortex [2], while infinite-dimensional examples include the Vlasov equation [3,4] and the Euler equation for the ideal fluid [5–9]. Lie–Poisson brackets naturally define a Poisson structure (i.e., a symplectic structure) on the dual of a Lie algebra. For the rigid body, the Lie algebra is the one associated with the rotation group, $SO(3)$, while for the Kida vortex moment reduction the underlying group is $SO(2,1)$. For the two-dimensional ideal fluid, the relevant Lie algebra corresponds to the group of volume-preserving diffeomorphisms on the fluid domain.

We will classify low-order bracket extensions and find their Casimir invariants. An extension is simply a new Lie bracket, derived from a base algebra (for example, $SO(3)$), and defined on $n$-tuples of that algebra. We are ruling out extensions where the brackets that appear are not of the same form as that of the base algebra. We are thus omitting some brackets [5,10], but the brackets we are considering are amenable to a general classification.

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PII: S0167-2789(99)00155-4

The method of extension yields interesting and physically relevant algebras. Using this method we can describe finite-dimensional systems of several variables and infinite- dimensional systems of several fields. For finite-dimensional systems an example is the two vector model of the heavy top [11]. For infinite-dimensional systems there are models with two [12–14], three [12,15,16], and four [17] fields. Knowing the bracket allows one to find the Casimir invariants of the system [18–20]. These are quantities which commute with every functional on the Poisson manifold, and thus are conserved by the dynamics for any Hamiltonian. They are useful for analyzing the constraints in the system [21] and for establishing stability criteria (see for example [22–24] and the reviews [25] and [26]).

The outline of this paper is as follows. In Section 2, we review the general theory behind Lie–Poisson brackets. We give examples of physical systems of Lie–Poisson type, both finite- and infinite-dimensional. We introduce the concept of Lie algebra extensions and derive some of their basic properties. Section 3 is devoted to the more abstract treatment of extensions through the theory of Lie algebra cohomology [27–29]. We define some terminology and special extensions such as the semidirect sum and the Leibniz extension. In Section 4, we use the cohomology techniques to treat the specific type of extension with which we are concerned, brackets over $n$-tuples. We give an explicit classification of low-order extensions. By classifying we mean reducing — through coordinate changes — all possible brackets to independent normal forms. We find that the normal forms are relatively few and involve no free parameters — at least for low-order extensions. In Section 5, we turn to the problem of finding the Casimir invariants of the brackets, those functionals that commute with every other functional in the algebra. We derive some general techniques for doing so that apply to extensions of any order. Some explicit examples are derived, including the Casimir invariants of a particular model of magnetohydrodynamics (MHD). These are also given a physical interpretation. A formula for the invariants of Leibniz extenions of any order is also derived. Then in Section 6 we use the classification of Section 4 to derive the Casimir invariants for low-order extensions. Finally in Section 7 we offer some concluding remarks and discuss future directions.

## 2. Lie–Poisson brackets

Lie–Poisson brackets define a natural Poisson structure on duals of Lie algebras. Physically, they often arise in the *reduction* of a system. For our purposes, a reduction is a mapping of the dynamical variables of a system to a smaller set of variables, such that the transformed Hamiltonian and bracket depend only on the smaller set of variables. (For a more detailed mathematical treatment, see for example [30–34].) The simplest example of a reduction is the case in which a cyclic variable is eliminated, but more generally a reduction exists as a consequence of an underlying symmetry of the system. For instance, the Lie–Poisson bracket for the rigid body is obtained from a reduction of the canonical Euler angle description using the rotational symmetry of the system [11]. The Euler equation for the two-dimensional ideal fluid is obtained from a reduction of the Lagrangian description of the fluid, which has a relabeling symmetry [26,35–37].

Here, we shall take a more abstract viewpoint: we do not assume that the Lie–Poisson bracket is obtained from a reduction, though it is always possible to do so by the method of Clebsch variables [26]. Rather we proceed directly from a given Lie algebra to build a Lie–Poisson bracket. The choice of algebra can be guided by the symmetries of the system. After deriving the basic theory behind Lie–Poisson brackets in Section 2.1, we will show some explicit examples in Section 2.2. We then describe general Lie algebra extensions in Section 2.3.

### 2.1. Lie–Poisson brackets on duals of Lie Algebras

We begin by taking the Lie algebra $\mathfrak{g}$ associated with some Lie group. The Lie group might be chosen to reflect the symmetries of a physical system. There will be a Lie bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ associated with $\mathfrak{g}$. Consider the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ with respect to the pairing $\langle , \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$. Then for real-valued functionals $F$ and $G$, that is, $F, G : \mathfrak{g}^* \to \mathbb{R}$, and $\xi \in \mathfrak{g}^*$, we can define

$$\{F, G\}_{\pm}(\xi) = \pm \left\langle \xi, \left[ \frac{\delta F}{\delta \xi}, \frac{\delta G}{\delta \xi} \right] \right\rangle. \tag{2.1}$$

The sign choice comes from whether we are considering right invariant (+) or left invariant (−) functions on the cotangent bundle of the Lie group [9,34], but for our purposes we simply choose the sign as needed. The functional derivative $\delta F/\delta \xi$ is defined by

$$\delta F[\xi; \delta\xi] := \left. \frac{\mathrm{d}}{\mathrm{d}\epsilon} F[\xi + \epsilon\delta\xi] \right|_{\epsilon=0} =: \left\langle \delta\xi, \frac{\delta F}{\delta \xi} \right\rangle. \tag{2.2}$$

We shall refer to the bracket [ , ] as the inner bracket and to the bracket { , } as the Lie–Poisson bracket. The dual $\mathfrak{g}^*$ together with the Lie–Poisson bracket is a Poisson manifold; that is, the bracket { , } is a Lie algebra structure on real-valued functionals that is a derivation in each of its arguments. For finite-dimensional groups, Eq. (2.1) was first written down by Lie [38] and was rediscovered by Berezin [39]; it is also closely related to work of Arnold [40], Kirillov [41], Kostant [42], and Souriau [43].

The bracket in $\mathfrak{g}$ is the same as the adjoint action of $\mathfrak{g}$ on itself: $[\alpha, \beta] = \mathrm{ad}_\alpha \beta$, where $\alpha, \beta \in \mathfrak{g}$. From this we define the coadjoint action $\mathrm{ad}^\dagger_\alpha$ of $\mathfrak{g}$ on $\mathfrak{g}^*$ by

$$\langle \mathrm{ad}^\dagger_\alpha \xi, \beta \rangle := \langle \xi, \mathrm{ad}_\alpha \beta \rangle, \tag{2.3}$$

where $\xi \in \mathfrak{g}^*$. We also define the coadjoint bracket $[ , ]^\dagger : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ by $[\alpha, \xi]^\dagger := \mathrm{ad}^\dagger_\alpha \xi$, so that

$$\langle [\alpha, \xi]^\dagger, \beta \rangle := \langle \xi, [\alpha, \beta] \rangle; \tag{2.4}$$

the bracket $[, ]^\dagger$ satisfies the identity

$$\langle [\alpha, \xi]^\dagger, \beta \rangle = -\langle [\beta, \xi]^\dagger, \alpha \rangle.$$

Since the inner bracket is Lie, it satisfies the Jacobi identity, and consequently the form given by (2.1) for the Lie–Poisson bracket will automatically satisfy the Jacobi identity ([44], p. 614).

Given a Hamiltonian $H : \mathfrak{g}^* \to \mathbb{R}$, the equation of motion for $\xi \in \mathfrak{g}^*$ is

$$\dot{\xi} = \{\xi, H\} = \pm \left\langle \xi, \left[ \varDelta, \frac{\delta H}{\delta \xi} \right] \right\rangle = \mp \left\langle \left[ \frac{\delta H}{\delta \xi}, \xi \right]^\dagger, \varDelta \right\rangle = \mp \left[ \frac{\delta H}{\delta \xi}, \xi \right]^\dagger, \tag{2.5}$$

where $\varDelta$ is a Kronecker or Dirac delta, or a combination of both for an infinite-dimensional system of several fields.

### 2.2. Examples of Lie–Poisson systems

We will say that a physical systems can be described by a given Lie–Poisson bracket and Hamiltonian if its equations of motion can be written as (2.5); the system is then said to be Hamiltonian of the Lie–Poisson type. We give four examples: the first is finite-dimensional (the free rigid body, Section 2.2.1) and the second is infinite-dimensional (Euler's equation for the ideal fluid, Section 2.2.2). The third and fourth examples are also infinite-dimensional and serve to introduce the concept of extension. They are low–beta reduced magnetohydrodynamics (MHD) in Section 2.2.3 and compressible reduced MHD in Section 2.2.4. These last two examples are meant to illustrate the physical relevance of Lie algebra extensions.

*2.2.1. The free rigid body*

The classic example of a Lie–Poisson bracket is obtained by taking for $\mathfrak{g}$ the Lie algebra of the rotation group $SO(3)$. If the $\hat{\mathbf{e}}_{(i)}$ denote a basis of $\mathfrak{g} = so(3)$, the Lie bracket is given by

$$\left[\hat{\mathbf{e}}_{(i)}, \hat{\mathbf{e}}_{(j)}\right] = c_{ij}^{k}\, \hat{\mathbf{e}}_{(k)},$$

where $c_{ij}^{k} = \varepsilon_{ijk}$ are the structure constants of the algebra, in this case the totally antisymmetric symbol. Using as a pairing the usual contraction between upper and lower indices, with (2.1) we are led to the Lie–Poisson bracket

$$\{f, g\} = -c_{ij}^{k}\, \ell_k \frac{\partial f}{\partial \ell_i} \frac{\partial g}{\partial \ell_j},$$

where the three-vector $\ell$ is in $\mathfrak{g}^*$, and we have chosen the minus sign in (2.1). The coadjoint bracket is obtained using (2.3),

$$[\beta, \ell]_i^{\dagger} = -c_{ij}^{k}\, \beta^j\, \ell_k.$$

If we use this coadjoint bracket and insert the Hamiltonian

$$H = \tfrac{1}{2}(I^{-1})^{ij}\, \ell_i\, \ell_j$$

in (2.5) we obtain

$$\dot{\ell}_m = \{\ell_m, H\} = c_{mj}^{k}(I^{-1})^{jp}\, \ell_k\, \ell_p.$$

Notice how the moment of inertia tensor $I$ plays the role of a metric — it allows us to build a quadratic form (the Hamiltonian) from two elements of $\mathfrak{g}^*$. If we take $I = \mathrm{diag}(I_1, I_2, I_3)$, we recover Euler's equations for the motion of the free rigid body

$$\dot{\ell}_1 = \left(\frac{1}{I_2} - \frac{1}{I_3}\right) \ell_2\, \ell_3,$$

and cyclic permutations of 1,2,3. The $\ell_i$ are the angular momenta about the axes and the $I_i$ are the principal moments of inertia. This result is naturally appealing because we expect the rigid body equations to be invariant under the rotation group, hence the choice of $SO(3)$ for $G$.

*2.2.2. The two-dimensional ideal fluid*

Consider now an ideal fluid with the flow taking place over a two- dimensional domain $\Omega$. Let $\mathfrak{g}$ be the infinite-dimensional Lie algebra associated with the Lie group of volume-preserving diffeomorphisms of $\Omega$. In two spatial dimensions this is the same as the group of canonical transformations on $\Omega$. The bracket in $\mathfrak{g}$ is the canonical bracket

$$[a, b] = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial b}{\partial x} \frac{\partial a}{\partial y}. \tag{2.6}$$

We formally identify $\mathfrak{g}$ and $\mathfrak{g}^*$ and use as the pairing $\langle , \rangle$ the usual integral over the fluid domain,

$$\langle F, G\rangle = \int_{\Omega} F(\mathbf{x})\, G(\mathbf{x})\, \mathrm{d}^2 x,$$

where $\mathbf{x} := (x, y)$. For infinite-dimensional spaces, there are functional analytic issues about whether we can make this identification, and take $\mathfrak{g}^{**} = \mathfrak{g}$. We will assume here that these relationships hold formally. See Marsden et al. [32] for references on this subject and the book by Audin [45] for a treatment of the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$.

Assuming appropriate boundary conditions for simplicity, we get $[\,,]^\dagger = -[\,,]$ from (2.4). (Otherwise the coadjoint bracket would involve extra boundary terms.) Take the vorticity $\omega$ as the field variable $\xi$ and write for the Hamiltonian

$$H[\omega] = -\tfrac{1}{2}\langle \omega, \nabla^{-2}\omega\rangle,$$

where

$$(\nabla^{-2}\omega)(\mathbf{x}) := \int_\Omega K(\mathbf{x}|\mathbf{x}')\,\omega(\mathbf{x}')\,\mathrm{d}^2x',$$

and $K$ is Green's function for the Laplacian. The Green's function plays the role of a metric since it maps an element of $\mathfrak{g}^*$ (the vorticity $\omega$) into an element of $\mathfrak{g}$ to be used in the right slot of the pairing. This relationship is only weak: the mapping $K$ is not surjective, and thus the metric cannot formally be inverted (it is called *weakly nondegenerate*). When we have identified $\mathfrak{g}$ and $\mathfrak{g}^*$ we shall often drop the comma in the pairing and write

$$H[\omega] = -\tfrac{1}{2}\langle \omega\,\phi\rangle = \tfrac{1}{2}\langle |\nabla\phi|^2\rangle,$$

where $\omega = \nabla^2\phi$ defines the streamfunction $\phi$. We work out the evolution equation for $\omega$ explicitly:

$$\begin{aligned}\dot\omega(\mathbf{x}) = \{\omega, H\} &= \int_\Omega \omega(\mathbf{x}')\left[\frac{\delta\omega(\mathbf{x})}{\delta\omega(\mathbf{x}')}, \frac{\delta H}{\delta\omega(\mathbf{x}')}\right]\mathrm{d}^2x' = \int_\Omega \omega(\mathbf{x}')[\delta(\mathbf{x}-\mathbf{x}'), -\phi(\mathbf{x}')]\,\mathrm{d}^2x' \\ &= \int_\Omega \delta(\mathbf{x}-\mathbf{x}')[\omega(\mathbf{x}'), \phi(\mathbf{x}')]\,\mathrm{d}^2x' = [\omega(\mathbf{x}), \phi(\mathbf{x})].\end{aligned}$$

This is Euler's equation for a two-dimensional ideal fluid. We could also have written this result down directly from (2.5) using $[,]^\dagger = -[,]$.

### *2.2.3. Low–beta reduced MHD*

This example will illustrate the concept of a Lie algebra extension, the main topic of this paper. Essentially, the idea is to use an algebra of $n$-tuples, which we call an extension, to describe a physical system with more than one dynamical variable. As in Section 2.2.2 we consider a flow taking place over a two-dimensional domain $\Omega$. The Lie algebra $\mathfrak{g}$ is again taken to be that of volume-preserving diffeomorphisms on $\Omega$, but now we consider also the vector space $V$ of real-valued functions on $\Omega$ (an Abelian Lie algebra under addition). The *semidirect sum* of $\mathfrak{g}$ and $V$ is a new Lie algebra whose elements are two-tuples $(\alpha, v)$ with a bracket defined by

$$[(\alpha, v), (\beta, w)] := ([\alpha, \beta], [\alpha, w] - [\beta, v]), \tag{2.7}$$

where $\alpha$ and $\beta \in \mathfrak{g}$, $v$ and $w \in V$. This is a Lie algebra, so we can use the prescription of Section 2.1 to build a Lie–Poisson bracket,

$$\{F, G\} = \int_\Omega \left(\omega\left[\frac{\delta F}{\delta\omega}, \frac{\delta G}{\delta\omega}\right] + \psi\left(\left[\frac{\delta F}{\delta\omega}, \frac{\delta G}{\delta\psi}\right] - \left[\frac{\delta G}{\delta\omega}, \frac{\delta F}{\delta\psi}\right]\right)\right)\mathrm{d}^2x.$$

Let $\omega = \nabla^2\phi$, where $\phi$ is the electric potential, $\psi$ is the magnetic flux, and $J = \nabla^2\psi$ is the current. (We use the same symbol for the electric potential as for the streamfunction of Section 2.2.2 since they play a similar role.) The pairing used is a dot product of the vectors followed by an integral over the fluid domain (again identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ as in Section 2.2.2). The Hamiltonian

$$H[\omega; \psi] = \frac{1}{2}\int_\Omega \left(|\nabla\phi|^2 + |\nabla\psi|^2\right)\mathrm{d}^2x$$

with the above bracket leads to the equations of motion

$$\dot{\omega} = [\omega, \phi] + [\psi, J], \qquad \dot{\psi} = [\psi, \phi].$$

This is a model for low-beta reduced MHD [12,46,47]. It is obtained by an expansion in the inverse aspect ratio $\epsilon$ of a tokamak, with $\epsilon$ small. This is called low beta since the plasma beta (the ratio of plasma pressure to magnetic pressure) is of order $\epsilon^2$. With a strong toroidal magnetic field, the dynamics are then approximately two-dimensional.

Benjamin [13] used a system with a similar Lie–Poisson structure, but for waves in a density-stratified fluid. Semidirect sum structures are ubiquitous in advective systems: one variable (in this example, $\phi$) "drags" the others along [21].

