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Hamiltonian Description of Shear Flow

N.J. Balmforth and P.J. Morrison

1 Introduction

In traditional mechanics courses we learn to solve ideal problems, ones without friction-like effects, by transforming to normal coordinates. Such problems are typically Hamiltonian and the transformation theory of Hamiltonian systems is exploited. If the system under consideration describes linear motion about a stable equilibrium point, then in terms of normal coordinates the Hamiltonian has the following normal form:

$$H(p, q) = \sum_k \frac{\nu_k}{2}\left(q_k^2 + p_k^2\right) = \sum_k \nu_k J_k. \tag{1.1}$$

Here (q, p) denotes a set of canonical variables and the ν_k's are the natural frequencies of oscillation of the system. In (1.1) the Hamiltonian is written in two ways: as a sum over independent simple harmonic oscillators and as a sum over the action variables, J_k, with their conjugate variables being absent from the Hamiltonian, i.e. these variables are ignorable. It is well-known that the equations that describe ideal fluid flow are Hamiltonian (see e.g. Morrison 1998, 1982 and references therein) and thus one would expect that a similar transformation to normal coordinates should exist for inviscid fluid problems. Indeed this is the case and we demonstrate this here for the problem of shear flow in a channel.

This shear flow problem is complicated by the fact that it possesses a continuous spectrum associated with the presence of critical levels. Continuous spectra cannot occur in finite degree-of-freedom systems and so new tools are required. We will describe these new tools and use them to show that the transformation to normal coordinates can be constructed using the singular eigenfunctions associated with the continuous spectrum. In particular, we show that this normal coordinate transformation is a linear integral transform that is a generalization of the Hilbert transform.

For convenience we consider flow profiles for which our shear flow problem is stable and possesses only a continuous spectrum. That is, we consider profiles for which there is no discrete component to the spectrum. According to Rayleigh's criterion this is guaranteed to be the case if the profile contains no inflection points. A more general criterion for assuring this type of spectrum is given in Balmforth & Morrison (1999), which treats necessary and sufficient conditions for stability.

Stability criteria like those of Rayleigh (1880), Fjørtoft (1950), and Arnol'd (1965, 1966) will be seen to be essentially energy criteria. The finite degree-of-freedom version of these criteria would amount to showing that the Hamiltonian for a linear problem, $H = App + Bpq + Cqq$, is a definite quadratic form. Thus all of these criteria are versions of Dirichlet's energy criterion for stability of Hamiltonian systems. Researchers in fluid and plasma dynamics have noticed that Rayleigh-like criteria exist for a variety of fluid and plasma systems, ranging from Drift–Rossby dynamics to magnetohydrodynamics. While such stability criteria may be useful, they are incomplete. A complete solution would be to transform to the normal coordinates or the normal form Hamiltonian analogous to (1.1), where the oscillating individual degrees of freedom are separated. Accomplishing this for the shear flow problem is the major goal of the present paper.

The ν_k's that appear in the Hamiltonian of (1.1) need not be positive numbers. Indeed, a degree of freedom for which $\nu_k < 0$ is referred to as a negative energy mode. Negative energy modes oscillate, just as ordinary stable oscillations, but their energy, as given by the Hamiltonian, is negative. Negative energy modes are important because typically they are destabilized upon adding dissipation and they can give rise to explosive instability when nonlinearity is introduced. Also, the signature of a mode, as defined by $\mathrm{sgn}(\nu_k)$, plays an important role in Hamiltonian bifurcation theory. In this paper we show how to define signature for the continuous spectrum.

Just as all stable finite degree-of-freedom Hamiltonian systems can be put into the form of (1.1), with appropriate signature, one would expect similar generality for infinite systems with continuous spectra. Indeed, there exists a normal form for systems of this type. The essential ideas and calculations of the present paper have been done previously for the Vlasov–Poisson system in Morrison & Pfirsch (1992), Morrison & Shadwick (1994), Shadwick (1995), and Morrison (2000). It appears that all Hamiltonian systems that possess a Hamiltonian formulation in terms of a kind of Poisson bracket known as a Lie–Poisson bracket (see e.g. Morrison 1998 and Thiffeault & Morrison 2000) possess the transformation to normal form, in the manner described here. (The essential ideas in the shear flow context were first given in Balmforth & Morrison 1995a and 1995b, and then extended to include the β-effect in Vanneste 1996.)

In the remainder of this Introduction we gather together some known material about shear flow. Most of this material will be needed in subsequent sections, but some is included to add insight and to place our new results in historical context. Readers who are well-versed in the specifics of shear flow may wish to skim through this part and proceed to Section 2. Specifically, in Section 1.1 we review some general properties of shear flow dynamics. In this subsection we set our notation while describing the problem, and we briefly discuss constants of motion and the notion of dynamical accessibility. Section 1.2 contains facts about the associated eigenvalue problem. Properties of Rayleigh's equation are reviewed and it is shown why pure oscillations make up a continuous spectrum and are associated with singular eigenfunctions. In Section 2, with the singular eigenfunctions as motivation, we introduce the integral transform. In addition we show that the transform possesses an inverse and we present some transform identities that are then used to solve the shear flow problem. In Section 3 we interpret the transform in the Hamiltonian context. In this section we review the Hamiltonian structure for Euler's equation, and then derive a canonical Hamiltonian description for the shear flow problem, which we solve by means of finding a canonical transformation generated by a mixed variable generating functional. We discuss the concept of modal signature, and how it relates to negative energy modes and stability. Lastly in Section 3, we show that Rayleigh-like criteria are essentially energy arguments. Finally, we conclude in Section 4.