### 2.2.4. Compressible reduced MHD

In general there are other, more general ways to extend Lie algebras besides the semidirect sum. The model derived by Hazeltine et al. [17,48] for two-dimensional compressible reduced MHD (CRMHD) is an example. This model has four fields, and as for the system in Section 2.2.3 it is also obtained from an expansion in the inverse aspect ratio of a tokamak. It includes compressibility and finite ion Larmor radius effects. The Hamiltonian is

$$H[\omega, v, p, \psi] = \frac{1}{2}\int_{\Omega} \left( |\nabla\phi|^2 + v^2 + \frac{(p - 2\beta_i\, x)^2}{\beta_i} + |\nabla\psi|^2 \right) \mathrm{d}^2 x, \tag{2.8}$$

where $v$ is the parallel ion velocity, $p$ the pressure, and $\beta_i$ is a parameter that measures compressibility. The other variables are as in Section 2.2.3. The coordinate $x$ points outward from the center of the tokamak in the horizontal plane and $y$ is the vertical coordinate. The motion is made two-dimensional by the strong toroidal magnetic field. The bracket we will use is

$$\begin{aligned}\{F, G\} = \int_{\Omega} \Bigg( &\omega \left[\frac{\delta F}{\delta\omega}, \frac{\delta G}{\delta\omega}\right] + v \left( \left[\frac{\delta F}{\delta\omega}, \frac{\delta G}{\delta v}\right] + \left[\frac{\delta F}{\delta v}, \frac{\delta G}{\delta\omega}\right] \right) + p \left( \left[\frac{\delta F}{\delta\omega}, \frac{\delta G}{\delta p}\right] + \left[\frac{\delta F}{\delta p}, \frac{\delta G}{\delta\omega}\right] \right) \\ &+ \psi \left( \left[\frac{\delta F}{\delta\omega}, \frac{\delta G}{\delta\psi}\right] + \left[\frac{\delta F}{\delta\psi}, \frac{\delta G}{\delta\omega}\right] \right) - \beta_i\, \psi \left( \left[\frac{\delta F}{\delta p}, \frac{\delta G}{\delta v}\right] + \left[\frac{\delta F}{\delta v}, \frac{\delta G}{\delta p}\right] \right) \Bigg) \mathrm{d}^2 x.\end{aligned} \tag{2.9}$$

Together, this bracket and the Hamiltonian (2.8) lead to the equations

$$\dot{\omega} = [\omega, \phi] + [\psi, J] + 2[p, x], \quad \dot{v} = [v, \phi] + [\psi, p] + 2\beta_i[x, \psi], \quad \dot{p} = [p, \phi] + \beta_i[\psi, v], \quad \dot{\psi} = [\psi, \phi],$$

which reduce to the example of Section 2.2.3 in the limit $v = p = \beta_i = 0$ (when compressibility effects are unimportant).

It is far from clear that the Jacobi identity for (2.9) is satisfied. A direct verification is straightforward (if tedious), but we shall see in Section 2.3 that there is an easier way.

## 2.3. General algebra extensions

We wish to generalize the types of bracket used in Sections 2.2.3 and 2.2.4. We build an algebra extension by forming an $n$-tuple of elements of a single Lie algebra $\mathfrak{g}$,

$$\alpha := (\alpha_1, \ldots, \alpha_n), \tag{2.10}$$

where $\alpha_i \in \mathfrak{g}$. The most general bracket on this $n$-tuple space obtained from a linear combination of the one in $\mathfrak{g}$ has components

$$[\alpha, \beta]_\lambda = \sum_{\mu,\nu=1}^{n} W_\lambda^{\mu\nu}\, [\alpha_\mu, \beta_\nu], \qquad \lambda = 1, \ldots, n, \tag{2.11}$$

where the $W_\lambda^{\mu\nu}$ are constants. (From now on we will assume that repeated indices are summed unless otherwise noted.) Since the bracket in $\mathfrak{g}$ is antisymmetric the $W$'s must be symmetric in their upper indices,

$$W_\lambda^{\mu\nu} = W_\lambda^{\nu\mu}. \tag{2.12}$$

This bracket must also satisfy the Jacobi identity

$$[\alpha, [\beta, \gamma]]_\lambda + [\beta, [\gamma, \alpha]]_\lambda + [\gamma, [\alpha, \beta]]_\lambda = 0, \qquad \lambda = 1, \ldots, n.$$

The first term can be written as

$$[\alpha, [\beta, \gamma]]_\lambda = W_\lambda^{\sigma\tau}\, W_\sigma^{\mu\nu}\, [\alpha_\tau, [\beta_\mu, \gamma_\nu]],$$

which when added to the other two gives

$$W_\lambda^{\sigma\tau}\, W_\sigma^{\mu\nu}([\alpha_\tau, [\beta_\mu, \gamma_\nu]] + [\beta_\tau, [\gamma_\mu, \alpha_\nu]] + [\gamma_\tau, [\alpha_\mu, \beta_\nu]]) = 0.$$

We cannot yet make use of the Jacobi identity in $\mathfrak{g}$: the subscripts of $\alpha$, $\beta$, and $\gamma$ are different in each term so they represent different elements of $\mathfrak{g}$. We first relabel the sums and then make use of the Jacobi identity in $\mathfrak{g}$ to obtain

$$(W_\lambda^{\sigma\tau}\, W_\sigma^{\mu\nu} - W_\lambda^{\sigma\nu}\, W_\sigma^{\tau\mu})\, [\alpha_\tau, [\beta_\mu, \gamma_\nu]] + (W_\lambda^{\sigma\mu}\, W_\sigma^{\nu\tau} - W_\lambda^{\sigma\nu}\, W_\sigma^{\tau\mu})\, [\beta_\mu, [\gamma_\nu, \alpha_\tau]] = 0.$$

This identity is satisfied if and only if

$$W_\lambda^{\sigma\tau}\, W_\sigma^{\mu\nu} = W_\lambda^{\sigma\nu}\, W_\sigma^{\tau\mu}, \tag{2.13}$$

which together with (2.12) implies that the quantity $W_\lambda^{\sigma\tau}\, W_\sigma^{\mu\nu}$ is symmetric in all three free upper indices. If we write the $W$'s as $n$ matrices $W^{(\nu)}$ with rows labeled by $\lambda$ and columns by $\mu$,

$$\left[W^{(\nu)}\right]_\lambda^{\ \mu} := W_\lambda^{\mu\nu}, \tag{2.14}$$

then (2.13) says that those matrices pairwise commute:

$$W^{(\nu)}\, W^{(\sigma)} = W^{(\sigma)}\, W^{(\nu)}. \tag{2.15}$$

Eqs. (2.12) and (2.15) form a necessary and sufficient condition: a set of $n$ commuting matrices of size $n \times n$ satisfying the symmetry given by (2.12) can be used to make a good Lie algebra bracket. From this Lie bracket we can build a Lie–Poisson bracket using the prescription of (2.1) to obtain

$$\{F, G\}_\pm(\xi) = \pm \sum_{\lambda,\mu,\nu=1}^{n} W_\lambda^{\mu\nu} \left\langle \xi^\lambda,\ \left[\frac{\delta F}{\delta \xi^\mu}, \frac{\delta G}{\delta \xi^\nu}\right]\right\rangle.$$

We now return to the two extension examples of Sections 2.2.3 and 2.2.4 and examine them in light of the general extension concept introduced here.

### *2.3.1. Low-beta reduced MHD*

For this example we have $(\xi^0, \xi^1) = (\omega, \psi)$, with

$$W^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad W^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The reason why we start labeling at 0 will become clearer in Section 4.4. The two $W^{(\mu)}$ commute since $W^{(0)} = I$, the identity. The tensor $W$ also satisfies the symmetry property (2.12). Hence, the bracket is a good Lie algebra bracket.

*2.3.2. Compressible reduced MHD*

We have $n = 4$ and take $(\xi^0, \xi^1, \xi^2, \xi^3) = (\omega, v, p, \psi)$, so the tensor $W$ is given by

$$W^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad W^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta_i & 0 \end{pmatrix},$$
$$W^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\beta_i & 0 & 0 \end{pmatrix}, \quad W^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.16}$$

It is easy to verify that these matrices commute and that the tensor $W$ satisfies the symmetry property (2.12), so that the Lie–Poisson bracket given by (2.9) satisfies the Jacobi identity. (See Section 4.4 for an explanation of why the labeling is chosen to begin at zero.)

## 3. Extension of a Lie algebra

In this section we review the theory of Lie algebra cohomology and its application to extensions. This is useful for shedding light on the methods used in Section 4 for classifying extensions. However, the mathematical details presented in this section can be skipped without seriously compromising the flavor of the classification scheme of Section 4.

### 3.1. Cohomology of Lie algebras

We now introduce the abstract formalism of Lie algebra cohomology. Historically there were two different reasons for the development of this theory. One, known as the Chevalley–Eilenberg formulation [27], was developed from de Rham cohomology. de Rham cohomology concerns the relationship between exact and closed differential forms, which is determined by the global properties (topology) of a differentiable manifold. A Lie group is a differentiable manifold and so has an associated de Rham cohomology. If invariant differential forms are used in the computation, one is led to the cohomology of Lie algebras presented in this section [28,29,49]. The second motivation is the one that concerns us: we will show in Section 3.2 that the extension problem — the problem of enumerating extensions of a Lie algebra — can be related to the cohomology of Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra, and let the vector space $V$ over the field $K$ (which we take to be the real numbers later) be a left $\mathfrak{g}$-module,[1] that is, there is an operator $\rho : \mathfrak{g} \times V \to V$ such that

$$\rho_\alpha(v + v') = \rho_\alpha\, v + \rho_\alpha\, v', \quad \rho_{\alpha+\alpha'} v = \rho_\alpha v + \rho_{\alpha'} v, \quad \rho_{[\alpha,\alpha']} v = [\rho_\alpha, \rho_{\alpha'}]\, v, \tag{3.1}$$

for $\alpha, \alpha' \in \mathfrak{g}$ and $v, v' \in V$. The operator $\rho$ is known as a left action. A $\mathfrak{g}$-module gives a representation of $\mathfrak{g}$ on $V$.

An $n$-dimensional $V$-valued cochain $\omega_n$ for $\mathfrak{g}$, or just $n$-cochain for short, is a skew-symmetric $n$-linear mapping

$$\omega_n : \mathfrak{g} \times \mathfrak{g} \overset{\leftarrow n \rightarrow}{\times \ldots} \times \mathfrak{g} \to V.$$

[1] When $V$ is a right $\mathfrak{g}$-module, we have $\rho_{[\alpha,\alpha']} = -[\rho_\alpha, \rho_{\alpha'}]$. The results of this section can be adapted to a right action by changing the sign every time a commutator appears.

Cochains are Lie algebra cohomology analogues of differential forms on a manifold. Addition and scalar multiplication of $n$-cochains are defined in the obvious manner by

$$(\omega_n + \omega_n')(\alpha_1, \dots, \alpha_n) := \omega_n(\alpha_1, \dots, \alpha_n) + \omega_n'(\alpha_1, \dots, \alpha_n), \quad (a\omega_n)(\alpha_1, \dots, \alpha_n) := a\omega_n(\alpha_1, \dots, \alpha_n),$$

where $\alpha_1, \dots, \alpha_n \in \mathfrak{g}$ and $a \in K$. The set of all $n$-cochains thus forms a vector space over the field $K$ and is denoted by $C^n(\mathfrak{g}, V)$. The 0-cochains are defined to be just the elements of $V$, so that $C^0(\mathfrak{g}, V) = V$.

The coboundary operator is the map between cochains,

$$s_n : C^n(\mathfrak{g}, V) \to C^{n+1}(\mathfrak{g}, V),$$

defined by

$$\begin{aligned}(s_n\omega_n)(\alpha_1, \dots, \alpha_{n+1}) &:= \sum_{i=1}^{n+1} (-)^{i+1} \rho_{\alpha_i} \omega_n(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_{n+1}) \\ &+ \sum_{\substack{j,k=1 \\ j<k}}^{n+1} (-)^{j+k} \omega_n([\alpha_j, \alpha_k], \alpha_1, \dots, \hat{\alpha}_j, \dots, \hat{\alpha}_k, \dots, \alpha_{n+1}),\end{aligned}$$

where the caret means an argument is omitted. We shall often drop the $n$ subscript on $s_n$, deducing it from the dimension of the cochain on which $s$ acts.

We shall make use mostly of the first few cases:

$$(s\omega_0)(\alpha_1) = \rho_{\alpha_1} \omega_0, \tag{3.2}$$

$$(s\omega_1)(\alpha_1, \alpha_2) = \rho_{\alpha_1} \omega_1(\alpha_2) - \rho_{\alpha_2}\omega_1(\alpha_1) - \omega_1([\alpha_1, \alpha_2]), \tag{3.3}$$

$$\begin{aligned}(s\,\omega_2)(\alpha_1, \alpha_2, \alpha_3) &= \rho_{\alpha_1}\omega_2(\alpha_2, \alpha_3) + \rho_{\alpha_2}\,\omega_2(\alpha_3, \alpha_1) + \rho_{\alpha_3}\,\omega_2(\alpha_1, \alpha_2) \\ &- \omega_2([\alpha_1, \alpha_2], \alpha_3) - \omega_2([\alpha_2, \alpha_3], \alpha_1) - \omega_2([\alpha_3, \alpha_1], \alpha_2).\end{aligned} \tag{3.4}$$

It is easy to verify that $s\omega_n$ defines an $(n+1)$-cochain, and it is straightforward (if tedious) to show that $s_{n+1}s_n = s^2 = 0$. For this to be true, the homomorphism property of $\rho$ is crucial.

An $n$-cocycle is an element $\omega_n$ of $C^n(\mathfrak{g}, V)$ such that $s_n\omega_n = 0$. An $n$-coboundary $\omega_{\text{cob}}$ is an element of $C^n(\mathfrak{g}, V)$ for which there exists an element $\omega_{n-1}$ of $C^{n-1}(\mathfrak{g}, V)$ such that $\omega_{\text{cob}} = s\omega_{n-1}$. Note that all coboundaries are cocycles, but not vice versa.

Let

$$Z^n_\rho(\mathfrak{g}, V) := \ker s_n$$

be the vector subspace of all $n$-cocycles, $Z^n_\rho(\mathfrak{g}, V) \subset C^n(\mathfrak{g}, V)$, and let

$$B^n_\rho(\mathfrak{g}, V) := \operatorname{range} s_{n-1}$$

be the vector subspace of all $n$-coboundaries, $B^n_\rho(\mathfrak{g}, V) \subset C^n(\mathfrak{g}, V)$. The $n$th cohomology group of $\mathfrak{g}$ with coefficients in $V$ is defined to be the quotient vector space:

$$H^n_\rho(\mathfrak{g}, V) := Z^n_\rho(\mathfrak{g}, V)/B^n_\rho(\mathfrak{g}, V). \tag{3.5}$$

Note that for $n > \dim \mathfrak{g}$, we have $H^n_\rho(\mathfrak{g}, V) = Z^n_\rho(\mathfrak{g}, V) = B^n_\rho(\mathfrak{g}, V) = 0$.

### 3.2. Application of cohomology to extensions

In Section 2.3 we gave a definition of extension that is specific to our problem. We will now define extensions in a more abstract manner. We then show how the cohomology of Lie algebras of Section 3.1 is related to the problem of classifying extensions. In Section 4 we will return to the more concrete concept of extension, of the form given in Section 2.3.

Let $f_i : \mathfrak{g}_i \to \mathfrak{g}_{i+1}$ be a collection of Lie algebra homomorphisms,

$$\cdots \to \mathfrak{g}_i \xrightarrow{f_i} \mathfrak{g}_{i+1} \xrightarrow{f_{i+1}} \mathfrak{g}_{i+2} \to \cdots .$$

The sequence $f_i$ is called an exact sequence of Lie algebra homomorphisms if

$$\operatorname{range} f_i = \ker f_{i+1}.$$

Let $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{a}$ be Lie algebras. The algebra $\mathfrak{h}$ is said to be an *extension* of $\mathfrak{g}$ by $\mathfrak{a}$ if there is a short exact sequence of Lie algebra homomorphisms

$$0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0. \tag{3.6}$$

The homomorphism $i$ is an insertion (injection), and $\pi$ is a projection (surjection). We shall distinguish brackets in the different algebras by appropriate subscripts. We also define $\tau : \mathfrak{g} \to \mathfrak{h}$ to be a linear mapping such that $\pi \circ \tau = 1_{|\mathfrak{g}}$ (the identity mapping in $\mathfrak{g}$). Note that $\tau$ is not unique, since the kernel of $\pi$ is not trivial. Let $\beta \in \mathfrak{h}$, $\eta \in \mathfrak{a}$; then

$$\pi[\beta, i\,\eta]_{\mathfrak{h}} = [\pi\beta, \pi\, i\eta]_{\mathfrak{g}} = 0,$$

using the homomorphism property of $\pi$ and $\pi \circ i = 0$, a consequence of the exactness of the sequence. Thus $[\beta, i\eta]_{\mathfrak{h}} \in \ker \pi = \operatorname{range} i$, and $i\,\mathfrak{a}$ is an ideal in $\mathfrak{h}$ since $[\beta, i\eta] \in i\,\mathfrak{a}$. Hence, we can form the quotient algebra $\mathfrak{h}/\mathfrak{a}$, with equivalence classes denoted by $\beta + \mathfrak{a}$. By exactness $\pi(\beta + \mathfrak{a}) = \pi\,\beta$, so $\mathfrak{g}$ is isomorphic to $\mathfrak{h}/\mathfrak{a}$ and we write $\mathfrak{g} = \mathfrak{h}/\mathfrak{a}$.

Though $i\,\mathfrak{a}$ is a subalgebra of $\mathfrak{h}$, $\tau\,\mathfrak{g}$ is not necessarily a subalgebra of $\mathfrak{h}$, for in general

$$[\tau\alpha, \tau\beta]_{\mathfrak{h}} \neq \tau[\alpha, \beta]_{\mathfrak{g}},$$

for $\alpha, \beta \in \mathfrak{g}$; that is, $\tau$ is not necessarily a homomorphism. The classification problem essentially resides in the determination of how much $\tau$ differs from a homomorphism. The cohomology machinery of Section 3.1 is the key to quantifying this difference, and we proceed to show this.

To this end, we use the algebra $\mathfrak{a}$ as the vector space $V$ of Section 3.1, so that $\mathfrak{a}$ will be a left $\mathfrak{g}$-module. We define the left action as

$$\rho_\alpha \eta := i^{-1}[\tau\alpha, i\eta]_{\mathfrak{h}} \tag{3.7}$$

for $\alpha \in \mathfrak{g}$ and $\eta \in \mathfrak{a}$. For $\mathfrak{a}$ to be a left $\mathfrak{g}$-module, we need $\rho$ to be a homomorphism, i.e., $\rho$ must satisfy (3.1). Therefore consider

$$\begin{aligned}[\rho_\alpha, \rho_\beta]\eta &= (\rho_\alpha\rho_\beta - \rho_\beta\rho_\alpha)\eta = \rho_\alpha i^{-1}[\tau\beta, i\eta]_{\mathfrak{h}} - \rho_\beta i^{-1}[\tau\alpha, i\eta]_{\mathfrak{h}} \\ &= i^{-1}\left[\tau\alpha, [\tau\beta, i\eta]_{\mathfrak{h}}\right]_{\mathfrak{h}} - i^{-1}\left[\tau\beta, [\tau\alpha, i\eta]_{\mathfrak{h}}\right]_{\mathfrak{h}},\end{aligned}$$

which upon using the Jacobi identity in $\mathfrak{h}$ becomes

$$\begin{aligned}[\rho_\alpha,\rho_\beta]\eta &= i^{-1}\left[[\tau\alpha,\tau\beta]_{\mathfrak{h}},i\eta\right]_{\mathfrak{h}} = i^{-1}\left[\tau[\alpha,\beta]_{\mathfrak{g}},i\eta\right]_{\mathfrak{h}} + i^{-1}\left[\left([\tau\alpha,\tau\beta]_{\mathfrak{h}}-\tau[\alpha,\beta]_{\mathfrak{g}}\right),i\eta\right]_{\mathfrak{h}}\\ &= \rho_{[\alpha,\beta]_{\mathfrak{g}}}\eta + i^{-1}\left[\left([\tau\alpha,\tau\beta]_{\mathfrak{h}}-\tau[\alpha,\beta]_{\mathfrak{g}}\right),i\eta\right]_{\mathfrak{h}}.\end{aligned} \tag{3.8}$$

By applying $\pi$ on the expression in parentheses of the last term of (3.8), we see that it vanishes and so is in $\ker\pi$, and by exactness it is also in $i\mathfrak{a}$. Thus the $\mathfrak{h}$ commutator above involves two elements of $i\mathfrak{a}$. We define $\omega:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{a}$ by

$$\omega(\alpha,\beta) := i^{-1}\left([\tau\alpha,\tau\beta]_{\mathfrak{h}}-\tau[\alpha,\beta]_{\mathfrak{g}}\right). \tag{3.9}$$

The mapping $i^{-1}$ is well defined on $i\mathfrak{a}$. Eq. (3.8) becomes

$$[\rho_\alpha,\rho_\beta]\eta = \rho_{[\alpha,\beta]_{\mathfrak{g}}}\eta + [\omega(\alpha,\beta),\eta]_{\mathfrak{a}}. \tag{3.10}$$

Therefore, $\rho$ satisfies the homomorphism property if either of the following is true:

1. $\mathfrak{a}$ is Abelian,
2. $\tau$ is a homomorphism,

Condition (1) implies $[,]_{\mathfrak{a}}=0$, while condition (2) means

$$[\tau\alpha,\tau\beta]_{\mathfrak{h}} = \tau[\alpha,\beta]_{\mathfrak{g}},$$

which implies $\omega\equiv 0$. If either of these conditions is satisfied, $\mathfrak{a}$ with the action $\rho$ is a left $\mathfrak{g}$-module. We treat these two cases separately in Sections 3.3 and 3.4, respectively.

### *3.3. Extension by an Abelian Lie Algebra*

In this section we assume that the homomorphism condition (1) at the end of Section 3.2 is met. Therefore, $\mathfrak{a}$ is a left $\mathfrak{g}$-module, and we can define $\mathfrak{a}$-valued cochains on $\mathfrak{g}$. In particular, $\omega$ defined by (3.9) is a 2-cochain, $\omega\in C^2(\mathfrak{g},\mathfrak{a})$, that measures the “failure” of $\tau$ to be a homomorphism. We now show, moreover, that $\omega$ is a 2-cocycle, $\omega\in Z^2_\rho(\mathfrak{g},\mathfrak{a})$. By using (3.4),

$$\begin{aligned}(s\omega)(\alpha,\beta,\gamma) &= \rho_\alpha\omega(\beta,\gamma)+\rho_\beta\omega(\gamma,\alpha)+\rho_\gamma\omega(\alpha,\beta)-\omega([\alpha,\beta]_{\mathfrak{g}},\gamma)-\omega([\beta,\gamma]_{\mathfrak{g}},\alpha)-\omega([\gamma,\alpha]_{\mathfrak{g}},\beta),\\ &= i^{-1}\left(\left[\tau\alpha,[\tau\beta,\tau\gamma]_{\mathfrak{h}}\right]_{\mathfrak{h}}+\text{cyc. perm.}\right)+i^{-1}\tau\left(\left[[\alpha,\beta]_{\mathfrak{g}},\gamma\right]_{\mathfrak{g}}+\text{cyc. perm.}\right)=0.\end{aligned}$$

The first parenthesis vanishes by the Jacobi identity in $\mathfrak{h}$, the second by the Jacobi identity in $\mathfrak{g}$, and the other terms were canceled in pairs. Hence, $\omega$ is a 2-cocycle.

Two extensions $\mathfrak{h}$ and $\mathfrak{h}'$ are equivalent if there exists a Lie algebra isomorphism $\sigma$ such that the diagram

$$\begin{tikzcd} & & \mathfrak{h} \arrow[dd, "\sigma"] \arrow[dr, "\pi"] & & \\ 0 \arrow[r] & \mathfrak{a} \arrow[ur, "i"] \arrow[dr, "i'"'] & & \mathfrak{g} \arrow[r] & 0 \\ & & \mathfrak{h}' \arrow[ur, "\pi'"'] & & \end{tikzcd} \tag{3.11}$$

is commutative, that is if $\sigma\circ i = i'$ and $\pi=\pi'\circ\sigma$.

There will be an injection $\tau$ associated with $\pi$ and a $\tau'$ associated with $\pi'$, such that $\pi\circ\tau = 1_{|\mathfrak{g}} = \pi'\circ\tau'$. The linear map $\nu=\sigma^{-1}\tau'-\tau$ must be from $\mathfrak{g}$ to $i\mathfrak{a}$, so $i^{-1}\nu\in C^1(\mathfrak{g},\mathfrak{a})$. Consider $\rho$ and $\rho'$ respectively defined using $\tau$, $i$ and $\tau'$, $i'$ by (3.7). Then

$$(\rho_\alpha - \rho'_\alpha)\eta = i^{-1}[\tau\alpha, i\eta]_{\mathfrak{h}} - i'^{-1}[\tau'\alpha, i'\eta]_{\mathfrak{h}'}, = i^{-1}[\tau\alpha, i\eta]_{\mathfrak{h}} - i^{-1}[(\nu+\tau)\alpha, i\eta]_{\mathfrak{h}}, = -i^{-1}[\nu\alpha, i\eta]_{\mathfrak{h}} = 0, \tag{3.12}$$

since $\mathfrak{a}$ is Abelian. Hence $\tau$ and $\tau'$ define the same $\rho$. Now consider the 2-cocycles $\omega$ and $\omega'$ defined from $\tau$ and $\tau'$ by (3.9). We have

$$\begin{aligned}\omega'(\alpha,\beta) - \omega(\alpha,\beta) &= i'^{-1}\left([\tau'\alpha, \tau'\beta]_{\mathfrak{h}'} - \tau'[\alpha,\beta]_{\mathfrak{g}}\right) - i^{-1}\left([\tau\alpha,\tau\beta]_{\mathfrak{h}} - \tau[\alpha,\beta]_{\mathfrak{g}}\right), \\ &= i^{-1}\left([\tau\alpha,\nu\beta]_{\mathfrak{h}} + [\nu\alpha,\tau\beta]_{\mathfrak{h}} - \nu[\alpha,\beta]_{\mathfrak{g}}\right), = \rho_\alpha(i^{-1}\nu\beta) - \rho_\beta(i^{-1}\nu\alpha) - i^{-1}\nu[\alpha,\beta]_{\mathfrak{g}}.\end{aligned}$$

Comparing this with (3.3), we see that

$$\omega' - \omega = s(i^{-1}\nu), \tag{3.13}$$

so $\omega$ and $\omega'$ differ by a coboundary. Hence they represent the same element in $H^2_\rho(\mathfrak{g},\mathfrak{a})$. Equivalent extensions uniquely define an element of the second cohomology group $H^2_\rho(\mathfrak{g},\mathfrak{a})$. Note that this is true in particular for $\mathfrak{h} = \mathfrak{h}'$, $\sigma = 1$, so that the element of $H^2_\rho(\mathfrak{g},\mathfrak{a})$ is independent of the choice of $\tau$.

We are now ready to write down explicitly the bracket in $\mathfrak{h}$. We can represent an element $\alpha \in \mathfrak{h}$ as a two-tuple: $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_1 \in \mathfrak{g}$ and $\alpha_2 \in \mathfrak{a}$ ($\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{a}$ as a vector space). The injection $i$ is then $i\alpha_2 = (0, \alpha_2)$, the projection $\pi$ is $\pi(\alpha_1,\alpha_2) = \alpha_1$, and since the extension is independent of the choice of $\tau$ we take $\tau\alpha_1 = (\alpha_1, 0)$. By linearity

$$[\alpha,\beta]_{\mathfrak{h}} = [(\alpha_1,0),(\beta_1,0)]_{\mathfrak{h}} + [(0,\alpha_2),(0,\beta_2)]_{\mathfrak{h}} + [(\alpha_1,0),(0,\beta_2)]_{\mathfrak{h}} + [(0,\alpha_2),(\beta_1,0)]_{\mathfrak{h}}.$$

We know that $[(0,\alpha_2),(0,\beta_2)]_{\mathfrak{h}} = 0$ since $\mathfrak{a}$ is Abelian. By definition of the cocycle $\omega$, Eq. (3.9), we have

$$[(\alpha_1,0),(\beta_1,0)]_{\mathfrak{h}} = [\tau\alpha_1,\tau\beta_1]_{\mathfrak{h}} = i\omega(\alpha_1,\beta_1) + \tau[\alpha_1,\beta_1]_{\mathfrak{g}} = ([\alpha_1,\beta_1]_{\mathfrak{g}}, \omega(\alpha_1,\beta_1)).$$

Finally, by the definition of $\rho$, Eq. (3.7),

$$[(\alpha_1,0),(0,\beta_2)]_{\mathfrak{h}} = [\tau\alpha_1, i\beta_2]_{\mathfrak{h}} = \rho_{\alpha_1}\beta_2,$$

and similarly for $[(0,\alpha_2),(\beta_1,0)]_{\mathfrak{h}}$, with opposite sign. So the bracket is

$$[\alpha,\beta]_{\mathfrak{h}} = \left([\alpha_1,\beta_1]_{\mathfrak{g}}, \rho_{\alpha_1}\beta_2 - \rho_{\beta_1}\alpha_2 + \omega(\alpha_1,\beta_1)\right). \tag{3.14}$$

As a check we work out the Jacobi identity in $\mathfrak{h}$:

$$\begin{aligned}\left[\alpha,[\beta,\gamma]_{\mathfrak{h}}\right]_{\mathfrak{h}} &= \left([\alpha_1,[\beta,\gamma]_1]_{\mathfrak{g}}, \rho_{\alpha_1}[\beta,\gamma]_2 - \rho_{[\beta,\gamma]_1}\alpha_2 + \omega(\alpha_1,[\beta,\gamma]_1)\right) \\ &= \left(\left[\alpha_1,[\beta_1,\gamma_1]_{\mathfrak{g}}\right]_{\mathfrak{g}}, \rho_{\alpha_1}(\rho_{\beta_1}\gamma_2 - \rho_{\gamma_1}\beta_2 + \omega(\beta_1,\gamma_1)) - \rho_{[\beta_1,\gamma_1]_{\mathfrak{g}}}\alpha_2 + \omega(\alpha_1,[\beta_1,\gamma_1]_{\mathfrak{g}})\right).\end{aligned}$$

Upon adding permutations, the first component will vanish by the Jacobi identity in $\mathfrak{g}$. We are left with

$$\begin{aligned}\left[\alpha,[\beta,\gamma]_{\mathfrak{h}}\right]_{\mathfrak{h}} + \text{cyc. perm.} = {} & \left(0, \left(\rho_{\alpha_1}\rho_{\beta_1} - \rho_{\beta_1}\rho_{\alpha_1} - \rho_{[\alpha_1,\beta_1]_{\mathfrak{g}}}\right)\gamma_2 + \rho_{\alpha_1}\omega(\beta_1,\gamma_1) - \omega([\alpha_1,\beta_1]_{\mathfrak{g}},\gamma_1)\right) \\ & + \text{cyc. perm.},\end{aligned}$$

which vanishes by the the homomorphism property of $\rho$ and the fact that $\omega$ is a 2-cocycle, Eq. (3.4).

Eq. (3.14) is the most general form of the Lie bracket for extension by an Abelian Lie algebra. It turns out that the theory of extension by a non-Abelian algebra can be reduced to the study of extension by the center of $\mathfrak{a}$, which is Abelian [29]. We will not need this fact here, as the only extensions by non-Abelian algebras we will deal with are of the simpler type of Section 3.4. If the action $\rho$ vanishes but the cocycle $\omega$ does not, the extension is called *central*.

We have thus shown that equivalent extensions are enumerated by the second cohomology group $H^2_\rho(\mathfrak{g}, \mathfrak{a})$. The coordinate transformation $\sigma$ used in (3.11) to define equivalence of extensions preserves the form of $\mathfrak{g}$ and $\mathfrak{a}$ as subsets of $\mathfrak{h}$. However, we have the freedom to choose coordinate transformations which do transform these subsets. All we require is that the isomorphism $\sigma$ between $\mathfrak{h}$ and $\mathfrak{h}'$ be a Lie algebra isomorphism. We can represent this by the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{a} & \xrightarrow{i} & \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{g} & \to & 0 \\ & & & & \downarrow^{\sigma} & & & & \\ 0 & \to & \mathfrak{a}' & \xrightarrow{i} & \mathfrak{h}' & \xrightarrow{\pi} & \mathfrak{g}' & \to & 0. \end{array} \tag{3.15}$$

The primed and the unprimed extensions are not equivalent, but they are isomorphic [50]. Cohomology for us is not the whole story, since we are interested in isomorphic extensions, but it will guide our classification scheme. We discuss this point further in Section 4.3.

### 3.4. Semidirect and direct extensions

Assume now that $\omega$ defined by (3.9) is a coboundary. By (3.13) there exists an equivalent extension with $\omega \equiv 0$. For that equivalent extension, $\tau$ is a homomorphism and condition (ii) at the end of Section 3.2 is satisfied. Thus the sequence

$$\mathfrak{h} \xleftarrow{\tau} \mathfrak{g} \leftarrow 0$$

is an exact sequence of Lie algebra homomorphisms, as well as the sequence given by (3.6). We then say that the extension is a semidirect extension (or a semidirect sum of algebras) by analogy with the group case. More generally, we say that $\mathfrak{h}$ splits if it is isomorphic to a semidirect sum, which corresponds to $\omega$ being a coboundary, not necessarily zero. If $\mathfrak{a}$ is not Abelian, then (3.12) is not satisfied and two equivalent extensions (or two different choices of $\tau$) do not necessarily lead to the same $\rho$.

Representing elements of $\mathfrak{h}$ as 2-tuples, as in Section 3.3, we can derive the bracket in $\mathfrak{h}$ for a semidirect sum,

$$[\alpha, \beta]_{\mathfrak{h}} = \left([\alpha_1, \beta_1]_{\mathfrak{g}}, \rho_{\alpha_1}\beta_2 - \rho_{\beta_1}\alpha_2 + [\alpha_2, \beta_2]_{\mathfrak{a}}\right), \tag{3.16}$$

where we have not assumed $\mathfrak{a}$ Abelian. Verifying Jacobi for (3.16) we find the $\rho$ must also satisfy

$$\rho_{\alpha_1}[\beta_2, \gamma_2]_{\mathfrak{a}} = [\rho_{\alpha_1}\beta_2, \gamma_2]_{\mathfrak{a}} + [\beta_2, \rho_{\alpha_1}\gamma_2]_{\mathfrak{a}},$$

which is trivially satisfied if $\mathfrak{a}$ is Abelian, but in general this condition states that $\rho_\alpha$ is a derivation on $\mathfrak{a}$.

Now consider the case where $i^{-1}$ is a homomorphism and $\ker i^{-1} = \text{range}\,\tau$. Then the sequence

$$0 \rightleftharpoons \mathfrak{a} \overset{i^{-1}}{\rightleftharpoons} \mathfrak{h} \overset{\tau}{\rightleftharpoons} \mathfrak{g} \rightleftharpoons 0$$

is exact in both directions and, hence, both $i$ and $\pi = \tau^{-1}$ are bijections. The action of $\mathfrak{g}$ on $\mathfrak{a}$ is

$$\rho_\alpha \eta = i^{-1}[\tau\alpha, i\eta]_{\mathfrak{h}} = [i^{-1}\tau\alpha, \eta]_{\mathfrak{a}} = 0,$$

since by exactness $i^{-1} \circ \tau = 0$. This is called a direct sum. Note that in this case the role of $\mathfrak{g}$ and $\mathfrak{a}$ is interchangeable and they are both ideals in $\mathfrak{h}$. The bracket in $\mathfrak{h}$ is easily obtained from (3.16) by letting $\rho = 0$,

$$[\alpha, \beta]_{\mathfrak{h}} = \left([\alpha_1, \beta_1]_{\mathfrak{g}}, [\alpha_2, \beta_2]_{\mathfrak{a}}\right). \tag{3.17}$$

Semidirect and direct extensions play an important role in physics. A simple example of a semidirect extension structure is when $\mathfrak{g}$ is the Lie algebra $so(3)$ associated with the rotation group $SO(3)$ and $\mathfrak{a}$ is $\mathbb{R}^3$. Their semidirect sum is the algebra of the six parameter Euclidean group of rotations and translations. That algebra can be used in a Lie–Poisson bracket to describe the dynamics of the heavy top (see for example [11,51]). We have already discussed the semidirect sum in Section 2.2.3. The bracket (2.7) is a semidirect sum, with $\mathfrak{g}$ the algebra of the group of volume-preserving diffeomorphisms and $\mathfrak{a}$ the Abelian Lie algebra of functions on $\mathbb{R}^2$. The action is just the adjoint action $\rho_\alpha\, v := [\alpha, v]$ obtained by identifying $\mathfrak{g}$ and $\mathfrak{a}$.

A Lie–Poisson bracket built from a direct extension is just a sum of the separate brackets. The interaction between the variables can only come from the Hamiltonian or from constitutive equations. For example in the baroclinic instability model of two superimposed fluid layers with different potential vorticities the two layers are coupled through the potential vorticity relation [14].