1.1 Shear flow dynamics

The two-dimensional Euler equation that describes an incompressible and inviscid fluid can be written as

$$\frac{\partial \omega}{\partial t} + [\psi, \omega] = 0, \tag{1.2}$$

where the scalar vorticity, ω, is related to the streamfunction, ψ, through $\nabla^2 \psi = \omega$, and where

$$[\psi, \omega] := \mathbf{v} \cdot \nabla \omega = \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} \tag{1.3}$$

with $\mathbf{v} := (-\partial\psi/\partial y, \partial\psi/\partial x)$.

We are interested in analysing Euler's equations in a domain D that corresponds to an infinitely long channel of fixed width, $(x, y) \in \mathbb{R} \times [-1, 1]$, with the usual boundary condition that there be no flow normal to the boundaries located at $y = \pm 1$. With these assumptions, Poisson's equation can be rewritten as

$$\psi(x, y) = \int_D G(x, y, x', y')\omega(x', y')\, dy'\, dx', \tag{1.4}$$

where G is the Green's function, which implicitly contains the boundary condition.

By straightforward formal manipulation it can be shown that (1.2) conserves the energy functional

$$H[\omega] = \frac{1}{2}\int_D |\nabla \psi|\, dy\, dx, \tag{1.5}$$

a quantity that will be seen to be the Hamiltonian functional for the infinite degree-of-freedom Hamiltonian description given in Section 3.1. In addition to the energy, (1.2) formally conserves the momentum functionals

$$P_x[\omega] := \int_D y\omega(x,y,t)\,dy\,dx, \qquad P_y[\omega] := -\int_D x\omega(x,y,t)\,dy\,dx, \tag{1.6}$$

and the Casimir invariants

$$C[\omega] := \int_D \mathcal{C}(\omega)\,dy\,dx, \tag{1.7}$$

where $\mathcal{C}$ is an arbitrary function of the vorticity.

It is well-known that the solution to Euler's equations can be written as a rearrangement: $\omega = \overset{\circ}{\omega} \circ Z$, where $\circ$ denotes the composition of functions, Z represents the solution to the characteristic equations, and $\overset{\circ}{\omega}$ represents the initial vorticity. Equivalently, this can be written out as

$$\omega(x,y,t) = \overset{\circ}{\omega}(\overset{\circ}{x}(x,y,t), \overset{\circ}{y}(x,y,t)). \tag{1.8}$$

It is also well-known that not all rearrangements are allowed: only those that preserve the area measure $dx\,dy$. It can be shown directly that the Casimir invariants of (1.7) are preserved under such rearrangements (see Appendix B of Morrison & Pfirsch 1990).

In this paper we are interested in the linear dynamics about equilibrium configurations where the streamfunction, Ψ, and vorticity, Ω, satisfy

$$[\Omega, \Psi] = 0, \qquad \nabla^2\Psi = \Omega. \tag{1.9}$$

The subset of solutions of (1.9) that are of interest satisfy $\Omega(y) = -U'(y) = \Psi''(y)$, where prime denotes d/dy. The equilibrium velocity $U(y)$, the "flow profile," is in general an arbitrary function of y. Here we take this function to be monotonic for $y \in [-1,1]$; nonmonotonic profiles will be considered elsewhere.

Upon linearizing by writing $\psi = \Psi + \delta\psi$ and $\omega = \Omega + \delta\omega$ and expanding (1.2) to first order, we obtain

$$\frac{\partial \delta\omega}{\partial t} + U\frac{\partial \delta\omega}{\partial x} - U''\frac{\partial \delta\psi}{\partial x} = 0, \tag{1.10}$$

with $\nabla^2\delta\psi = \delta\omega$. The linear dynamics conserves the following functional:

$$H_L[\delta\omega] = \frac{1}{2}\int_D \left[\frac{U}{U''}(\delta\omega)^2 - \delta\omega\,\delta\psi\right] dy\,dx, \tag{1.11}$$

which corresponds physically to the total energy contained in a perturbation away from the equilibrium state. In Section 3.1 we will see that this functional is the Hamiltonian for the linear dynamics. The linear dynamics also conserves the momenta obtained upon linearizing (1.6).

Since the solution to Euler's equation is a rearrangement, i.e. has the form of (1.8), it is natural to restrict initial infinitesimal perturbations to be rearrangements of the equilibrium state. In Morrison & Pfirsch (1989, 1990, 1992) such restricted variations were called *dynamically accessible* variations and there the following formula was presented:

$$\delta\omega = [h, \omega], \tag{1.12}$$

where ω is any vorticity function, and h is an arbitrary function of (x, y). (This idea has recurred many times in the literature in different contexts, many of which are pointed out in Morrison 1998.)

Of interest here are variations about the shear flow equilibrium state that are to be initial conditions for (1.10). Thus $\delta\mathring{\omega} = [h, \Omega] = \Omega' \partial h/\partial x$. This implies, for nonsingular h's, that $\delta\mathring{\omega}$ has the same extrema as Ω. Also, it is easy to show that if a function is initially dynamically accessible, then under the dynamics of (1.10) it will be so for all time; i.e. $\delta\omega$ at any time can be written in the form of $[h, \Omega]$ for some function h.

Since the equilibria of interest are independent of the coordinate x, perturbation quantities can be Fourier transformed in that coordinate. We can then consider each Fourier mode independently and write $\delta\psi = \psi_k(y, t)\exp(ikx)$ and $\delta\omega = \omega_k(y, t)\exp(ikx)$ for the infinitesimal perturbations, where k is the streamwise wavenumber. The quantities $\delta\psi_k$ and $\delta\omega_k$ satisfy

$$\frac{\partial \omega_k}{\partial t} + ikU\omega_k - ikU''\psi_k = 0 \tag{1.13}$$

with

$$\psi_k(y, t) = \int_{-1}^{1} \mathcal{K}_k(y, y')\omega_k(y', t)\, dy', \tag{1.14}$$

where

$$\mathcal{K}_k(y, y') = \begin{cases} \sinh[k(y-1)]\sinh[k(y'+1)]/k\sinh[2k] & \text{for } \ y > y', \\ \sinh[k(y'-1)]\sinh[k(y+1)]/k\sinh[2k] & \text{for } \ y \le y'. \end{cases} \tag{1.15}$$

The boundary conditions are now that each ψ_k vanish at $y = \pm 1$.