## 4. Classification of extensions of a Lie algebra

In this section we return to the main problem introduced in Section 2.3: the classification of algebra extensions built by forming $n$-tuples of elements of a single Lie algebra $\mathfrak{g}$. The elements of this Lie algebra $\mathfrak{h}$ are written as $\alpha := (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathfrak{g}$, with a bracket defined by

$$[\alpha, \beta]_\lambda = W_\lambda^{\mu\nu}[\alpha_\mu, \beta_\nu], \tag{2.11}$$

where $W_\lambda^{\mu\nu}$ are constants. We will call $n$ the *order* of the extension. Recall (see Section 2.3) the $W$'s are symmetric in their upper indices,

$$W_\lambda^{\mu\nu} = W_\lambda^{\nu\mu}, \tag{2.12}$$

and commute,

$$W^{(\nu)} W^{(\sigma)} = W^{(\sigma)} W^{(\nu)}, \tag{2.15}$$

where the $n \times n$ matrices $W^{(\nu)}$ are defined by $[W^{(\nu)}]_\lambda^{\ \mu} := W_\lambda^{\nu\mu}$. Since the $W$'s are 3-tensors we can also represent their elements by matrices obtained by fixing the lower index,

$$W_{(\lambda)} := [W_{(\lambda)}]^{\mu\nu} := W_\lambda^{\mu\nu}, \tag{4.1}$$

which are symmetric but do not commute. Either collection of matrices, (2.14) or (4.1), completely describes the Lie bracket, and which one we use will be understood by whether the parenthesized index is up or down.

What do we mean by a classification? A classification is achieved if we obtain a set of normal forms for the extensions that are independent, that is, not related by linear transformations. We use linear transformations because they preserve the Lie–Poisson structure — they amount to transformations of the $W$ tensor. We thus begin by assuming the most general $W$ possible.

We first show in Section 4.1 how an extension can be broken down into a direct sum of degenerate subblocks (degenerate in the sense that the eigenvalues have multiplicity greater than unity). The classification scheme is thus reduced to the study of a single degenerate subblock. In Section 4.2 we couch our particular extension problem in terms of the Lie algebra cohomology language of Section 3.2 and apply the techniques therein. The limitations of this cohomology approach are investigated in Section 4.3, and we look at other coordinate transformations that do not necessarily preserve the extension structure of the algebra, as expressed in diagram (3.15). In Section 4.5 we introduce a particular type of extension, called the Leibniz extension, which is in a sense the “maximal” extension. Finally, in Section 4.6 we give an explicit classification of solvable extensions up to order four.

### 4.1. Direct sum structure

A set of commuting matrices can be put into simultaneous block-diagonal form by a coordinate transformation, each block corresponding to a degenerate eigenvalue [52]. Let us denote the change of basis by a matrix $M_{\bar\beta}^{\bar\alpha}$, with inverse $\left(M^{-1}\right)_{\bar\alpha}^{\beta}$, such that the matrix $\tilde W^{(\nu)}$, whose components are given by

$$\tilde W_{\bar\beta}^{\bar\alpha\nu} = (M^{-1})_{\bar\beta}^{\lambda} W_{\lambda}^{\mu\nu}\, M_{\mu}^{\bar\alpha},$$

is in block-diagonal form for all $\nu$. However, $W_{\lambda}^{\mu\nu}$ is a 3-tensor and so the third index is also subject to the coordinate change:

$$\bar W_{\bar\beta}^{\bar\alpha\bar\gamma} = \tilde W_{\bar\beta}^{\bar\alpha\nu}\, M_{\nu}^{\bar\gamma}.$$

This last step adds linear combinations of the $\tilde W^{(\nu)}$'s together, so the $\tilde W^{(\nu)}$'s and the $\bar W^{(\bar\gamma)}$'s have the same block-diagonal structure. Note that the $\bar W$ tensors are still symmetric in their upper indices, since this property is preserved by a change of basis. So from now on we just assume that we are working in a basis where the $W^{(\nu)}$'s are block-diagonal and symmetric in their upper indices; this symmetry means that if we look at a $W$ as a cube, then in the block-diagonal basis it consists of smaller cubes along the main diagonal. This is the 3-tensor equivalent of a block-diagonal matrix.

Block-diagonalization is the first step in the classification: each block of $W$ is associated with an ideal (hence, a subalgebra) in the full $n$-tuple algebra $\mathfrak{g}$. Hence, by the definition of Section 3.4 the algebra $\mathfrak{g}$ is a direct sum of the subalgebras associated with each block. Each of these subalgebras can be studied independently, so from now on we assume that we have $n$ commuting matrices, each with $n$-fold degenerate eigenvalues. The eigenvalues can, however, be different for each matrix.

Such a set of commuting matrices can be put into lower-triangular form by a coordinate change, and again the transformation of the third index preserves this structure (though it changes the eigenvalue of each matrix). The eigenvalue of each matrix lies on the diagonal; we denote the eigenvalue of $W^{(\mu)}$ by $\Lambda^{(\mu)}$. The matrix $W_{(1)}$, which as prescribed by (4.1) consists of the first row of the lower-triangular matrices $W^{(\mu)}$, is given by

$$W_{(1)} = \begin{pmatrix} \Lambda^{(1)} & 0 & 0 & \cdots & 0 \\ \Lambda^{(2)} & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ \Lambda^{(n)} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Evidently, the symmetry of $W_{(1)}$ requires

$$\Lambda^{(\nu)} = \theta\delta_1^{\nu},$$

that is, all the matrices $W^{(\mu)}$ are nilpotent (their eigenvalues vanish) except for $W^{(1)}$ when $\theta \neq 0$. If this first eigenvalue is nonzero then it can be scaled to $\theta = 1$ by the coordinate transformation $M_{\nu}^{\bar\alpha} = \theta^{-1}\delta_{\nu}^{\bar\alpha}$. We will use the symbol $\theta$ to mean a variable that can take the value 0 or 1.

### 4.2. Connection to cohomology

We now bring together the abstract notions of Section 3 with the $n$-tuple extensions of Section 2.3. It is shown in Section 4.2.1 that we need only classify the case of $\theta = 0$. This case will be seen to correspond to solvable extensions, which we classify in Section 4.2.2.

*4.2.1. Preliminary splitting*

Assume we are in the basis described at the end of Section 4.1, and for now, suppose $\theta = 1$. The set of elements of the form $\beta = (0, \beta_2, \dots, \beta_n)$ is a nilpotent ideal in $\mathfrak{h}$ that we denote by $\mathfrak{a}$ ($\mathfrak{a}$ is thus a solvable subalgebra [53]). Hence, we can construct the algebra $\mathfrak{g} = \mathfrak{h}/\mathfrak{a}$, so that $\mathfrak{h}$ is an extension of $\mathfrak{g}$ by $\mathfrak{a}$. If $\mathfrak{g}$ is semisimple, then $\mathfrak{a}$ is the radical of $\mathfrak{h}$ (the maximal solvable ideal). It is easy to see that the elements of $\mathfrak{g}$ are of the form $\alpha = (\alpha_1, 0, \dots, 0)$. We will now see that $\mathfrak{h}$ splits; that is, there exist coordinates in which $\mathfrak{h}$ is manifestly the semidirect sum of $\mathfrak{g}$ and the (in general non-Abelian) algebra $\mathfrak{a}$.

In Appendix A we give a lower-triangular coordinate transformation that makes $W^{(1)} = I$, the identity matrix. Assuming we have effected this transformation, the mappings $i$, $\pi$, and $\tau$ of Section 3.2 are given by

$$\begin{aligned}
&i : \mathfrak{a} \to \mathfrak{h}, \qquad i(\alpha_2, \dots, \alpha_n) = (0, \alpha_2, \dots, \alpha_n),\\
&\pi : \mathfrak{h} \to \mathfrak{g}, \qquad \pi(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha_1,\\
&\tau : \mathfrak{g} \to \mathfrak{h}, \qquad \tau(\alpha_1) = (\alpha_1, 0, \dots, 0),
\end{aligned}$$

and the cocycle of (3.9) is

$$i\omega(\alpha, \beta) = [\tau\alpha, \tau\beta]_{\mathfrak{h}} - \tau[\alpha, \beta]_{\mathfrak{g}} = [(\alpha_1, 0, \dots, 0), (\beta_1, 0, \dots, 0)]_{\mathfrak{h}} - ([\alpha_1, \beta_1], 0, \dots, 0) = 0.$$

Since $\omega \equiv 0$, the extension is a semidirect sum (see Section 3.4). The coordinate transformation that made $W^{(1)} = I$ removed a coboundary, making the above cocycle vanish identically. For the case where $\mathfrak{g}$ is finite-dimensional and semisimple, we have an explicit demonstration of the Levi decomposition theorem: any finite-dimensional [2] Lie algebra $\mathfrak{h}$ (of characteristic zero) with radical $\mathfrak{a}$ is the semidirect sum of a semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{a}$ [53].

*4.2.2. Solvable extensions*

Above we assumed the eigenvalue $\theta$ of the first matrix was unity; however, if this eigenvalue vanishes, then we have a solvable algebra of $n$-tuples to begin with. Since $n$ is arbitrary we can study these two solvable cases together.

Thus, we now suppose $\mathfrak{h}$ is a solvable Lie algebra of $n$-tuples (we reuse the symbols $\mathfrak{h}$, $\mathfrak{g}$, and $\mathfrak{a}$ to parallel the notation of Section 3.1), where all of the the $W^{(\mu)}$'s are lower-triangular with zeros along the diagonal. Note that $W^{(n)} = 0$, so the set of elements of the form $\alpha = (0, \dots, 0, \alpha_n)$ forms an Abelian subalgebra of $\mathfrak{h}$. In fact, this subalgebra is an ideal. Now assume $\mathfrak{h}$ contains an Abelian ideal of order $n - m$ (the order of this ideal is at least 1), which we denote by $\mathfrak{a}$. The elements of $\mathfrak{a}$ can always be cast in the form

$$\alpha = (0, \dots, 0, \alpha_{m+1}, \dots, \alpha_n)$$

via a coordinate transformation that preserves the lower-triangular, nilpotent form of the $W^{(\mu)}$.

We also denote by $\mathfrak{g}$ the algebra of $m$-tuples with the bracket

$$[(\alpha_1, \dots, \alpha_m), (\beta_1, \dots, \beta_m)]_{\mathfrak{g}\lambda} = \sum_{\mu,\nu=1}^{m} W_{\lambda}^{\mu\nu} [\alpha_\mu, \beta_\nu], \quad \lambda = 1, \dots, m.$$

It is trivial to show that $\mathfrak{g} = \mathfrak{h}/\mathfrak{a}$, so that $\mathfrak{h}$ is an extension of $\mathfrak{g}$ by $\mathfrak{a}$. Since $\mathfrak{a}$ is Abelian we can use the formalism of Section 3.1 (the other case we used above was for $\mathfrak{a}$ non-Abelian but where the extension was semidirect). The injection and projection maps are given by

[2] The inner bracket can be infinite dimensional, but the order of the extension is finite.

$$i : \mathfrak{a} \to \mathfrak{h}, \qquad i(\alpha_{m+1}, \dots, \alpha_n) = (0, \dots, 0, \alpha_{m+1}, \dots, \alpha_n),$$
$$\pi : \mathfrak{h} \to \mathfrak{g}, \qquad \pi(\alpha_1, \alpha_2, \dots, \alpha_n) = (\alpha_1, \dots, \alpha_m),$$
$$\tau : \mathfrak{g} \to \mathfrak{h}, \qquad \tau(\alpha_1, \dots, \alpha_m) = (\alpha_1, \dots, \alpha_m, 0, \dots, 0).$$

From the definition of the action, (3.7), we have for $\alpha \in \mathfrak{g}$ and $\eta \in \mathfrak{a}$,

$$\begin{aligned} i\rho_\alpha \eta &= [\tau\alpha, i\eta]_{\mathfrak{h}} = [(\alpha_1, \dots, \alpha_m, 0, \dots, 0), (0, \dots, 0, \eta_{m+1}, \dots, \eta_n)]_{\mathfrak{h}} \\ &= \sum_{\mu=1}^{m} \sum_{\nu=m+1}^{n-1} (0, \dots, 0, W_{m+2}^{\mu\nu}[\alpha_\mu, \eta_\nu], \dots, W_n^{\mu\nu}[\alpha_\mu, \eta_\nu]). \end{aligned} \tag{4.2}$$

In addition to the action, the solvable extension is also characterized by the cocycle defined in (3.9),

$$\begin{aligned} i\omega(\alpha, \beta) &= [\tau\alpha, \tau\beta]_{\mathfrak{h}} - \tau[\alpha, \beta]_{\mathfrak{g}} = [(\alpha_1, \dots, \alpha_m, 0, \dots, 0), (\beta_1, \dots, \beta_m, 0, \dots, 0)]_{\mathfrak{h}} \\ &\quad -\tau[(\alpha_1, \dots, \alpha_m), (\beta_1, \dots, \beta_m)]_{\mathfrak{g}} \quad = \sum_{\mu,\nu=1}^{m} (0, \dots, 0, W_{m+1}^{\mu\nu}[\alpha_\mu, \beta_\nu], \dots, W_n^{\mu\nu}[\alpha_\mu, \beta_\nu]). \end{aligned} \tag{4.3}$$

We can illustrate which parts of the $W$'s contribute to the action and which to the cocycle by writing

$$W_{(\lambda)} = \left(\begin{array}{c|c} \mathbf{w}_\lambda & \mathbf{r}_\lambda \\ \hline \mathbf{r}_\lambda^{\mathrm{T}} & \mathbf{0} \end{array}\right), \quad \lambda = m+1, \dots, n, \tag{4.4}$$

where the $\mathbf{w}_\lambda$'s are $m \times m$ symmetric matrices that determine the cocycle $\omega$ and the $\mathbf{r}_\lambda$'s are $m \times (n-m)$ matrices that determine the action $\rho$. The $(n-m) \times (n-m)$ zero matrix on the bottom right of the $W_{(\lambda)}$'s is a consequence of $\mathfrak{a}$ being Abelian.

The algebra $\mathfrak{g}$ is completely characterized by the $W_{(\lambda)}$, $\lambda = 1, \dots, m$. Hence we can look for the maximal Abelian ideal of $\mathfrak{g}$ and repeat the procedure we used for the full $\mathfrak{h}$. It is straightforward to show that although coordinate transformations of $\mathfrak{g}$ might change the cocycle $\omega$ and the action $\rho$, they will not alter the *form* of Eq. (4.4).

Recall that in Section 3.1 we defined 2-coboundaries as 2-cocycles obtained from 1-cochains by the coboundary operator, $s$. The 2-coboundaries turned out to be removable obstructions to a semidirect sum structure. Here the coboundaries are associated with the parts of the $W_{(\lambda)}$ that can be removed by (a restricted class of) coordinate transformations, as shown below.

Let us explore the connection between 1-cochains and coboundaries in the present context. Since a 1-cochain is just a linear mapping from $\mathfrak{g}$ to $\mathfrak{a}$, for $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathfrak{g}$ we can write this as

$$\omega_\mu^{(1)}(\alpha) = -\sum_{\lambda=1}^{m} k_\mu^\lambda \alpha_\lambda, \qquad \mu = m+1, \dots, n, \tag{4.5}$$

where the $k_\mu^\lambda$ are arbitrary constants. To find the form of a 2-coboundary we act on the 1-cochain (4.5) with the coboundary operator; using (3.3) and (4.2) we obtain

$$\begin{aligned} \omega_\lambda^{\mathrm{cob}}(\alpha, \beta) &= (s\omega^{(1)})(\alpha, \beta) = \rho_\alpha \omega^{(1)}(\beta) + \rho_\beta \omega^{(1)}(\alpha) - \omega^{(1)}([\alpha, \beta]_{\mathfrak{g}}), \\ &= \sum_{\mu=1}^{m} \sum_{\nu=m+1}^{n} W_\lambda^{\mu\nu} \left[\alpha_\mu, \omega_\nu^{(1)}(\beta)\right] - \sum_{\mu=1}^{m} \sum_{\nu=m+1}^{n} W_\lambda^{\mu\nu} \left[\beta_\mu, \omega_\nu^{(1)}(\alpha)\right] + \sum_{\mu,\nu,\sigma=1}^{m} k_\lambda^\sigma W_\sigma^{\mu\nu}[\alpha_\mu, \beta_\nu]. \end{aligned} \tag{4.6}$$

After inserting (4.5) into (4.6) and relabeling, we obtain the general form of a 2-coboundary

$$\omega_\lambda^{\mathrm{cob}}(\alpha,\beta)=\sum_{\mu,\nu=1}^{m} V_\lambda^{\mu\nu}[\alpha_\mu,\beta_\nu],\quad \lambda=m+1,\ldots,n,$$

where

$$V_\lambda^{\mu\nu}:=\sum_{\tau=1}^{m}k_\lambda^\tau W_\tau^{\mu\nu}-\sum_{\sigma=m+1}^{n}(k_\sigma^\mu W_\lambda^{\nu\sigma}+k_\sigma^\nu W_\lambda^{\mu\sigma}). \tag{4.7}$$

To see how coboundaries are removed, consider the lower-triangular coordinate transformation

$$[M_\sigma^{\bar\tau}]=\left(\begin{array}{c|c}\mathbf{I}&\mathbf{0}\\ \hline \mathbf{k}&c\mathbf{I}\end{array}\right),$$

where $\sigma$ labels rows. This transformation subtracts $V_{(\lambda)}$ from $W_{(\lambda)}$ for $\lambda>m$ and leaves the first $m$ of the $W_{(\lambda)}$'s unchanged. In other words, if $\bar W$ is the transformed $W$,

$$\bar W_{(\lambda)}=\begin{cases} W_{(\lambda)}, & \lambda=1,\ldots,m,\\ \left(\begin{array}{c|c} c^{-1}(\mathbf{w}_\lambda-\mathbf{V}_\lambda) & \mathbf{r}_\lambda\\ \hline \mathbf{r}_\lambda^{\mathrm{T}} & \mathbf{0}\end{array}\right), & \lambda=m+1,\ldots,n.\end{cases} \tag{4.8}$$

We have also included in this transformation an arbitrary scale factor $c$. Since by (4.3) the block in the upper-left characterizes the cocycle, we see that the transformed cocycle is the cocycle characterized by $\mathbf{w}_\lambda$ minus the coboundary characterized by $\mathbf{V}_\lambda$.