The channel is infinite in x, so k is really a continuous variable, even though it appears above as a subscript. The label k corresponds to the eigenvalue of a second continuous spectrum, that associated with a regular problem in an infinite domain. Later in this paper we will consider k to be fixed at some value and we will not be concerned with this latter continuous spectrum.

1.2 Eigenmodes and Rayleigh's equation

We now discuss the associated eigenvalue problem, i.e. we investigate eigenmode solutions that are characterized by an exponential time dependence with a phase velocity c that corresponds to the eigenvalue. Upon writing

$$\psi_k(y,t) = \psi_k(y,c)\exp(-ikct), \qquad \omega_k(y,t) = \omega_k(y,c)\exp(-ikct), \tag{1.16}$$

(1.13) and (1.14) become

$$(U-c)\omega_k - U''\psi_k = 0, \tag{1.17}$$

and

$$\psi_k(y,c) = \int_{-1}^{1} \mathcal{K}_k(y,y')\omega_k(y',c)\,dy', \tag{1.18}$$

respectively. (Note, we will distinguish the functions ω_k and ψ_k of (1.13) and (1.14) from those of (1.17) and (1.18) by explicitly displaying the argument, when there is a possibility of confusion.)

Equation (1.17) is Rayleigh's equation, which is generally written as follows:

$$(U-c)(\psi_k'' - k^2\psi_k) = U''\psi_k. \tag{1.19}$$

If Rayleigh's equation possesses solutions with c being a complex number, then such solutions come in growing and decaying mode pairs and they are always regular (square integrable) and discrete. Rayleigh (1880) (see also Rayleigh 1896) showed that these complex modes can only appear when the velocity profile contains an inflection point, i.e. a point y_I such that $U''(y_I) = 0$. This follows easily: simple manipulation of (1.17) leads to

$$-\int_{-1}^{1} (|\psi_k'|^2 + k^2|\psi_k|^2)\,dy = \int_{-1}^{1} \frac{U''}{U-c}|\psi_k|^2\,dy, \tag{1.20}$$

which has the imaginary part

$$c_i \int_{-1}^{1} \frac{U''}{|U-c|^2}|\psi_k|^2\,dy = 0. \tag{1.21}$$

Thus, either $c_i = 0$ and there is no complex mode, or the integral must vanish. A necessary condition for the integral to vanish is that U'' change sign within the domain. Hence, an inflection point is necessary for instability.

Fjørtoft's (1950) generalization of Rayleigh's criterion goes further by using both (1.21) and (1.20). Evidently,

$$-\int_{-1}^{1} \frac{[U-U(y_I)]U''}{|U-c|^2}|\psi_k|^2\,dy = \int_{-1}^{1} (|\psi_k'|^2 + k^2|\psi_k|^2)\,dy \geq 0. \tag{1.22}$$

Thus, not only must there be an inflection point, but at that point, $U''/(U-U_I)$ must be negative for instability, where $U_I = U(y_I)$. Equivalently, for instability $U'''(y_I)/U'(y_I) < 0$, or, since $\Omega = -U'$, the vorticity must be a minimum at the (sole) inflection point.

Both Rayleigh's and Fjørtoft's criteria are necessary conditions for instability. A necessary and sufficient condition for instability was given in Balmforth & Morrison (1999), which we will not describe here. However, all of the equilibrium flow profiles treated in the present paper are stable, and this can be guaranteed by Rayleigh's criterion, Fjørtoft's criterion, or the criterion of Balmforth & Morrison (1999). Henceforth we assume $c \in \mathbb{R}$ and in the remainder of this section we discuss neutral modes. We show why they constitute a continuous spectrum, why they are singular, and then we construct the singular eigenfunctions associated with them.

If $c \in \mathbb{R}$, then Rayleigh's equation is a second-order differential equation with singular points located where the phase velocity matches the equilibrium velocity, $c = U(y)$. Such points are commonly referred to as critical levels. If the flow profile is monotonic, then $c = U(y)$ has a single solution, $y = y_c$. This is the situation considered here. For general, nonmonotonic flow profiles, there may be more than one critical level for a given mode, and this introduces complications that will be treated in a later publication.

In the vicinity of the critical level it is straightforward to obtain two Frobenius series solutions of the form,

$$\psi_k^{(g)}(y, y_c) = [U(y) - c]\varphi_k^{(1)}(y, c) \tag{1.23}$$

and

$$\psi_k^{(b)}(y, y_c) = [U(y) - c] \log |U(y) - c| \varphi_k^{(2)}(y, c) + \varphi_k^{(3)}(y, c), \tag{1.24}$$

where the φ_k's are regular functions that do not in general vanish at the critical level. The 'good' solution, $\psi_k^{(g)}$, cannot satisfy the boundary condition, and this is, in part, the reason for the occurrence of the continuous spectrum. The proof of this uses a classical integral relation of the early nineteenth century (Green's transform) to understand the extension of the spectrum in the complex plane (see e.g. Hille 1976), which has been reconsidered in more recent literature (see e.g. Barston 1964 and Rosencrans & Sattinger 1966). Because the argument is not difficult, we include it here.