The special case we will encounter most often is when the maximal Abelian ideal of $\mathfrak{h}$ simply consists of elements of the form $(0,\ldots,0,\alpha_n)$. For this case $m=n-1$, and the action vanishes since $W_n^{\mu n}=0$ (the extension is central). The cocycle $\omega$ is entirely determined by $W_{(n)}$. The form of the coboundary is reduced to

$$V_n^{\mu\nu}=\sum_{\tau=1}^{n-1}k_n^\tau W_\tau^{\mu\nu}, \tag{4.9}$$

that is, a linear combinations of the first $(n-1)$ matrices. Thus it is easy to see at a glance which parts of the cocycle characterized $W_{(n)}$ can be removed by lower-triangular coordinate transformations.

### *4.3. Further coordinate transformations*

In the previous section we restricted ourselves to lower-triangular coordinate transformations, which in general preserve the lower-triangular structure of the $W^{(\mu)}$. But when the matrices are relatively sparse, there exist non-lower-triangular coordinate transformations that nonetheless preserve the lower-triangular structure. As alluded to in Section 3.3, these transformations are outside the scope of cohomology theory, which is restricted to transformations that preserve the exact form of the action and the algebras $\mathfrak{g}$ and $\mathfrak{a}$, as shown by (4.8). In other words, cohomology theory classifies extensions *given* $\mathfrak{g}$, $\mathfrak{a}$, and $\rho$. We need not obey this restriction. We can allow non-lower-triangular coordinate transformations as long as they preserve the lower-triangular structure of the $W^{(\mu)}$'s.

We now discuss a particular class of such transformations that will be useful in Section 4.6. Consider the case where both the algebra of $(n-1)$-tuples $\mathfrak{g}$ and that of 1-tuples $\mathfrak{a}$ are Abelian. Then the possible (solvable) extensions, in lower-triangular form, are characterized by $W_{(\lambda)}=0$, $\lambda=1,\ldots,n-1$, with $W_{(n)}$ arbitrary (except for $W_n^{\mu n}=0$). Let us apply a coordinate change of the form

$$M=\left(\begin{array}{c|c}\mathbf{m}&\mathbf{0}\\ \hline \mathbf{0}&c\end{array}\right),$$

where $\mathbf{m}$ is an $(n-1)\times(n-1)$ nonsingular matrix and $c$ is again a nonzero scale factor. Denoting by $\bar{W}$ the transformed $W$, we have

$$\bar{W}_{(\lambda)} = \begin{cases} 0, & \lambda = 1,\dots,n-1, \\ \left(\begin{array}{c|c} c^{-1}\mathbf{m}^{\mathrm{T}}\mathbf{w}_\lambda\mathbf{m} & \mathbf{0} \\ \hline \mathbf{0} & 0 \end{array}\right), & \lambda = n. \end{cases} \tag{4.10}$$

This transformation does not change the lower-triangular form of the extension, even if $\mathbf{m}$ is not lower-triangular. The manner in which $\mathbf{w}_n$ is transformed by $M$ is very similar to that of a (possibly singular) metric tensor: it can be diagonalized and rescaled such that all its eigenvalues are 0 or $\pm 1$. We can also change the overall sign of the eigenvalues using $c$ (something that cannot be done for a metric tensor). Hence, we shall order the eigenvalues such that the $+1$'s come first, followed by the $-1$'s, and finally by the 0's. We will show in Section 4.6 how the negative eigenvalues can be eliminated to harmonize the notation.

### 4.4. *Appending a semisimple part*

In Section 4.2 we showed that because of the Levi decomposition theorem we only needed to classify the solvable part of the extension for a given degenerate block. Most physical applications have a semisimple part ($\theta = 1$); when this is so, we shall label the matrices by $W^{(0)}, W^{(1)}, \dots, W^{(n)}$, where they are now of size $n+1$ and $W^{(0)}$ is the identity.[3] Thus the matrices labeled by $W^{(1)}, \dots, W^{(n)}$ will always form a solvable subalgebra. This explains the labeling in Sections 2.3.1 and 2.3.2.

If the extension has a semisimple part ($\theta = 1$, or equivalently $W^{(0)} = I$), we shall refer to it as *semidirect*. This was the case treated in Section 4.2.1. If the extension is not semidirect, then it is solvable (and contains $n$ matrices instead of $n+1$).

Given a solvable algebra of $n$-tuples we can carry out in some sense the inverse of the Levi decomposition and append a semisimple part to the extension. Effectively, this means that the $n\times n$ matrices $W^{(1)}, \dots, W^{(n)}$ are made $n+1\times n+1$ by adding a row and column of zeros. Then we simply append the matrix $W^{(0)} = I$ to the extension. In this manner we construct a semisimple extension from a solvable one. This is useful since we will be classifying solvable extensions, and afterwards we will want to recover their semidirect counterpart.

The extension obtained by appending a semisimple part to the completely Abelian algebra of $n$-tuples will be called *pure semidirect*. It is characterized by $W^{(0)} = I$, and $W_\lambda^{\mu\nu} = 0$ for $\mu, \nu > 0$.

### 4.5. *Leibniz extension*

A particular extension that we shall consider is called the Leibniz extension [54]. For the solvable case this extension has the form

$$W^{(1)} := N = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \cdots & \cdots & \\ & & & 1 & 0 \end{pmatrix} \tag{4.11}$$

[3] The term semisimple is not quite precise: if the base algebra is not semisimple then neither is the extension. However, we will use the term to distinguish the different cases.

or $W_\lambda^{\mu 1} = \delta_{\lambda-1}^{\mu}$, $\lambda > 1$; i.e. the first matrix is an $n \times n$ Jordan block. In this case the other matrices, in order to commute with $W^{(1)}$, must be in striped lower-triangular form [52]. After using the symmetry of the upper indices the matrices can be reduced to

$$W^{(\nu)} = (N)^{\nu}, \tag{4.12}$$

where on the right-hand side the $\nu$ denotes an exponent, not a superscript. An equivalent way of characterizing the Leibniz extension is

$$W_\lambda^{\mu\nu} = \delta_\lambda^{\mu+\nu}, \quad \mu, \nu, \lambda = 1, \dots, n. \tag{4.13}$$

The tensor $\delta$ is an ordinary Kronecker delta. Note that neither (4.12) nor (4.13) are covariant expressions, reflecting the coordinate-dependent nature of the Leibniz extension.

The Leibniz extension is in some sense a "maximal" extension: it is the only extension that has $W_{(\lambda)} \neq 0$ for all $\lambda = 2, \dots, n$ (up to coordinate transformations). Its uniqueness will become clear in Section 4.6, and is discussed in Thiffeault [55].

To construct the semidirect Leibniz extension, we append $W^{(0)} = I$, a square matrix of size $n+1$, to the solvable Leibniz extension above, as described in Section 4.4.

### 4.6. Low-order extensions

We now classify the algebra extensions of low order. As demonstrated in Section 4.2 we only need to classify solvable algebras, which means that $W^{(n)} = 0$ for all cases. We will do the classification up to order $n = 4$. For each case we first write down the most general set of lower-triangular matrices $W^{(\nu)}$ (we have already used the fact that a set of commuting matrices can be lower-triangularized) with the symmetry $W_\lambda^{\mu\nu} = W_\lambda^{\nu\mu}$ built in. Then we look at what sort of restrictions the commutativity of the matrices places on the elements. Finally, we eliminate coboundaries for each case by the methods of Sections 4.2 and 4.3. This requires coordinate transformations, but we usually will not bother using new symbols and just assume the transformation was effected.

Note that, due to the lower-triangular structure of the extensions, the classification found for an $m$-tuple algebra applies to the first $m$ elements of an $n$-tuple algebra, $n > m$. Thus, $W_{(n)}$ is the cocycle that contains all of the new information not included in the previous $m = n - 1$ classification. These comments will become clearer as we proceed.

We shall call an order $n$ extension *trivial* if $W_{(n)} \equiv 0$, so that the cocycle appended to the order $n-1$ extension contributes nothing to the bracket.

#### 4.6.1. n=1

This case is Abelian, with the only possible element $W_1^{11} = 0$.

#### 4.6.2. n=2

The most general lower-triangular form for the matrices is

$$W^{(1)} = \begin{pmatrix} 0 & 0 \\ W_2^{11} & 0 \end{pmatrix}, \quad W^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If $W_2^{11} \neq 0$, then we can rescale it to unity. Hence we let $W_2^{11} := \theta_1$, where $\theta_1 = 0$ or 1. The case $\theta_1 = 0$ is the Abelian case, while for $\theta = 1$ we have the $n = 2$ Leibniz extension (Section 4.5). Thus for $n = 2$ there are only two possible algebras. The cocycle which we have added at this stage is characterized by $\theta_1$.

*4.6.3. n=3*

Using the result of Section 4.6.2, the most general lower-triangular form is

$$W^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ \theta_1 & 0 & 0 \\ W_3^{11} & W_3^{21} & 0 \end{pmatrix}, \quad W^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ W_3^{21} & W_3^{22} & 0 \end{pmatrix},$$

and $W^{(3)} = 0$. These satisfy the symmetry condition (2.12), and the requirement that the matrices commute leads to the condition

$$\theta_1 W_3^{22} = 0.$$

The symmetric matrix representing the cocycle is

$$W_{(3)} = \begin{pmatrix} W_3^{11} & W_3^{21} & 0 \\ W_3^{21} & W_3^{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.14}$$

If $\theta_1 = 1$, then $W_3^{22}$ must vanish. Then, by (4.9) we can remove from $W_{(3)}$ a multiple of $W_{(2)}$, and therefore we may assume $W_3^{11}$ vanishes. A suitable rescaling allows us to write $W_3^{21} = \theta_2$, where $\theta_2 = 0$ or 1. The cocycle for the case $\theta_1 = 1$ is thus

$$W_{(3)} = \begin{pmatrix} 0 & \theta_2 & 0 \\ \theta_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $\theta_2 = 1$ we have the Leibniz extension (Section 4.5).

If $\theta_1 = 0$, we have the case discussed in Section 4.3. For this case we can diagonalize and rescale $W_{(3)}$ such that

$$W_{(3)} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $(\lambda_1, \lambda_2)$ can be $(1, 1)$, $(1, 0)$, $(0, 0)$, or $(1, -1)$. This last case, as alluded to at the end of Section 4.3, can be transformed so that it corresponds to $\theta_1 = 0, \theta_2 = 1$. The choice $(1, 0)$ can be transformed to the $\theta_1 = 1, \theta_2 = 0$ case. Finally for $(\lambda_1, \lambda_2) = (1, 1)$ we can use the complex transformation

$$\xi^1 \to \frac{1}{\sqrt{2}}\left(\xi^1 + \xi^2\right), \quad \xi^2 \to -\frac{i}{\sqrt{2}}\left(\xi^1 - \xi^2\right), \quad \xi^3 \to \xi^3,$$

to transform to the $\theta_1 = 0, \theta_2 = 1$ case.

We allow complex transformations in our classification because we are chiefly interested in finding Casimir invariants for Lie–Poisson brackets. If we disallow complex transformations, the final classification would contain a few more members. The use of complex transformations will be noted as we proceed.

There are thus four independent extensions for $n = 3$, corresponding to

$$(\theta_1, \theta_2) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

These will be referred to as Cases 1–4, respectively. Cases 1 and 3 have $\theta_2 = 0$, and so are trivial ($W_{(3)} = 0$). Case 2 is the solvable part of the compressible reduced MHD bracket (Section 2.3.2). Case 4 is the solvable Leibniz extension.

*4.6.4. n=4*

Proceeding as before and using the result of Sections 4.6.2 and 4.6.3, we now know that we need only write

$$W_{(4)} = \begin{pmatrix} W_4^{11} & W_4^{21} & W_4^{31} & 0 \\ W_4^{21} & W_4^{22} & W_4^{32} & 0 \\ W_4^{31} & W_4^{32} & W_4^{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.15}$$

The matrices $W_{(1)}$, $W_{(2)}$, and $W_{(3)}$ are given by their $n = 3$ analogues padded with an extra row and column of zeros (owing to the lower-triangular form of the matrices). The requirement that the matrices $W^{(1)} \cdots W^{(4)}$ commute leads to the conditions

$$\theta_2 W_4^{33} = 0, \quad \theta_2 W_4^{31} = \theta_1 W_4^{22}, \quad \theta_2 W_4^{32} = 0, \quad \theta_1 W_4^{32} = 0. \tag{4.16}$$

There are four cases to look at, corresponding to the possible values of $\theta_1$ and $\theta_2$.

**Case 1** ($\theta_1 = 0, \theta_2 = 0$). This is the unconstrained case discussed in Section 4.3, that is, all the commutation relations (4.16) are automatically satisfied. We can diagonalize to give

$$W_{(4)} = \begin{pmatrix} \lambda'_1 & 0 & 0 & 0 \\ 0 & \lambda'_2 & 0 & 0 \\ 0 & 0 & \lambda'_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$(\lambda'_1, \lambda'_2, \lambda'_3) \in \{(1,1,1), (1,1,0), (1,0,0), (0,0,0), (1,1,-1), (1,-1,0)\},$$

so there are six distinct cases. The exact form of the transformation is unimportant, but the $(1, 1, 0)$ extension can be mapped to Case 2 (the transformation is complex, $(1, 0, 0)$ can be mapped to Case 3(a), and $(1, -1, 0)$ can be mapped to Case 2. Finally the $(1, 1, 1)$ extension can be mapped to the $(1, 1, -1)$ case by a complex transformation. After transforming that $(1, 1, -1)$ case, we are left with

$$W_{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

These will be called Cases 1(a) and 1(b).

**Case 2** ($\theta_1 = 0, \theta_2 = 1$). The commutation relations (4.16) reduce to $W_4^{31} = W_4^{32} = W_4^{33} = 0$, and we have

$$W_{(4)} = \begin{pmatrix} W_4^{11} & W_4^{21} & 0 & 0 \\ W_4^{21} & W_4^{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can remove $W_4^{21}$ because it is a coboundary (in this case a multiple of $W_{(3)}$). We can also rescale appropriately to obtain four possible extensions: $W_{(4)} = 0$, and

$$W_{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Again, the form of the transformation is unimportant, but it turns out that the first of the above extensions can be mapped to Case 3(c), and the second and third to Case 3(b). This last transformation is complex. Thus there is only one independent possibility, the trivial extension $W_{(4)} = 0$.

**Case 3** ($\theta_1 = 1, \theta_2 = 0$). We can remove $W_4^{11}$ using a coordinate transformation. From the commutation requirement (4.16) we obtain $W_4^{22} = W_4^{32} = 0$. We are left with $W_{(3)} = 0$ and

$$W_{(4)} = \begin{pmatrix} 0 & W_4^{21} & W_4^{31} & 0 \\ W_4^{21} & 0 & 0 & 0 \\ W_4^{31} & 0 & W_4^{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the fact that elements of the form $(0, \alpha_2, 0, \alpha_4)$ are an Abelian ideal of this bracket, we find that $W_4^{33} W_4^{31} = 0$. Using an upper-triangular transformation we can also make $W_4^{21} W_4^{31} = 0$. After suitable rescalings we find there are five cases. One of these,

$$W_{(4)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

may be mapped to Case 4 (below) with $\theta_3 = 0$. We are thus left with four cases: the trivial extensions, $W_{(4)} = 0$, and

$$W_{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We will refer to these four extensions as Cases 3(a)–(d), respectively (Case 3(a) is the trivial extension).

**Case 4** ($\theta_1 = 1, \theta_2 = 1$). The elements $W_4^{11}$ and $W_4^{21}$ are coboundaries that can be removed by a coordinate transformation. From (4.16) we have $W_4^{33} = W_4^{32} = 0$, $W_4^{22} = W_4^{31} =: \theta_3$, so that

$$W_{(4)} = \begin{pmatrix} 0 & 0 & \theta_3 & 0 \\ 0 & \theta_3 & 0 & 0 \\ \theta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $\theta_3 = 1$ we have the Leibniz extension. The two cases will be referred to as Case 4(a) for $\theta_3 = 0$ and 4(b) for $\theta_3 = 1$.

Table 1 summarizes the results. There are are total of nine independent $n = 4$ extensions, four of which are trivial ($W_{(4)} = 0$). As noted in Section 4.5 only the Leibniz extension, Case 4(b), has nonvanishing $W_{(i)}$ for all $1 < i \leq n$.

The surprising fact is that even to order four the normal forms of the extensions involve no free parameters: all entries in the coefficients of the bracket are either zero or one. There is no obvious reason this should hold true if

Table 1
Enumeration of the independent extensions up to $n = 4$ (we have $W_{(1)} = 0$ for all the cases, and we have left out a row and a column of zeros at the end of each matrix; we have also omitted cases 1–4(a), for which $W_{(4)} = 0$)

| Case | $W_{(2)}$ | $W_{(3)}$ | $W_{(4)}$ b | c | d |
|---|---|---|---|---|---|
| 1 | (0) | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | | |
| 2 | (0) | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | | | |
| 3 | (1) | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| 4 | (1) | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | | |

we try to classify extensions of order $n > 4$. It would be interesting to find out, but the classification scheme used in this paper becomes prohibitive at such high order. The problem is that some of the transformations used to relate extensions cannot be systematically derived and were obtained by educated guessing.

## 5. Casimir invariants for extensions

In this section we will use the bracket extensions of Section 4 to make Lie–Poisson brackets, following the prescription of Section 2. In Section 5.1 we write down the general form of the Casimir condition (the condition under which a functional is a Casimir invariant) for a general class of inner brackets. Then in Section 5.2 we see how the Casimirs separate for a direct sum of algebras, the case discussed in Section 4.1. Section 5.3 discusses the particular properties of Casimirs of solvable extensions. In Section 5.4 we give a general solution to the Casimir problem and introduce the concept of *coextension*. Finally, in Section 5.5 we work out the Casimir invariants for some specific examples, including CRMHD and the Leibniz extension.