From the Frobenius solutions of (1.23) and (1.24) we see that $\psi_k^{(g)}$ is regular and vanishes at the critical point, while $\psi_k^{(b)}$ is constant to leading order, but has a discontinuous derivative at the critical level unless U'' vanishes there. (Note (1.24) can be rewritten as $\psi_k^{(b)} \sim 1 + U''(y_c)(y - y_c) \log |y - y_c|$.) Smooth eigensolutions can therefore be built if either there is no critical level within the channel, or if the eigenmodes contain only $\psi_k^{(g)}$ near $y = y_c$. However, the argument based on the classical integral relation indicates that such smooth solutions cannot satisfy the boundary conditions.

In terms of the displacement ξ_k, defined by $(U - c)\xi_k = \psi_k$, Rayleigh's equation becomes

$$\frac{d}{dy}\left[(U-c)^2\frac{d\xi_k}{dy}\right] - k^2(U-c)^2\xi_k = 0, \tag{1.25}$$

which has both a regular and a singular solution at $y = y_c$. Multiplying this equation by ξ_k^* and integrating over the interval $[y_1, y_2]$ gives

$$\left[(U-c)^2\xi_k^*\xi_k'\right]_{y_1}^{y_2} = \int_{y_1}^{y_2} [U(y) - c]^2 \left[|\xi_k'|^2 + k^2|\xi_k|^2\right] dy. \tag{1.26}$$

If there is no critical level (y_c is outside the channel), we select $y_1 = -1$ and $y_2 = 1$, then both sides of (1.26) must vanish and so there is no nontrivial solution. This establishes that only neutral modes with critical layers exist. If we now select $y_1 = -1$ and $y_2 = y_c$ or $y_1 = y_c$ and $y_2 = 1$, then we observe that, unless ξ_k is singular at the critical point, the solution is again trivial. This indicates that ξ_k must be singular at the critical layer. Equivalently, ψ_k must contain $\psi_k^{(b)}$ near $y = y_c$, so the streamfunction has no derivative there (unless $U''(y_c) = 0$). Consequently, the neutral modes are generically singular. However, such a mode exists for each phase velocity that satisfies $U = c$, and thus we have a continuous spectrum.

The eigenfunctions associated with these singular modes can be described by following Van Kampen (1955) and rewriting (1.17) as a distribution (generalized function)

$$\omega_k = \mathcal{P}\left(\frac{U''\psi_k}{U - C}\right) + \mathcal{C}_k\delta(U - c), \tag{1.27}$$

where $\mathcal{P}$ signifies the Cauchy principal value, $\mathcal{C}_k$ is (as yet) arbitrary, and δ is Dirac's delta function. (Van Kampen considered a solution in the form of (1.27) in the context of plasma oscillations; the solution was previously used in quantum scattering theory and perhaps first appeared in a paper by Rice 1929, according to Van Kampen's 1951 interpretation. In fluid mechanics the special case of constant shear flow was considered in Eliassen *et al.* 1953.)

By writing the vorticity of the eigenfunction in the form of (1.27), we explicitly display the singular nature of the mode. Its vorticity is composed of two pieces. There is a divergent, global component that corresponds to advection of the underlying flow by the perturbation, and relies on the presence of a mean vorticity gradient. For a linear velocity profile only the second term is present, and this represents a line vortex positioned at the critical level.

Eliminating ω_k between (1.17) and (1.18) yields the inhomogeneous integral equation

$$\psi_k(y, y_c) = \mathcal{P}\int_{-1}^{1} \mathcal{K}_k(y, y')\frac{U''(y')\psi_k(y', y_c)}{U(y') - c}\,dy + \frac{\mathcal{C}_k}{|U_c'|}\mathcal{K}_k(y, y_c), \tag{1.28}$$

where $U'_c = U'(y_c)$. The kernel of this integral equation is singular at the critical level, but we have yet to specify $\mathcal{C}_k$.

Because our problem is linear, the amplitude of the eigenfunction is indeterminate up to scaling. We determine this amplitude through an integral normalization. One that is especially convenient is (Kamp 1991, Kowalski & Feldman 1961, Sattinger 1966, Sedláček 1971)

$$\Xi_k = \int_{-1}^{1} \omega_k \, dy, \tag{1.29}$$

where Ξ_k is the modal amplitude, a quantity that can be taken to be any desired function of the eigenvalue c (or y_c). (In calculations that will be reported elsewhere we have found it acceptable to set Ξ_k to unity.) Inserting the singular eigenfunction (1.27) into (1.29) gives

$$\Xi_k = \frac{\mathcal{C}_k}{|U'_c|} + \mathcal{P} \int_{-1}^{1} \frac{U''(y')\psi_k(y', y_c)}{U(y') - U(y_c)} \, dy', \tag{1.30}$$

which is an equation for the unknown quantity $\mathcal{C}_k$. With this selection for $\mathcal{C}_k$ the integral equation (1.28) becomes

$$\psi_k(y, y_c) = f_k(y) + \int_{-1}^{1} \mathcal{F}_k(y, y'; y_c)\psi_k(y', y_c) \, dy', \tag{1.31}$$

where the inhomogeneous term $f_k(y) := \Xi_k \mathcal{K}_k(y, y_c)$ and the kernel is defined by

$$\mathcal{F}_k(y, y'; y_c) = \left[\frac{\mathcal{K}_k(y, y') - \mathcal{K}_k(y, y_c)}{U(y') - U(y_c)} \right] U''(y'). \tag{1.32}$$

This kernel is regular at the critical level (accordingly we have omitted the principal-value symbol), and it depends on a ‘parameter’, y_c. Thus, the choice of normalization (1.29) converts the singular integral equation (1.28) into the regular integral equation (1.31).