### *5.1. Casimir condition*

A generalized Casimir invariant (or Casimir for short) is a function $C : \mathfrak{g}^* \to \mathbb{R}$ for which

$$\{F, C\} \equiv 0,$$

for all $F : \mathfrak{g}^* \to \mathbb{R}$. Using (2.1) and (2.4), we can write this as

$$\left\langle \xi, \left[\frac{\delta F}{\delta \xi}, \frac{\delta C}{\delta \xi}\right] \right\rangle = -\left\langle \left[\frac{\delta C}{\delta \xi}, \xi\right]^{\dagger}, \frac{\delta F}{\delta \xi} \right\rangle.$$

Since this vanishes for all $F$ we conclude

$$\left[\frac{\delta C}{\delta \xi}, \xi\right]^{\dagger} = 0. \tag{5.1}$$

To figure out the coadjoint bracket corresponding to (2.11), we write

$$\langle \xi, [\alpha, \beta] \rangle = \langle \xi^\lambda, W_\lambda^{\mu\nu} [\alpha_\mu, \beta_\nu] \rangle,$$

which after using the coadjoint bracket of $\mathfrak{g}$ becomes

$$\langle [\beta, \xi]^\dagger, \alpha \rangle = \left\langle W_\lambda^{\mu\nu} \left[\beta_\nu, \xi^\lambda\right]^\dagger, \alpha_\mu \right\rangle$$

so that

$$[\beta, \xi]^{\dagger\nu} = W_\lambda^{\mu\nu} \left[\beta_\mu, \xi^\lambda\right]^\dagger.$$

We can now write the Casimir condition (5.1) for the bracket by extension as

$$W_\lambda^{\mu\nu} \left[ \frac{\delta C}{\delta \xi^\mu}, \xi^\lambda \right]^\dagger = 0, \quad \nu = 0, \ldots, n. \tag{5.2}$$

We now specialize the bracket to the case of most interested to us, where the inner bracket is of canonical form (2.6). As we saw in Section 2, this is the bracket for 2D fluid flows. The construction we give here has a finite-dimensional analogue, where one uses the Cartan–Killing form to map vectors to covectors, but we will not pursue this here (see [55]). Further, we assume that the form of the Casimir invariants is

$$C[\xi] = \int_\Omega \mathcal{C}(\xi(\mathbf{x}))\, \mathrm{d}^2 x, \tag{5.3}$$

and thus, since $\mathcal{C}$ does not contain derivatives of $\xi$, functional derivatives of $C$ can be written as ordinary partial derivatives of $\mathcal{C}$. We can then rewrite (5.2) as

$$W_\lambda^{\mu\nu} \frac{\partial^2 \mathcal{C}}{\partial \xi^\mu \partial \xi^\sigma} \left[\xi^\sigma, \xi^\lambda\right] = 0, \quad \nu = 0, \ldots, n. \tag{5.4}$$

In the canonical case where the inner bracket is like (2.6) the $\left[\xi^\sigma, \xi^\lambda\right]$ are independent and antisymmetric in $\lambda$ and $\sigma$. Thus a necessary and sufficient condition for the Casimir condition to be satisfied is

$$W_\lambda^{\mu\nu} \frac{\partial^2 \mathcal{C}}{\partial \xi^\mu \partial \xi^\sigma} = W_\sigma^{\mu\nu} \frac{\partial^2 \mathcal{C}}{\partial \xi^\mu \partial \xi^\lambda}, \tag{5.5}$$

for $\lambda, \sigma, \nu = 0, \ldots, n$. Sometimes we shall abbreviate this as

$$W_\lambda^{\mu\nu} \mathcal{C}_{,\mu\sigma} = W_\sigma^{\mu\nu} \mathcal{C}_{,\mu\lambda}, \tag{5.6}$$

that is, any subscript $\mu$ on $\mathcal{C}$ following a comma indicates differentiation with respect to $\xi^\mu$. (5.6) is trivially satisfied when $\mathcal{C}$ is a linear function of the $\xi$'s. That solution usually follows from special cases of more general solutions, and we shall only mention it in Section 5.4.2 where it is the only solution.

An important result is immediate from (5.6) for a semidirect extension. Whenever the extension is semidirect we shall label the variables $\xi^0, \xi^1, \ldots, \xi^n$, because the subset $\xi^1, \ldots, \xi^n$ then forms a solvable subalgebra (see Section 4.4 for terminology). For a semidirect extension, $W^{(0)}$ is the identity matrix, and thus (5.6) gives

$$\delta_\lambda^\mu \mathcal{C}_{,\mu\sigma} = \delta_\sigma^\mu \mathcal{C}_{,\mu\lambda}, \qquad \mathcal{C}_{,\lambda\sigma} = \mathcal{C}_{,\sigma\lambda},$$

which is satisfied because we can interchange the order of differentiation. Hence, $\nu = 0$ does not lead to any conditions on the Casimir. However, the variables $\mu, \lambda, \sigma$ still take values from 0 to $n$ in (5.6).

### 5.2. Direct sum

For the direct sum we found in Section 4.1 that if we look at the 3-tensor $W$ as a cube, then it "blocks out" into smaller cubes, or subblocks, along its main diagonal, each subblock representing a subalgebra. We denote each subblock of $W_\lambda^{\mu\nu}$ by $W_{i\lambda}^{\mu\nu}$, $i = 1, \dots, r$, where $r$ is the number of subblocks. We can rewrite (2.1) as

$$\{A, B\} = \sum_{i=1}^{r} \left\langle \xi_i^\lambda, W_{i_\lambda}^{\mu\nu} \left[ \frac{\delta A}{\delta \xi_i^\mu}, \frac{\delta B}{\delta \xi_i^\nu} \right] \right\rangle =: \sum_{i=1}^{r} \{A, B\}_i,$$

where $i$ labels the different subblocks and the greek indices run over the size of the $i$th subblock. Each of the subbrackets $\{,\}_i$ depends on different fields. In particular, if the functional $C$ is a Casimir, then, for any functional $F$

$$\{F, C\} = \sum_{i=1}^{r} \{F, C\}_i = 0 \Rightarrow \{F, C\}_i = 0, \qquad i = 1, \dots, r.$$

The solution for this is

$$C[\xi] = C_1[\xi_1] + \cdots + C_r[\xi_r], \quad \text{where} \{F, C_i\}_i = 0, \quad i = 1, \dots, r,$$

that is, the Casimir is just the sum of the Casimir for each subbracket. Hence, the question of finding the Casimirs can be treated separately for each component of the direct sum. We thus assume we are working on a single degenerate subblock, as we did for the classification in Section 4, and henceforth we drop the subscript $i$.

There is a complication when a single (degenerate) subblock has more than one simultaneous eigenvector. By this we mean $k$ vectors $u^{(a)}$, $a = 1, \dots, k$, such that

$$W_\lambda^{\mu(\nu)} u_\mu^{(a)} = \Lambda^{(\nu)} u_\lambda^{(a)}.$$

Note that lower-triangular matrices always have at least the simultaneous eigenvector $u_\mu = \delta_\mu^n$. Let $\eta^{(a)} := u_\rho^{(a)} \xi^\rho$, and consider a form $\mathcal{C}(\eta^{(1)}, \dots, \eta^{(k)})$ for the Casimir. Then

$$W_\lambda^{\mu(\nu)} \frac{\partial^2 \mathcal{C}}{\partial \xi^\mu \partial \xi^\sigma} = W_\lambda^{\mu(\nu)} \sum_{a,b=1}^{k} u_\mu^{(a)} u_\sigma^{(b)} \frac{\partial^2 \mathcal{C}}{\partial \eta^{(a)} \partial \eta^{(b)}} = \Lambda^{(\nu)} \sum_{a,b=1}^{k} u_\lambda^{(a)} u_\sigma^{(b)} \frac{\partial^2 \mathcal{C}}{\partial \eta^{(a)} \partial \eta^{(b)}}.$$

Because the eigenvalue $\Lambda^{(\nu)}$ does not depend on $a$ (the block was assumed to have degenerate eigenvalues), the above expression is symmetric in $\lambda$ and $\sigma$. Hence, the Casimir condition (5.5) is satisfied.

The reason this is introduced here is that if a degenerate block splits into a direct sum, then it will have several simultaneous eigenvectors. The Casimir invariants $\mathcal{C}^{(a)}\left(\eta^{(a)}\right)$ and $\mathcal{C}^{(b)}\left(\eta^{(b)}\right)$ corresponding to each eigenvector, instead of adding as $\mathcal{C}^{(a)}\left(\eta^{(a)}\right) + \mathcal{C}^{(b)}\left(\eta^{(b)}\right)$, will combine into one function, $\mathcal{C}\left(\eta^{(a)}, \eta^{(b)}\right)$, a more general functional dependence. However, these situations with more than one eigenvector are not limited to direct sums. For instance, they occur in semidirect sums. In Section 6 we will see examples of both cases.

### 5.3. Local Casimirs for solvable extensions

In the solvable case, when all the $W^{(\mu)}$'s are lower-triangular with vanishing eigenvalues, a special situation occurs. If we consider the Casimir condition (5.4), we notice that derivatives with respect to $\xi^n$ do not occur at all, since $W^{(n)} = 0$. Hence the functional

$$C[\xi] = \int_\Omega \xi^n(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') \, \mathrm{d}^2 x' = \xi^n(\mathbf{x})$$

is conserved. The variable $\xi^n(\mathbf{x})$ is *locally* conserved. It cannot have any dynamics associated with it. This holds true for any other simultaneous null eigenvectors the extension happens to have, but for the solvable case $\xi^n$ is always such a vector (provided the matrices have been put in lower-triangular form, of course).

Hence there are at most $n-1$ dynamical variables in an order $n$ solvable extension. An interesting special case occurs when the only nonvanishing $W_{(\mu)}$ is for $\mu = n$. Then the Lie–Poisson bracket is

$$\{F, G\} = \sum_{\mu,\nu=1}^{n-1} W_n^{\mu\nu} \int_\Omega \xi^n(\mathbf{x}) \left[\frac{\delta F}{\delta \xi^\mu(\mathbf{x})}, \frac{\delta G}{\delta \xi^\nu(\mathbf{x})}\right] \mathrm{d}^2 x,$$

where $\xi^n(\mathbf{x})$ is some function of our choosing. This bracket is not what we would normally call Lie–Poisson because $\xi^n(\mathbf{x})$ is not dynamical. It gives equations of motion of the form

$$\dot{\xi}^\nu = W_n^{\nu\mu} \left[\frac{\delta H}{\delta \xi^\mu}, \xi^n\right],$$

which can be used to model, for example, advection of scalars in a specified flow given by $\xi^n(\mathbf{x})$. This bracket occurs naturally when a Lie–Poisson bracket is linearized [26,34].

### 5.4. *Solution of the Casimir problem*

We now proceed to find the solution to (5.4). We assume that all the $W^{(\mu)}$, $\mu = 0, \ldots, n$, are in lower-triangular form, and that the matrix $W^{(0)}$ is the identity matrix. Although this is the semidirect form of the extension, we will see that we can also recover the Casimir invariants of the solvable part. We assume $\nu > 0$ in (5.4), since $\nu = 0$ does not lead to a condition on the Casimir (Section 5.1). Therefore $W_\lambda^{n\nu} = 0$. Thus, we separate the Casimir condition into a part involving indices ranging from $0, \ldots, n-1$ and a part that involves only $n$. The condition

$$\sum_{\mu,\sigma,\lambda=0}^{n} W_\lambda^{\mu\nu} \mathcal{C}_{,\mu\sigma} \left[\xi^\lambda, \xi^\sigma\right] = 0, \quad \nu > 0,$$

becomes

$$\sum_{\lambda=0}^{n} \left( \sum_{\mu,\sigma=0}^{n-1} W_\lambda^{\mu\nu} \mathcal{C}_{,\mu\sigma} \left[\xi^\lambda, \xi^\sigma\right] + \sum_{\mu=0}^{n-1} W_\lambda^{\mu\nu} \mathcal{C}_{,\mu n} \left[\xi^\lambda, \xi^n\right] \right) = 0,$$

where we have used $W_\lambda^{n\nu} = 0$ to limit the sum on $\mu$. Separating the sum on $\lambda$ gives

$$\sum_{\lambda=0}^{n-1} \left( \sum_{\mu,\sigma=0}^{n-1} W_\lambda^{\mu\nu} \mathcal{C}_{,\mu\sigma} \left[\xi^\lambda, \xi^\sigma\right] + \sum_{\mu=0}^{n-1} W_\lambda^{\mu\nu} \mathcal{C}_{,\mu n} \left[\xi^\lambda, \xi^n\right] \right) + \sum_{\mu,\sigma=0}^{n-1} W_n^{\mu\nu} \mathcal{C}_{,\mu\sigma} [\xi^n, \xi^\sigma] + \sum_{\mu=0}^{n-1} W_n^{\mu\nu} \mathcal{C}_{,\mu n} [\xi^n, \xi^n] = 0.$$

The last sum vanishes because $[\xi^n, \xi^n] = 0$. Now we separate the condition into semisimple and solvable parts,

$$\sum_{\mu=1}^{n-1}\left(\sum_{\lambda,\sigma=0}^{n-1} W_{\lambda}^{\mu\nu}\mathcal{C}_{,\mu\sigma}\left[\xi^{\lambda},\xi^{\sigma}\right]-\sum_{\sigma=0}^{n-1}W_{\sigma}^{\mu\nu}\mathcal{C}_{,\mu n}[\xi^{n},\xi^{\sigma}]+\sum_{\sigma=0}^{n-1}W_{n}^{\mu\nu}\mathcal{C}_{,\mu\sigma}[\xi^{n},\xi^{\sigma}]\right)$$
$$+\sum_{\lambda,\sigma=0}^{n-1} W_{\lambda}^{0\nu}\mathcal{C}_{,0\sigma}\left[\xi^{\lambda},\xi^{\sigma}\right]\quad-\sum_{\sigma=0}^{n-1}W_{\sigma}^{0\nu}\mathcal{C}_{,0n}[\xi^{n},\xi^{\sigma}]+\sum_{\sigma=0}^{n-1}W_{n}^{0\nu}\mathcal{C}_{,0\sigma}[\xi^{n},\xi^{\sigma}]=0.$$

Using $W_{\sigma}^{0\nu}=\delta_{\sigma}^{\nu}$, we can separate the conditions into a part for $\nu=n$ and one for $0<\nu<n$. For $\nu=n$, the only term that survives is the last sum

$$\sum_{\sigma=0}^{n-1}\mathcal{C}_{,0\sigma}[\xi^{n},\xi^{\sigma}]=0.$$

Since the commutators are independent, we have the conditions,

$$\mathcal{C}_{,0\sigma}=0,\quad \sigma=0,\dots,n-1. \tag{5.7}$$

and for $0<\nu<n$,

$$\sum_{\mu=1}^{n-1}\left(\sum_{\lambda,\sigma=1}^{n-1} W_{\lambda}^{\mu\nu}\mathcal{C}_{,\mu\sigma}\left[\xi^{\lambda},\xi^{\sigma}\right]-\sum_{\sigma=1}^{n-1}W_{\sigma}^{\mu\nu}\mathcal{C}_{,\mu n}[\xi^{n},\xi^{\sigma}]+\sum_{\sigma=1}^{n-1}W_{n}^{\mu\nu}\mathcal{C}_{,\mu\sigma}[\xi^{n},\xi^{\sigma}]\right)-\mathcal{C}_{,0n}[\xi^{n},\xi^{\nu}]=0,$$

where we have used (5.7). Using independence of the inner brackets gives

$$\tilde{W}_{\lambda}^{\mu\nu}\mathcal{C}_{,\mu\sigma}=\tilde{W}_{\sigma}^{\mu\nu}\mathcal{C}_{,\mu\lambda}, \tag{5.8}$$

$$g^{\nu\mu}\mathcal{C}_{,\mu\sigma}=\tilde{W}_{\sigma}^{\nu\mu}\mathcal{C}_{,\mu n}+\delta_{\sigma}^{\nu}\mathcal{C}_{,0n}, \tag{5.9}$$

for $0<\sigma,\lambda,\nu,\mu<n$. From now on in this section repeated indices are summed, and all greek indices run from 1 to $n-1$ unless otherwise noted. We have written a tilde over the $W$'s to stress the fact that the indices run from 1 to $n-1$, so that the $\tilde{W}$ represent a solvable order $(n-1)$ subextension of $W$. This subextension does not include $W_{(n)}$. We have also made the definition

$$g^{\mu\nu}:=W_{n}^{\mu\nu}. \tag{5.10}$$

Eq. (5.8) is a Casimir condition: it says that $\mathcal{C}$ is also a Casimir of $\tilde{W}$. We now proceed to solve (5.9) for the case where $g$ is nonsingular. In Section 5.4.2 we will solve the singular $g$ case. We will see that in both cases (5.8) follows from Eq. (5.9).

*5.4.1. Nonsingular g*

The simplest case occurs when $g$ has an inverse, which we will call $g_{\mu\nu}$. Then (5.8) has the solution

$$\mathcal{C}_{,\tau\sigma}=A_{\tau\sigma}^{\mu}\mathcal{C}_{,\mu n}+g_{\tau\sigma}\mathcal{C}_{,0n}, \tag{5.11}$$

where

$$A_{\tau\sigma}^{\mu}:=g_{\tau\nu}\tilde{W}_{\sigma}^{\nu\mu}. \tag{5.12}$$

It is easily verified that $A_{\tau\sigma}^{\mu}=A_{\sigma\tau}^{\mu}$, as required by the symmetry of the left-hand side of (5.11).

In (5.11), it is clear that the $n$th variable is "special"; this suggests that we try the following form for the Casimir:

$$\mathcal{C}(\xi^0, \xi^1, \dots, \xi^n) = \sum_{i\geq 0} \mathcal{D}^{(i)}(\xi^0, \xi^1, \dots, \xi^{n-1}) f_i(\xi^n), \tag{5.13}$$

where $f$ is arbitrary and $f_i$ is the $i$th derivative of $f$ with respect to its argument. One immediate advantage of this form is that (5.8) follows from (5.9). Indeed, taking a derivative of (5.8) with respect to $\xi^\lambda$, inserting (5.13), and equating derivatives of $f$ leads to

$$g^{\nu\mu}\mathcal{D}^{(i)}_{,\mu\sigma\lambda} = \tilde{W}_\sigma^{\nu\mu}\mathcal{D}^{(i+1)}_{,\mu\lambda},$$

where we have used (5.7). Since the left-hand side is symmetric in $\lambda$ and $\sigma$ then so is the right-hand side, and (5.8) is satisfied.