2 The integral transform and coordinate change

Now we turn to the description of our integral transform solution. In Section 2.1 we give some motivation for the form of the transform, and then state it and its inverse. In Section 2.2 we show that the inverse stated in Section 2.1 is indeed the inverse, and in Section 2.3 identities associated with the transform are given. Finally in Section 2.4 these identities are used to obtain an expression for the solution.

2.1 Integral transform pair

Perhaps the easiest way to explain the origin of the integral transform is to consider it to be a coordinate change designed with a basis composed of a continuum of the singular eigenfunctions of (1.27). A superposition of the continuum of singular eigenfunctions, which we label by y_c, with amplitudes given by $\Lambda_k(y_c, t)$, suggests the following transform:

$$G_k[\Lambda_k](y,t) := \int_{-1}^{1} \mathcal{G}_k(y, y_c)\Lambda(y_c, t)\, dy_c \tag{2.1}$$

where

$$\mathcal{G}_k(y, y_c) := \epsilon_k^{(r)}(y)\delta(y - y_c) + \mathcal{P}\frac{U''(y)\psi_k(y, y_c)}{U(y) - U(y_c)}, \tag{2.2}$$

$$\epsilon_k^{(r)}(y) := 1 - \mathcal{P}\int_{-1}^{1} \frac{U''(y')\psi_k(y', y)}{U(y') - U(y)} dy', \tag{2.3}$$

and $\psi_k(y, y_c)$ is a solution to (1.31). The transform (2.1) takes the function $\Lambda_k(y_c, t)$ into a function $\omega_k(y,t) := G_k[\Lambda_k](y,t)$.

Alternatively, upon inserting (2.2) into (2.1) we obtain the following equivalent form for the integral transform:

$$G_k[\Lambda_k](y,t) := \epsilon_k^{(r)}(y)\Lambda_k(y_c, t) + \mathcal{P}\int_{-1}^{1} \frac{U''(y)\psi_k(y, y_c)}{U(y) - U(y_c)}\Lambda_k(y_c, t)\, dy_c, \tag{2.4}$$

whence we see that G_k is the sum of a simple multiplicative piece plus a piece that is a generalization of the Hilbert transform (Titchmarsh 1937, Stein & Weiss 1971).

We will show that the inverse of (2.1), subject to the conditions of monotonicity of U and no discrete spectrum is given by the following integral transform:

$$\hat{G}_k[\omega_k](y_c, t) := \int_{-1}^{1} \hat{\mathcal{G}}_k(y_c, y)\omega_k(y,t) dy, \tag{2.5}$$

where

$$\hat{\mathcal{G}}_k(y_c, y) = \frac{1}{|\epsilon_k(y_c)|^2}\left[\epsilon_k^{(r)}(y_c)\delta(y - y_c) + \mathcal{P}\frac{U''(y_c)\psi_k(y, y_c)}{U(y) - U(y_c)}\right], \tag{2.6}$$

and where $|\epsilon_k(y)|^2 := (\epsilon_k^{(r)})^2 + (\epsilon_k^{(i)})^2$ with

$$\epsilon_k^{(i)} := -\pi\psi_k(y_c, y_c)\frac{U''(y)}{U'(y)}. \tag{2.7}$$

Note that on the right-hand sides of (2.1) and (2.5), the functions $\Lambda_k(y_c, t)$ and $\omega_k(y, t)$ are arbitrary functions (within the function space that defines the domain of the transforms, which we will not specify here) of y_c and y, respectively, and the explicit time dependence appears only as a parameter. We include this time dependence to emphasize that the integral transforms are coordinate changes; the actual time dependence will be determined by the equation of motion.

If $\hat{G}_k$ is to be the inverse of the transform G_k, then $\hat{G}_k[G_k[\Lambda_k]] \equiv \Lambda_k$, which follows if

$$\int_{-1}^{1} \hat{\mathcal{G}}_k(y_c', y)\mathcal{G}_k(y, y_c)\, dy = \delta(y_c - y_c'). \tag{2.8}$$

This inverse relation can also be viewed as a relation that verifies the completeness of the continuous spectrum. We verify (2.8) in the next section. In a similar fashion we have the reciprocal completeness relation,

$$\int_{-1}^{1} \hat{\mathcal{G}}_k(y_c, y')\mathcal{G}_k(y, y_c)\, dy_c = \delta(y - y'). \tag{2.9}$$

2.2 Transform inverse

Now we show that the transform (2.1) is the inverse of (2.5). In addition we give some other identities that will be of use to us in Section 2.4 and in Section 3. Under mild restriction on the profiles U, many rigorous results can be obtained. For example, using generalization of techniques associated with Hilbert transform theory it can be shown that the transforms are bounded linear operators on L_p spaces. We will not pursue this here, but instead direct the reader to Morrison (2000) where the corresponding proofs are given in the context of the Vlasov–Poisson equation.

If we substitute the explicit forms of $\hat{\mathcal{G}}_k(y_c, y)$ and $\mathcal{G}(y, y_c')$ into the integral on the left-hand side of equation (2.8), we find

$$\begin{aligned}\int_{-1}^{1} \hat{\mathcal{G}}_k(y_c', y)\mathcal{G}_k(y, y_c)\, dy \quad &= \quad \epsilon_k^{(r)}(y_c)^2\delta(y_c - y_c') \\ &+ \frac{U''(y_c)}{U(y_c) - U(y_c')}\left[\epsilon_k^{(r)}(y_c)\psi_k(y_c, y_c') - \epsilon_k^{(r)}(y_c')\psi_k(y_c', y_c)\right] \\ &+ \frac{U''(y_c')}{|\epsilon_k(y_c')|}\mathcal{P}\int_{-1}^{1} \frac{U''(y)\psi_k(y, y_c)\psi_k(y, y_c')}{[U(y) - U(y_c)][U(y) - U(y_c')]}\, dy. \end{aligned} \tag{2.10}$$