Now, inserting the form of the Casimir (5.13) into the solution (5.11), we can equate derivatives of $f$ to obtain, for $\tau, \sigma = 1, \dots, n-1$,

$$\mathcal{D}^{(0)}_{,\tau\sigma} = 0, \quad \tau, \sigma = 1, \dots, n-1, \tag{5.14}$$

$$\mathcal{D}^{(i)}_{,\tau\sigma} = A^\mu_{\tau\sigma}\mathcal{D}^{(i-1)}_{,\mu} + g_{\tau\sigma}\mathcal{D}^{(i-1)}_{,0}, \quad i \geq 1. \tag{5.15}$$

The first condition, together with (5.7), says that $\mathcal{D}^{(0)}$ is linear in $\xi^0, \dots, \xi^{n-1}$. There are no other conditions on $\mathcal{D}^{(0)}$, so we can obtain $n$ independent solutions by choosing

$$\mathcal{D}^{(0)\nu} = \xi^\nu, \quad \nu = 0, \dots, n-1. \tag{5.16}$$

The equation for $\mathcal{D}^{(1)\nu}$ is

$$\mathcal{D}^{(1)\nu}_{,\tau\sigma} = \begin{cases} g_{\tau\sigma}, & \nu = 0, \\ A^\nu_{\tau\sigma}, & \nu = 1, \dots, n-1. \end{cases} \tag{5.17}$$

Thus $\mathcal{D}^{(1)\nu}$ is a quadratic polynomial (the arbitrary linear part does not yield an independent Casimir, so we set it to zero). Note that $\mathcal{D}^{(1)\nu}$ does not depend on $\xi^0$ since $\tau, \sigma = 1, \dots, n-1$. Hence, for $i > 1$ we can drop the $\mathcal{D}^{(i-1)}_{,0}$ term in (5.14). Taking derivatives of (5.14), we obtain

$$\mathcal{D}^{(i)\nu}_{,\tau_1\tau_2\dots\tau_{(i+1)}} = A^{\mu_1}_{\tau_1\tau_2}A^{\mu_2}_{\mu_1\tau_3}\cdots A^{\mu_{(i-1)}}_{\mu_{(i-2)}\tau_i}\mathcal{D}^{(1)\nu}_{,\mu_{(i-1)}\tau_{(i+1)}}. \tag{5.18}$$

We know the series will terminate because the $\tilde{W}^{(\mu)}$, and hence the $A_{(\mu)}$, are nilpotent. The solution to (5.18) is

$$\mathcal{D}^{(i)\nu} = \frac{1}{(i+1)!}\mathsf{D}^{(i)\nu}_{\tau_1\tau_2\dots\tau_{(i+1)}}\xi^{\tau_1}\xi^{\tau_2}\cdots\xi^{\tau_{(i+1)}}, \quad i > 1, \tag{5.19}$$

where the constants $\mathsf{D}$ are defined by

$$\mathsf{D}^{(i)\nu}_{\tau_1\tau_2\dots\tau_{(i+1)}} := A^{\mu_1}_{\tau_1\tau_2}A^{\mu_2}_{\mu_1\tau_3}\cdots A^{\mu_{(i-1)}}_{\mu_{(i-2)}\tau_i}\mathcal{D}^{(1)\nu}_{,\mu_{(i-1)}\tau_{(i+1)}}. \tag{5.20}$$

In summary, the $\mathcal{D}^{(i)}$'s of (5.13) are given by (5.16), (5.17) and (5.19).

Because the left-hand side of (5.18) is symmetric in all its indices, we require

$$A^\mu_{\tau\sigma}A^\nu_{\mu\lambda} = A^\mu_{\tau\lambda}A^\nu_{\mu\sigma}, \quad i > 1. \tag{5.12}$$

This is automatically satisfied for the nonsingular $g$ case [55]. Comparing this to (2.13), we see that the $A$'s satisfy all the properties of an extension, except with the dual indices. Thus we call the $A$'s the *coextension* of $\tilde{W}$ with respect to $g$. Essentially $g$ serves the role of a metric that allows us to raise and lower indices.

For a solvable extension we simply restrict $\nu > 0$ and the above treatment still holds. We conclude that the Casimirs of the solvable part of a semidirect extension are Casimirs of the full extension. We have also shown, for the case of nonsingular $g$, that the number of independent Casimirs is equal to the order of the extension.

### 5.4.2. Singular $g$

In general, $g$ is singular and thus has no inverse. However, it always has a (symmetric and unique) pseudoinverse $g_{\mu\nu}$ such that

$$g_{\mu\sigma}g^{\sigma\tau}g_{\tau\nu} = g_{\mu\nu}, \tag{5.22}$$

$$g^{\mu\sigma}g_{\sigma\tau}g^{\tau\nu} = g^{\mu\nu}. \tag{5.23}$$

The pseudoinverse is also known as the strong generalized inverse or the Moore–Penrose inverse [56]. It follows from (5.22) and (5.23) that the matrix operator

$$P^{\nu}_{\tau} := g^{\nu\kappa}g_{\kappa\tau}$$

projects onto the range of $g$. The system (5.9) only has a solution if the following solvability condition is satisfied:

$$P^{\nu}_{\tau}(\tilde{W}_{\sigma}^{\tau\mu}\mathcal{C}_{,\mu n} + \delta^{\tau}_{\sigma}\mathcal{C}_{,0n}) = \tilde{W}_{\sigma}^{\nu\mu}\mathcal{C}_{,\mu n} + \delta^{\nu}_{\sigma}\mathcal{C}_{,0n}, \tag{5.24}$$

that is, the right-hand side of Eq. (5.9) must live in the range of $g$.

If $\mathcal{C}_{,0n} \neq 0$, the quantity $\tilde{W}_{\sigma}^{\nu\mu}\mathcal{C}_{,\mu n} + \delta^{\nu}_{\sigma}\mathcal{C}_{,0n}$ has rank equal to $n$, because the quantity $\tilde{W}_{\sigma}^{\nu\mu}\mathcal{C}_{,\mu n}$ is lower-triangular (it is a linear combination of lower-triangular matrices). Hence the projection operator must also have rank $n$. But then this implies that $g$ has rank $n$ and so is nonsingular, which contradicts the hypothesis of this section. Hence, $\mathcal{C}_{,0n} = 0$ for the singular $g$ case, which together with (5.7) means that a Casimir that depends on $\xi^0$ can only be of the form $\mathcal{C} = f(\xi^0)$. However, since $\xi^0$ is not an eigenvector of the $W^{(\mu)}$'s, the only possibility is $\mathcal{C} = \xi^0$, the trivial linear case mentioned in Section 5.1.

The solvability condition (5.24) can thus be rewritten as

$$(P^{\nu}_{\tau}\tilde{W}_{\sigma}^{\tau\mu} - \tilde{W}_{\sigma}^{\nu\mu})\mathcal{C}_{,\mu n} = 0. \tag{5.25}$$

An obvious choice would be to require $P^{\nu}_{\tau}\tilde{W}_{\sigma}^{\tau\mu} = \tilde{W}_{\sigma}^{\nu\mu}$, but this is too strong. We will derive a weaker requirement shortly.

By an argument similar to that of Section 5.4.1, we now assume $\mathcal{C}$ is of the form

$$\mathcal{C}(\xi^1,\dots,\xi^n) = \sum_{i\geq 0}\mathcal{D}^{(i)}(\xi^1,\dots,\xi^{n-1})f_i(\xi^n), \tag{5.26}$$

where again $f_i$ is the $i$th derivative of $f$ with respect to its argument. As in Section 5.4.1, we only need to show (5.9) and (5.8) will follow. The number of independent solutions of (5.9) is equal of the rank of $g$. The choice

$$\mathcal{D}^{(0)\nu} = P^{\nu}_{\rho}\xi^{\rho}, \quad \nu = 1,\dots,n-1, \tag{5.27}$$

provides the right number of solutions because the rank of $P$ is equal to the rank of $g$. It also properly specializes to (5.16) when $g$ is nonsingular, for then $P^{\nu}_{\rho} = \delta^{\nu}_{\rho}$.

The solvability condition (5.25) with this form for the Casimir becomes

$$(P^{\nu}_{\tau}\tilde{W}_{\sigma}^{\tau\mu} - \tilde{W}_{\sigma}^{\nu\mu})\mathcal{D}^{(i)\nu}_{,\mu} = 0, \qquad i \geq 0. \tag{5.28}$$

For $i = 0$ the condition can be shown to simplify to

$$P^{\nu}_{\tau}\tilde{W}_{\sigma}^{\tau\mu} = \tilde{W}_{\sigma}^{\nu\tau}P^{\mu}_{\tau},$$

or to the equivalent matrix form

$$P\,\tilde{W}_{(\sigma)} = \tilde{W}_{(\sigma)}\,P, \tag{5.29}$$

since $P$ is symmetric [56].

Eq. (5.9) becomes

$$g^{\kappa\mu}\mathcal{D}^{(0)\nu}_{,\mu\sigma} = 0, \quad g^{\kappa\mu}\mathcal{D}^{(i)\nu}_{,\mu\sigma} = \tilde{W}_\sigma^{\kappa\mu}\mathcal{D}^{(i-1)\nu}_{,\mu},\ i > 0.$$

If (5.25) is satisfied, we know this has a solution given by

$$\mathcal{D}^{(i)\nu}_{,\lambda\sigma} = g_{\lambda\rho}\tilde{W}_\sigma^{\rho\mu}\mathcal{D}^{(i-1)\nu}_{,\mu} + (\delta_\lambda^\mu - g_{\lambda\rho}g^{\rho\mu})\mathcal{E}^{(i-1)\nu}_{\mu\sigma},\ i > 0,$$

where $\mathcal{E}$ is arbitrary, and $(\delta_\lambda^\mu - g_{\lambda\rho}g^{\rho\mu})$ projects onto the null space of $g$. The left-hand side is symmetric in $\lambda$ and $\sigma$, but not the right-hand side. We can symmetrize the right-hand side by an appropriate choice of the null eigenvector,

$$\mathcal{E}^{(i)\nu}_{\lambda\sigma} := g_{\sigma\rho}\tilde{W}_\lambda^{\rho\mu}\mathcal{D}^{(i)\nu}_{,\mu}, \qquad i \geq 0,$$

in which case

$$\mathcal{D}^{(i)\nu}_{,\lambda\sigma} = A^\mu_{\lambda\sigma}\mathcal{D}^{(i-1)\nu}_{,\mu}, \qquad i > 0,$$

where

$$A^\nu_{\lambda\sigma} := g_{\sigma\rho}\tilde{W}_\lambda^{\rho\nu} + g_{\lambda\rho}\tilde{W}_\sigma^{\rho\nu} - g_{\lambda\rho}g_{\sigma\kappa}g^{\rho\mu}\tilde{W}_\mu^{\kappa\nu}, \tag{5.30}$$

which is symmetric in $\lambda$ and $\sigma$. Eq. (5.30) also reduces to (5.12) when $g$ is nonsingular, for then the null eigenvector vanishes. The full solution is thus given in the same manner as (5.18) by

$$\mathcal{D}^{(i)\nu} = \frac{1}{(i+1)!}D^{(i)\nu}_{\tau_1\tau_2\ldots\tau_{(i+1)}}\xi^{\tau_1}\xi^{\tau_2}\cdots\xi^{\tau_{(i+1)}}, \qquad i > 0, \tag{5.31}$$

where the constants $D$ are defined by

$$D^{(i)\nu}_{\tau_1\tau_2\ldots\tau_{(i+1)}} := A^{\mu_1}_{\tau_1\tau_2}A^{\mu_2}_{\mu_1\tau_3}\cdots A^{\mu_{(i-1)}}_{\mu_{(i-2)}\tau_i}A^{\mu_i}_{\mu_{(i-1)}\tau_{(i+1)}}P^\nu_{\mu_i}, \tag{5.32}$$

and $\mathcal{D}^{(0)}$ is given by (5.27).

The $A$'s must still satisfy the coextension condition (5.12). Unlike the nonsingular case this condition does not follow directly and is an extra requirement in addition to the solvability condition (5.28). Note that only the $i = 0$ case, Eq. (5.29), needs to be satisfied, for then (5.28) follows. Both these conditions are coordinate-dependent, and this is a drawback. Nevertheless, we have found in obtaining the Casimir invariants for the low-order brackets that if these conditions are not satisfied, then the extension is a direct sum and the Casimirs can be found by the method of Section 5.2. However, this has not been proven rigorously.

### 5.5. *Examples*

We now illustrate the methods developed for finding Casimirs with a few examples. First we treat our prototypical case of CRMHD, and give a physical interpretation of invariants. Then, we derive the Casimir invariants for Leibniz extensions of arbitrary order. Finally, we give an example involving a singular $g$.

*5.5.1. Compressible reduced MHD*

The $W$ tensors representing the bracket for CRMHD (see Section 2.2.4) were given in Section 2.3.2. We have $n=3$, so from (5.10) we get

$$g=\begin{pmatrix}0 & -\beta_i\\ -\beta_i & 0\end{pmatrix},\qquad g^{-1}=\begin{pmatrix}0 & -\beta_i^{-1}\\ -\beta_i^{-1} & 0\end{pmatrix}.$$

In this case, the coextension is trivial: all three matrices $A^{(\nu)}$ defined by (5.12) vanish. Using (5.13) and (5.16), with $\nu=1$ and 2, the Casimirs for the solvable part are

$$\mathcal{C}^1=\xi^1 g(\xi^3)=vg(\psi),\qquad \mathcal{C}^2=\xi^2 h(\xi^3)=ph(\psi),$$

and the Casimir associated with the eigenvector $\xi^3$ is

$$\mathcal{C}^3=k(\xi^3)=k(\psi).$$

Since $g$ is nonsingular we also get another Casimir from the semidirect sum part,

$$\mathcal{C}^0=\xi^0 f(\xi^3)-\frac{1}{\beta_i}\xi^1\xi^2 f'(\xi^3)=\omega f(\psi)-\frac{1}{\beta_i}pvf'(\psi).$$

The physical interpretation of the invariant $\mathcal{C}^3$ is given in Morrison [24] and Thiffeault and Morrison [21]. This invariant implies the preservation of contours of $\psi$, so that the value $\psi_0$ on a contour labels that contour for all times. This is a consequence of the lack of dissipation and the divergence-free nature of the velocity. Substituting $\mathcal{C}^3(\psi)=\psi^k$ we also see that all the moments of the magnetic flux are conserved. By choosing $\mathcal{C}^3(\psi)=\Theta(\psi(\mathbf{x})-\omega_0)$, a heavyside function, and inserting into (5.3), it follows that the area inside of any $\psi$-contour is conserved.

To understand the Casimirs $\mathcal{C}^1$ and $\mathcal{C}^2$, we also let $g(\psi)=\Theta(\psi-\psi_0)$ in $\mathcal{C}^1$. In this case, we have

$$C^1[v;\psi]=\int_\Omega v\,g(\psi)\,\mathrm{d}^2x=\int_{\Psi_0} v(\mathbf{x})\,\mathrm{d}^2x,$$

where $\Psi_0$ represents the (not necessarily connected) region of $\Omega$ enclosed by the contour $\psi=\psi_0$ and $\partial\Psi_0$ is its boundary. By the interpretation we gave of $\mathcal{C}^3$, the contour $\partial\Psi_0$ moves with the fluid. So the total value of $v$ inside of a $\psi$-contour is conserved by the flow. The same is true of the pressure $p$. (See Thiffeault and Morrison [21] for an interpretation of these invariants in terms of relabeling symmetries, and a comparison with the rigid body.)

The total pressure and parallel velocity inside of any $\psi$-contour are preserved. To understand $\mathcal{C}^4$, we use the fact that $\omega=\nabla^2\phi$ and integrate by parts to obtain

$$C^4[w,v,p,\psi]=-\int_\Omega\left(\nabla\phi\cdot\nabla\psi+\frac{vp}{\beta_i}\right)f'(\psi)\,\mathrm{d}^2x.$$

The quantity in parentheses is thus invariant inside of any $\psi$-contour. It can be shown that this is a remnant of the conservation by the full MHD model of the cross helicity,

$$V=\int_\Omega \mathbf{v}\cdot\mathbf{B}\,\mathrm{d}^2x,$$

at second order in the inverse aspect ratio, while the conservation of $C^1[v;\psi]$ is a consequence of preservation of this quantity at first order. Here $\mathbf{B}$ is the magnetic field. The quantities $C^3[\psi]$ and $C^2[p;\psi]$ they are, respectively, the first and second order remnants of the preservation of helicity,

$$W = \int_\Omega \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}^2 x,$$

where $\mathbf{A}$ is the magnetic vector potential.

### 5.5.2. Leibniz extension

We first treat the nilpotent case. The Leibniz extension of Section 4.5 can be characterized by

$$W_\lambda^{\mu\nu} = \delta_\lambda^{\mu+\nu}, \qquad \mu, \nu, \lambda = 1, \ldots, n, \tag{4.13}$$

where the tensor $\delta$ is the ordinary Kronecker delta. Upon restricting the indices to run from 1 to $n-1$ (the tilde notation of Section 5.4), we have

$$g^{\mu\nu} = \tilde{W}_n^{\mu\nu} = \delta_n^{\mu+\nu}, \qquad \mu, \nu = 1, \ldots, n-1.$$

The matrix $g$ is nonsingular with inverse equal to itself: $g_{\mu\nu} = \delta^n_{\mu+\nu}$. The coextension of $\tilde{W}$ is thus

$$A^\mu_{\tau\sigma} = \sum_{\nu=1}^{n-1} g_{\tau\nu} \tilde{W}_\sigma^{\nu\mu} = \sum_{\nu=1}^{n-1} \delta^n_{\tau+\nu} \delta_\sigma^{\nu+\mu} = \delta^{\mu+n}_{\tau+\sigma}.$$

Eq. (5.20) becomes

$$\begin{aligned}\mathcal{D}^{(i)\nu}_{\tau_1\tau_2\ldots\tau_{(i+1)}} &= A^{\mu_1}_{\tau_1\tau_2} A^{\mu_2}_{\mu_1\tau_3} \cdots A^{\mu_{(i-1)}}_{\mu_{(i-2)}\tau_i} A^\nu_{\mu_{(i-1)}\tau_{(i+1)}} = \delta^{\mu_1+n}_{\tau_1+\tau_2} \delta^{\mu_2+n}_{\mu_1+\tau_3} \cdots \delta^{\mu_{(i-1)}+n}_{\mu_{(i-2)}+\tau_i} \delta^{\nu+n}_{\mu_{(i-1)}+\tau_{(i+1)}} \\ &= \delta^{\nu+in}_{\tau_1+\tau_2+\cdots+\tau_{(i+1)}}, \qquad \nu = 1, \ldots, n-1,\end{aligned}$$

which, as required, is symmetric under interchage of the $\tau_i$. Using (5.13), (5.16), (5.17) and (5.19) we obtain the $n-1$ Casimir invariants

$$\mathcal{C}^\nu(\xi^1, \ldots, \xi^n) = \sum_{i\geq 0} \frac{1}{(i+1)!} \delta^{\nu+in}_{\tau_1+\tau_2+\cdots+\tau_{(i+1)}} \xi^{\tau_1} \cdots \xi^{\tau_{(i+1)}} f_i^\nu(\xi^n), \tag{5.33}$$

for $\nu = 1, \ldots, n-1$. The superscript $\nu$ on $f$ indicates that the arbitrary function is different for each Casimir, and recall the subscript $i$ denotes the $i$th derivative with respect to $\xi^n$. The $n$th invariant is simply $\mathcal{C}^\nu(\xi^n) = f^n(\xi^n)$, corresponding to the null eigenvector in the system. Thus there are $n$ independent Casimirs, as stated in Section 5.4.1.