To evaluate the integral of the final term of (2.10) we use the relation,

$$\begin{aligned}
&U''(y_c')\,\mathcal{P}\int_{-1}^{1}\frac{U''(y)\psi_k(y,y_c)\psi_k(y,y_c')}{[U(y)-U(y_c')][U(y)-U(y_c)]}\,dy \\
&= \epsilon_k^{(i)}(y_c)^2\delta(y_c-y_c') + \frac{U''(y_c')}{U(y_c)-U(y_c')}\mathcal{P}\int_{-1}^{1}U''(y)\psi_k(y,y_c')\psi_k(y,y_c) \\
&\quad\times\left[\frac{1}{U(y)-U(y_c)}-\frac{1}{U(y)-U(y_c')}\right]dy, \qquad (2.11)
\end{aligned}$$

which is a form of the Poincaré–Bertrand transposition formula (see e.g. Gakhov 1990). Using (2.11) and collecting together terms, (2.10) becomes

$$\int_{-1}^{1}\hat{\mathcal{G}}(y_c',y)\mathcal{G}_k(y,y_c)\,dy = \delta(y_c-y_c') + \frac{U''(y_c')}{U(y_c)-U(y_c')}\frac{\mathcal{I}_k(y_c,y_c')}{|\epsilon_k(y_c')|^2}, \qquad (2.12)$$

where

$$\begin{aligned}
\mathcal{I}_k(y_c,y_c') \;:=\;& \epsilon_k^{(r)}(y_c)\psi_k(y_c,y_c') + \mathcal{P}\int_{-1}^{1}\frac{U''(y)\psi_k(y,y_c)\psi_k(y,y_c')}{U(y)-U(y_c)}\,dy \\
&- \epsilon_k^{(r)}(y_c')\psi_k(y_c',y_c) - \mathcal{P}\int_{-1}^{1}\frac{U''(y)\psi_k(y,y_c')\psi_k(y,y_c)}{U(y)-U(y_c')}\,dy. \qquad (2.13)
\end{aligned}$$

Recognizing that $\mathcal{C}_k/|U_c'| = \epsilon_k^{(r)}$, (1.31) can be rewritten as

$$\psi_k(y,y_c) = \epsilon_k^{(r)}(y_c) + \mathcal{P}\int_{-1}^{1}\mathcal{K}_k(y,y')\frac{U''(y')\psi_k(y',y_c)}{U(y')-U(y_c)}\,dy'. \qquad (2.14)$$

Insertion of (2.14) into the first and third terms of (2.13) and into the second ψ_k of the integrands of the second and fourth terms, reveals that $\mathcal{I}_k(y_c,y_c') \equiv 0$. Hence, we have verified (2.8) and thus that (2.1) is the inverse of (2.5).

2.3 Transform identities

Just as Fourier and Laplace transforms possess many useful identities, there are a variety of identities possessed by G_k and $\hat{G}_k$. For later use we state two such transform identities that will be used in Section 2.4:

$$\hat{G}_k[U\omega_k](y_c,t) = U(y_c)\hat{G}_k[\omega_k](y_c,t) + \frac{U''(y_c)}{|\epsilon_k|^2(y_c)}\mathcal{P}\int_{-1}^{1}\omega_k(y,t)\psi_k(y,y_c)\,dy, \qquad (2.15)$$

and

$$\hat{G}_k[U''\psi_k](y_c,t) = \frac{U''(y_c)}{|\epsilon_k|^2(y_c)}\mathcal{P}\int_{-1}^{1}\omega_k(y,t)\psi_k(y,y_c)\,dy, \qquad (2.16)$$

where (2.16) is valid if ω_k is related to ψ_k according to (1.18). The validity of these identities can be demonstrated in a manner similar to our demonstration of (2.8).

For the record we state two additional orthogonality-like identities,

$$\int_{-1}^{1} \hat{\mathcal{G}}_k(y_c, y)\hat{\mathcal{G}}_k(y_c', y)U''(y)\, dy = \frac{U''(y_c)}{|\epsilon_k|^2(y_c)}\delta(y_c - y_c')$$

and

$$\int_{-1}^{1} \left[\frac{U(y)}{U''(y)}\mathcal{G}_k(y, y_c) - \psi_k(y, y_c)\right]\mathcal{G}_k(y, y_c')\, dy = \frac{U(y_c)}{U''(y_c)}|\epsilon_k|(y_c)\delta(y_c - y_c').$$

These last two identities will not be needed in the rest of this paper.

2.4 Solution

Now we are in a position to use the transform pair to solve (1.13). Let $\Lambda_k(y,t) := \hat{G}_k[\omega_k]$ and then $\omega_k(y,t) = G_k[\Lambda_k]$. Transforming (1.13) with $\hat{G}_k$ gives

$$\frac{\partial \Lambda_k}{\partial t} + ikc\Lambda_k = 0, \tag{2.17}$$

where use has been made of both (2.15) and (2.16), with the obvious cancellation, and we have set $c = U(y_c)$. This simple form is obtained because the transform has been designed especially for (1.13); it is a coordinate change in which the equation becomes trivial to solve. The solution to (2.17) is clearly $\Lambda_k = \mathring{\Lambda}_k e^{-ikct}$, which upon transforming back gives

$$\omega_k(y,t) = G\big[\hat{G}[\mathring{\omega}_k]e^{-ikct}\big]. \tag{2.18}$$

Here we have written $\mathring{\Lambda}_k$ in terms of $\mathring{\omega}_k(y) := \omega_k(y, t = 0)$ by means of $\mathring{\Lambda}_k = \hat{G}[\mathring{\omega}_k]$.