For the Leibniz semidirect sum case, since $g$ is nonsingular, there will be an extra Casimir given by (5.33) with $\nu = 0$, and the $\tau_i$ sums run from 0 to $n-1$. This is the same form as the $\nu = 1$ Casimir of the order $(n+1)$ nilpotent extension.

For the $i$th term in (5.33), the maximal value of any $\tau_j$ is achieved when all but one (say, $\tau_1$) of the $\tau_j$ are equal to $n-1$, their maximum value. In this case we have

$$\tau_1 + \tau_2 + \cdots + \tau_{i+1} = \tau_1 + i(n-1) = \nu + in,$$

so that $\tau_1 = i + \nu$. Hence, the $i$th term depends only on $(\xi^{\nu+i}, \ldots, \xi^n)$, and the $\nu$th Casimir depends on $(\xi^\nu, \ldots, \xi^n)$. Also,

$$\max(\tau_1 + \cdots + \tau_{i+1}) = (i+1)(n-1) = \nu + in,$$

which leads to max $i = n - \nu - 1$. Thus the sum Eq. (5.33) terminates, as claimed in Section 5.4.1. We rewrite (5.33) in the more complete form

Table 2
Casimir invariants for Leibniz extensions up to order $n = 5$ ($\nu = 1$) (the primes denote derivatives)

| $n$ | Invariant |
|---|---|
| 1 | $f(\xi^1)$ |
| 2 | $\xi^1 f(\xi^2)$ |
| 3 | $\xi^1 f(\xi^3) + \frac{1}{2}(\xi^2)^2 f'(\xi^3)$ |
| 4 | $\xi^1 f(\xi^4) + \xi^2\xi^3 f'(\xi^4) + \frac{1}{3!}(\xi^3)^3 f''(\xi^4)$ |
| 5 | $\xi^1 f(\xi^5) + (\xi^2\xi^4 + \frac{1}{2}(\xi^3)^2) f'(\xi^5) + \frac{1}{2}\xi^3(\xi^4)^2 f''(\xi^5) + \frac{1}{4!}(\xi^4)^4 f'''(\xi^5)$ |

$$\mathcal{C}^\nu(\xi^\nu, \dots, \xi^n) = \sum_{k=1}^{n-\nu} \frac{1}{k!} \delta^{\nu+(k-1)n}_{\tau_1+\tau_2+\dots+\tau_k} \xi^{\tau_1} \cdots \xi^{\tau_k} f^\nu_{k-1}(\xi^n),$$

for $\nu = 0, \dots, n$. Table 2 gives the $\nu = 1$ Casimirs up to order $n = 5$.

*5.5.3. Singular $g$*

Now consider $n = 4$ extension from Section 4.6.4, Case 3(c). We have

$$\tilde{W}_{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

with $\tilde{W}_{(1)} = \tilde{W}_{(3)} = 0$. The pseudo inverse of $g$ is $g^{-1} = g$ and the projection operator is

$$P^\nu_\tau := g^{\nu\kappa} g_{\kappa\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The solvability condition (5.29) is obviously satisfied. We build the coextension given by (5.30), which in matrix form is

$$A^{(\nu)} = \tilde{W}^{(\nu)} g^{-1} + (\tilde{W}^{(\nu)} g^{-1})^T - g^{-1} g \tilde{W}^{(\nu)} g^{-1},$$

to obtain

$$A^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A^{(2)} = A^{(3)} = 0.$$

These are symmetric and obviously satisfy (5.12), so we have a good coextension. Using (5.26), (5.27), (5.31) and (5.32) we can write, for $\nu = 1$ and 3,

$$\mathcal{C}^1 = \xi^1 f(\xi^4) + \xi^2\xi^3 f'(\xi^4), \qquad \mathcal{C}^3 = \xi^3 g(\xi^4).$$

This extension has two null eigenvectors, so from Section 5.2 we also have the Casimir $h(\xi^2, \xi^4)$. The functions $f$, $g$, and $h$ are arbitrary, and the prime denotes differentiation with respect to argument.

## 6. Casimir invariants for low-order extensions

Using the techniques developed so far, we now find the Casimir invariants for the low-order extensions classified in Section 4.6. We first find the Casimir invariants for the solvable extensions, since these are also invariants for the semidirect sum case. Then, we obtain the extra Casimir invariants for the semidirect case, when they exist.

Table 3
Casimir invariants for solvable extensions of order $n = 3$

| Case | Invariant |
|---|---|
| 1 | $\mathcal{C}(\xi^1, \xi^2, \xi^3)$ |
| 2 | $\xi^1 f(\xi^3) + \xi^2 g(\xi^3) + h(\xi^3)$ |
| 3 | $\xi^1 f(\xi^2) + g(\xi^2, \xi^3)$ |
| 4 | $\xi^1 f(\xi^3) + \frac{1}{2}(\xi^2)^2 f'(\xi^3) + \xi^2 g(\xi^3) + h(\xi^3)$ |

### *6.1. Solvable extensions*

Now we look for the Casimirs of solvable extensions. As mentioned in Section 5.3, the Casimirs associated with null eigenvectors (the only kind of eigenvector for solvable extensions) are actually conserved locally. We shall still write them in the form $\mathcal{C} = f(\xi^n)$, where $\mathcal{C}$ is as in (5.3), so they have the correct form as invariants for the semidirect case of Section 6.2 (for which they are no longer locally conserved).

#### *6.1.1. n=1*

Since the bracket is Abelian, any function $\mathcal{C} = \mathcal{C}(\xi^1)$ is a Casimir.

#### *6.1.2. n=2*

For the Abelian case we have $\mathcal{C} = \mathcal{C}(\xi^1, \xi^2)$. The only other case is the Casimir of the Leibniz extension,

$$\mathcal{C}(\xi^1, \xi^2) = \xi^1 f(\xi^2) + g(\xi^2).$$

#### *6.1.3. n=3*

As shown in Section 4.6.3, there are four cases. Case 1 is the Abelian case, for which any function $\mathcal{C} = \mathcal{C}(\xi^1, \xi^2, \xi^3)$ is a Casimir. Case 2 is essentially the solvable part of the CRMHD bracket, which we treated in Section 5.5.1. Case 3 is a direct sum of the Leibniz extension for $n = 2$, which has the bracket

$$[(\alpha_1, \alpha_2), (\beta_1, \beta_2)] = (0, [\alpha_1, \beta_1]),$$

with the Abelian algebra $[\alpha_3, \beta_3] = 0$. Hence, the Casimir invariant is the same as for the $n = 2$ Leibniz extension with the extra $\xi^3$ dependence of the arbitrary function (see Section 5.2). Finally, Case 4 is the Leibniz Casimir. These results are summarized in Table 3.

Cases 1 and 3 are trivial extensions, that is, the cocycle appended to the $n = 2$ case vanishes. The procedure of then adding $\xi^n$ dependence to the arbitrary function works in general.

#### *6.1.4. n=4*

As shown in Section 4.6.4, there are nine cases to consider. We shall proceed out of order, to group together similar Casimir invariants.

Cases 1(a), 2, 3(a), and 4(a) are trivial extensions, and as mentioned in Section 6.1.3 they involve only addition of $\xi^4$ dependence to their $n = 3$ equivalents. Case 3(b) is a direct sum of two $n = 2$ Leibniz extensions, so the Casimirs add.

Case 3(c) is the semidirect sum of the $n = 2$ Leibniz extension with an Abelian algebra defined by $[(\alpha_3, \alpha_4), (\beta_3, \beta_4)] = (0, 0)$, with action given by

$$\rho_{(\alpha_1,\alpha_2)}(\beta_3, \beta_4) = (0, [\alpha_1, \beta_3]).$$

The Casimir invariants for this extension were derived in Section 5.5.3.

Table 4
Casimir invariants for solvable extensions of order $n = 4$

| Case | Invariant |
|---|---|
| 1(a) | $\mathcal{C}(\xi^1, \xi^2, \xi^3, \xi^4)$ |
| 1(b) | $\xi^1 f(\xi^4) + \xi^2 g(\xi^4) + \xi^3 h(\xi^4) + k(\xi^4)$ |
| 2 | $\xi^1 f(\xi^3) + \xi^2 g(\xi^3) + h(\xi^3, \xi^4)$ |
| 3(a) | $\xi^1 f(\xi^2) + g(\xi^2, \xi^3, \xi^4)$ |
| 3(b) | $\xi^1 f(\xi^2) + \xi^3 g(\xi^4) + h(\xi^2, \xi^4)$ |
| 3(c) | $\xi^1 f(\xi^4) + \xi^2\xi^3 f'(\xi^4) + \xi^3 g(\xi^4) + h(\xi^2, \xi^4)$ |
| 3(d) | $\xi^1 f(\xi^4) + \frac{1}{2}(\xi^2)^2 f'(\xi^4) + \xi^3 g(\xi^4) + \xi^2 h(\xi^4) + k(\xi^4)$ |
| 4(a) | $\xi^1 f(\xi^3) + \frac{1}{2}(\xi^2)^2 f'(\xi^3) + \xi^2 g(\xi^3) + h(\xi^3, \xi^4)$ |
| 4(b) | $\xi^1 f(\xi^4) + \xi^2\xi^3 f'(\xi^4) + \frac{1}{3!}(\xi^3)^3 f''(\xi^4)$ |
| | $+\xi^2 g(\xi^4) + \frac{1}{2}(\xi^3)^2 g'(\xi^4) + \xi^3 h(\xi^4) + k(\xi^4)$ |

Case 3(d) has a nonsingular $g$, so the techniques of Section 5.4.1 can be applied directly.

Finally, Case 4(b) is the $n = 4$ Leibniz extension, the Casimir invariants of which were derived in Section 5.5.2. The invariants are all summarized in Table 4.

### 6.2. Semidirect extensions

Now that we have derived the Casimir invariants for solvable extensions, we look at extensions involving the semidirect sum of an algebra with these solvable extensions. We label the new variable (the one which acts on the solvable part) by $\xi^0$. In Section 5.4.1 we showed that the Casimirs of the solvable part were also Casimirs of the full extension. We also concluded that a necessary condition for obtaining a new Casimir (other than the linear case $\mathcal{C}(\xi^0) = \xi^0$) from the semidirect sum was that det $W_{(n)} \neq 0$. We go through the solvable cases and determine the Casimirs associated with the semidirect extension, if any exist.

#### 6.2.1. n=1

There is only one solvable extension, so upon appending a semidirect part we have

$$W_{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad W_{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since det $W_{(1)} \neq 0$, we expect another Casimir. In fact this extension is of the semidirect Leibniz type and has the same Casimir form as the $n = 2$ solvable Leibniz (Section 5.5.2) extension. Thus, the new Casimir is just $\xi^0 f(\xi^1)$.

#### 6.2.2. n=2

Of the two possible extensions only the Leibniz one satisfies det $W_{(2)} \neq 0$. The Casimir is thus

$$\mathcal{C}_{\mathrm{sd}} = \xi^0 f(\xi^2) + \tfrac{1}{2}(\xi^1)^2 f'(\xi^2).$$

#### 6.2.3. n=3

Cases 2 and 4 have a nonsingular $W_{(3)}$. The Casimir for Case 2 is

$$\mathcal{C}_{\mathrm{sd}} = \xi^0 f(\xi^3) + \xi^1\xi^2 f'(\xi^3),$$

and for Case 4 it is of the Leibniz form

$$\mathcal{C}_{\mathrm{sd}} = \xi^0 f(\xi^3) + \xi^1\xi^2 f'(\xi^3) + \frac{1}{3!}(\xi^2)^3 f''(\xi^3).$$

Table 5
Casimir invariants for semidirect extensions of order $n = 5$ (these extensions also possess the corresponding Casimir invariants in Table 4)

| Case | Invariant |
|---|---|
| 1b | $\xi^0 f(\xi^4) + (\xi^1\xi^3 + \frac{1}{2}(\xi^2)^2) f'(\xi^4)$ |
| 3d | $\xi^0 f(\xi^4) + (\xi^1\xi^2 + \frac{1}{2}(\xi^3)^2) f'(\xi^4) + \frac{1}{3!}(\xi^2)^3 f''(\xi^4)$ |
| 4b | $\xi^0 f(\xi^4) + (\xi^1\xi^3 + \frac{1}{2}(\xi^2)^2) f'(\xi^4) + \frac{1}{2}\xi^2(\xi^3)^2 f''(\xi^4) + \frac{1}{4!}(\xi^3)^4 f'''(\xi^4)$ |

#### 6.2.4. n=4

Cases 1(b), 3(d), and 4(b) have a nonsingular $W_{(4)}$. The Casimirs are shown in Table 5.

## 7. Discussion

Using the tools of Lie algebra cohomology, we have classified low-order extensions. We found that there were only a few normal forms for the extensions, and that they involved no free parameters. This is not expected to carry over to higher orders ($n > 4$). The classification includes the Leibniz extension, which is the maximal extension. One of the normal forms is the bracket appropriate to compressible reduced MHD [17,48].

We then developed techniques for finding the Casimir invariants of Lie–Poisson brackets formed from Lie algebra extensions. We introduced the concept of coextension, which allows one to explicitly write down the solution of the Casimirs. The coextension for the Leibniz extension can be found for arbitrary order, so that we were able obtain the corresponding Casimirs in general.

It would be interesting to generalize the classification scheme presented here to a completely general form of extension bracket [5,10]. Certainly the type of coordinate transformations allowed would be more limited, and perhaps one cannot go any futher than cohomology theory allows.

The interpretation of the Casimir invariants can be pushed further, both in a mathematical and a physical sense. Mathematically, a precise geometrical relation between cocycles and the form of the Casimirs could be formulated. The cocycle and Casimirs should yield information about the holonomy of the system. For this one must study the extensions in the framework of their principal bundle description [29]. Physically we would like to attach a more precise physical meaning to these conserved quantities. The invariants associated with simultaneous eigenvectors can be regarded as constraining the associated field variable to move with the fluid elements [24]. The field variable can also be interpreted as partially labeling a fluid element. Some attempt has been made at formulating the Casimir invariants of brackets in such a manner [6,21], and an interpretation of cocycles in the context of dynamical accessiblity has been offered [55].

Sufficient conditions for stability can be obtained via the energy–Casimir method [22,24,25], or the related technique of dynamical accessibility [26,57]. In both these case, we can make use of the coextension to derive the stability conditions for Lie–Poisson bracket extensions and a large class of Hamiltonians [55].

## Acknowledgements

The authors thank Tom Yudichak for his comments and suggestions. This work was supported by the US Department of Energy under contract no. DE-FG03-96ER-54346. J-LT also acknowledges the support from Fonds pour la Formation de Chercheurs et l'Aide à la Recherche du Canada.

## Appendix A. Proof of $W^{(1)} = I$

Out goal is to demonstrate that through a series of lower-triangular coordinate transformations we can make $W^{(1)}$ (which has an $n$-fold degenerate eigenvalue equal to unity) equal to the identity matrix, while preserving the lower-triangular nilpotent form of $W^{(2)}, \ldots, W^{(n)}$.

We first show that we can always make a series of coordinate transformations to make $W_\lambda^{11} = \delta_\lambda^1$. First note that if the coordinate transformation $M$ is of the form $M = I + L$, where $I$ is the identity and $L$ is lower-triangular nilpotent, then $\tilde{W}^{(1)} = M^{-1} W^{(1)} M$ still has eigenvalue 1, and for $\mu > 1$ the $\tilde{W}^{(\mu)} = M^{-1} W^{(\mu)} M$ are still nilpotent.

For $\lambda > 1$ we have

$$\bar{W}_\lambda^{11} = \tilde{W}_\lambda^{11} + \tilde{W}_\lambda^{1\nu} L_\nu^1 = \tilde{W}_\lambda^{11} + \sum_{\nu=2}^{\lambda-1} \tilde{W}_\lambda^{1\nu} L_\nu^1 + L_\lambda^1, \tag{A.1}$$

where we used $\tilde{W}_\lambda^{1\lambda} = 1$. Owing to the triangular structure of the set of (A.1) we can always solve for the $L_\lambda^1$ to make $\bar{W}_\lambda^{11}$ vanish. This proves the first part.

We now show by induction that if $W_\lambda^{11} = \delta_\lambda^1$, as proved above, then $W^{(1)}$ is the identity matrix. For $\lambda = 1$ the result is trivial. Assume that $W_\mu^{1\nu} = \delta_\mu^\nu$, for $\mu < \lambda$. Setting two of the free indices to one, Eq. (2.13) can be written as

$$W_\lambda^{\mu 1} W_\mu^{1\sigma} = W_\lambda^{\mu\sigma} W_\mu^{11} = W_\lambda^{\mu\sigma} \delta_\mu^1 = W_\lambda^{1\sigma}.$$

Since $W^{(1)}$ is lower-triangular the index $\mu$ runs from 2 to $\lambda$ (since we are assuming $\lambda > 1$):

$$\sum_{\mu=2}^{\lambda} W_\lambda^{\mu 1} W_\mu^{1\sigma} = W_\lambda^{1\sigma},$$

and this can be rewritten, for $\sigma < \lambda$,

$$\sum_{\mu=2}^{\lambda-1} W_\lambda^{\mu 1} W_\mu^{1\sigma} = 0.$$

Finally, we use the inductive hypothesis

$$\sum_{\mu=2}^{\lambda-1} W_\lambda^{\mu 1} \delta_\mu^\sigma = W_\lambda^{\sigma 1} = 0,$$

which is valid for $\sigma < \lambda$. Hence, $W_\lambda^{\sigma 1} = \delta_\lambda^\sigma$ and we have proved the result. ($W_\lambda^{\lambda 1}$ must be equal to one since it lies on the diagonal and we have already assumed degeneracy of eigenvalues.)

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