The solution (2.18) is exact, but not entirely explicit because it requires $\psi_k(y, y_c)$, which in turn requires the solution of the integral equation (1.31). This is a fairly easy numerical exercise because (1.31) is a regular Fredholm equation. Most important is the fact that this calculation only needs to be done once for each flow profile, $U(y)$. After $\psi_k(y, y_c)$ is obtained, (2.18) gives the temporal dependence for all initial conditions. Over the past five years or so we have accumulated a variety of numerical results of this nature that will be reported elsewhere.

Before proceeding to the next section, where we show that the integral transform is in essence a canonical transformation to action-angle variables, we mention that the solution of (2.18) is an alternative to that obtained by using the Laplace transform to solve the initial value problem (see Case 1960, Engevik 1966, and Rosencrans & Sattinger 1966). Because both the Laplace transform solution and (2.18) are solutions they must be equal, and indeed with the wisdom of hindsight and some effort this can be shown. The solution (2.18) differs from the Laplace transform solution in that the integration involved in the transform G can be viewed as a sum over individual solutions varying as e^{-ikct}, while the Bromwich contour integral of the inverse Laplace transform solution is not a sum over solutions.

3 Hamiltonian interpretation

Now we interpret the solution of (2.18) in the Hamiltonian context. We begin in Section 3.1 by briefly reviewing the Hamiltonian description of Euler's equation in terms of the noncanonical Poisson bracket, the point of view adopted in Morrison & Greene (1980). In Section 3.2 the noncanonical Hamiltonian structure for the linearized shear flow dynamics is canonized (converted into the conventional canonical form) and then diagonalized (transformed into a set of uncoupled oscillators). In Section 3.3 modal signature and the ramifications of negative energy modes are discussed. In Section 3.4 we see that Rayleigh-type stability criteria amount to energy criteria.

3.1 Hamiltonian structure

The Hamiltonian structure possessed by equations that describe continuous media in terms of Eulerian variables has a Poisson bracket that is degenerate and not of the canonical form. This is because the variables that are usually used to describe fluids, as well as those of magnetohydrodynamics, kinetic theories, and other media theories, are not canonical variables. This formulation in terms of noncanonical variables, with associated noncanonical Poisson bracket, is employed here. However, we note that there are other forms that the Hamiltonian structure of fluid mechanics can assume; for example, the canonical form in terms of Lagrangian or material variables, which dates to the nineteenth century, and the Hamiltonian description in terms of a degenerate Lagrange bracket or two-form (see e.g. Appendix 2 of Arnol'd 1978). We direct the reader to Morrison (1998) where the Hamiltonian description of fluid dynamics is extensively reviewed.

In the present context, the upshot of the noncanonical Hamiltonian structure is that Euler's equation for the fluid can be written as follows:

$$\frac{\partial \omega}{\partial t} = \{\omega, H\} = [\omega, \psi], \tag{3.1}$$

where the noncanonical Poisson bracket is given by

$$\{A, B\} = \int_D \omega \left[\frac{\delta A}{\delta \omega}, \frac{\delta B}{\delta \omega}\right] dy\, dx, \tag{3.2}$$

the Hamiltonian functional H is given by (1.5), and

$$[\omega, \psi] := \frac{\partial \omega}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}.$$

In the bracket of (3.2), $A[\omega]$ and $B[\omega]$ are functionals of the dynamical variable ω and the functional derivative $\delta A/\delta\omega$ is defined by

$$\delta A = \int_D \frac{\delta A}{\delta \omega}\delta\omega \, dy \, dx. \tag{3.3}$$

It is evident that the bracket of (3.2) is not of canonical form for the following reasons: it depends explicitly upon ω, there is only a single dynamical field variable, ω, rather than a canonically conjugate pair, and it is degenerate. Degeneracy is evident because of the existence of the Casimir invariants of (1.7), which by definition satisfy $\{A, C\} = 0$ for all functionals A.

The linear dynamics obtains its Hamiltonian structure from that of the full dynamics given in (3.1) by linearizing. This procedure involves linearizing the bracket as well as the Hamiltonian. Upon inserting $\omega = \Omega + \delta\omega$ into the bracket and retaining the lowest order piece, we obtain

$$\{A, B\}_L = \int_D \Omega \left[\frac{\delta A}{\delta\delta\omega}, \frac{\delta B}{\delta\delta\omega}\right] dy \, dx, \tag{3.4}$$

where $\delta A/\delta\delta\omega$ is the functional derivative of a functional A of $\delta\omega$ with respect to the linear dynamical variable, $\delta\omega$. Because of the existence of the Casimir invariants, the Hamiltonian used in (3.1) is not unique: the quantity $F = H + C$, for any C, can serve as a Hamiltonian because $\{A, F\} = \{A, H\}$, for all functionals A. This nonuniqueness is exploited in obtaining the Hamiltonian for the linear dynamics. We choose C so that $\delta F/\delta\omega = 0$ is satisfied by the equilibrium vorticity of interest, Ω, and then expand F to second order in $\delta\omega$. This second order quantity is the expression $H_L[\delta\omega] := \delta^2 F$ given by (1.11), that serves as the Hamiltonian for the linear dynamics. With the Hamiltonian H_L and the bracket of (3.4), the linear dynamics of (1.10) has the Hamiltonian form

$$\frac{\partial \delta\omega}{\partial t} = \{\delta\omega, H_L\}_L.$$

We now turn to the task of transforming this Hamiltonian structure into a canonical, diagonal (action-angle) form.

3.2 Canonization and diagonalization

Canonization of the bracket of (3.4) is facilitated by introducing the Fourier representation:

$$\delta\omega(x, y) = \frac{1}{\sqrt{2\pi}} \int_R e^{ikx} \omega_k(y) \, dk, \qquad \omega_k(y) = \frac{1}{\sqrt{2\pi}} \int_R e^{-ikx} \delta\omega(x, y) \, dx. \tag{3.5}$$

Using (3.5) it is not difficult to show that the functional derivative transforms according to

$$\frac{\delta A}{\delta\delta\omega} = \frac{1}{\sqrt{2\pi}} \int_R e^{-ikx} \frac{\delta A}{\delta\omega_k} dk, \tag{3.6}$$

which in essence follows from the chain rule for functional derivatives. To see this we insert the first variation of (3.5) into the expression analogous to (3.3) for functionals of the variable $\delta\omega$,

$$\begin{aligned} \delta A &= \int_D \frac{\delta A}{\delta\delta\omega} \delta\delta\omega \, dy \, dx = \int_D \frac{\delta A}{\delta\omega_k} \delta\omega_k \, dy \, dk \\ &= \int_D \frac{\delta A}{\delta\omega_k} \frac{1}{\sqrt{2\pi}} \int_R e^{-ikx} \delta\delta\omega(x,y) \, dx \, dy \, dk. \end{aligned}$$

Interchanging the order of integration and equating coefficients of $\delta\delta\omega$ gives (3.6).

Inserting (3.6) and the corresponding formula for $\delta B/\delta\delta\omega$ into (3.4), integrating by parts, and mapping the $k<0$ region of integration to $k>0$, yields the following Poisson bracket:

$$\{A,B\}_L = \int_0^\infty \int_R ik\Omega' \left(\frac{\delta A}{\delta\omega_k} \frac{\delta B}{\delta\omega_{-k}} - \frac{\delta B}{\delta\omega_k} \frac{\delta B}{\delta\omega_{-k}} \right) dy \, dk, \tag{3.7}$$

where henceforth we denote the domain of integration of (3.7) by $D_k := \mathbb{R}^+ \times \mathbb{R}$.

Since the domain of integration is only over positive values of k, it is clear that ω_k and ω_{-k} are independent variables, and it is also clear from (3.7) that if it were not for the factor $ik\Omega'$, these variables would be canonical variables. Evidently a simple scaling is required for canonization, and we do this by introducing the new variables

$$q_k(y,t) := \omega_k(y,t), \qquad p_k(y,t) := \frac{\omega_{-k}(y,t)}{ik\Omega'}. \tag{3.8}$$

With this choice the bracket (3.7) obtains the canonical form

$$\{A,B\}_L = \int_{D_k} \left(\frac{\delta A}{\delta q_k} \frac{\delta B}{\delta p_k} - \frac{\delta B}{\delta q_k} \frac{\delta A}{\delta p_k} \right) dy \, dk, \tag{3.9}$$

where q_k and p_k are canonically conjugate variables.

The transformation of (3.8) appears to be singular at $k=0$ and at points (values of y) where $\Omega'=0$. As pointed out in Section 2, we assume $\Omega(y)$ supports no discrete eigenmodes, but this can be the case even when $\Omega'=0$, so singularity due to $\Omega'=0$ could be an issue. (Recall that Rayleigh's criterion is not necessary for linear stability.) However, the condition of dynamical accessibility discussed in Section 1.1 removes both the $k=0$ and the $\Omega'=0$ singularities. This follows because in terms of the Fourier variable ω_k, (1.12) becomes $\omega_k = ikh_k\Omega'$, and therefore ω_k vanishes at $k=0$ and at points where $\Omega' = 0$. The choice of canonical variables is designed so that the dynamics takes place on the constraint surface (symplectic leaf) determined by the equilibrium Ω, which in turn is determined by the value of the Casimir invariants. The singularities we are removing arise from the degeneracy in the original noncanonical bracket and the choice of coordinates (3.8) that removes them also removes this degeneracy.

The Hamiltonian for the linear dynamics is given by (1.11), which with the insertion of (3.5) becomes

$$\begin{aligned} H_L &= \int_{D_k} \left(\frac{U}{U''}\omega_k - \psi_k \right) \omega_{-k}\, dy\, dk, \\ &= \int_{D_k} \int_{D'_k} \omega_k(y) \mathcal{O}_{k,k'}(y|y') \omega_{k'}(y')\, dy\, dk\, dy'\, dk', \end{aligned} \tag{3.10}$$

where

$$\mathcal{O}_{k,k'}(y|y') := \frac{U}{U''} \delta_{k,-k'} \delta(y-y') - \mathcal{K}_k(y,y') \delta_{k,-k'}. \tag{3.11}$$

From (3.10) or (3.11) it is clear that this Hamiltonian is not diagonal; i.e. it does not possess a form that is the infinite dimensional generalization of either of the forms of (1.1). This remains true even after rewriting it in terms of the canonical variables of (3.8). However, we will see that a coordinate change that uses the integral transform of Section 2 achieves this diagonalization.

We wish to diagonalize the Hamiltonian while maintaining the Hamiltonian structure. This can be done by introducing the following *mixed variable generating functional*, the essence of which is determined by the integral transform of (2.5):

$$\mathcal{F}[q,P] = \int_{D_k} P_k(y)\, \hat{G}[q_k](y)\, dy\, dk. \tag{3.12}$$

The transformation to the new canonical variables (Q_k, P_k) are given by the following:

$$p_k(y) = \frac{\delta \mathcal{F}[q,P]}{\delta q_k(y)} = \hat{G}^\dagger[P_k](y), \qquad Q_k(y_c) = \frac{\delta \mathcal{F}[q,P]}{\delta P_k(y_c)} = \hat{G}[q_k](y_c). \tag{3.13}$$

That this generates a canonical transformation can be verified directly. By a chain rule procedure similar to that used in obtaining (3.6), we obtain

$$\frac{\delta A}{\delta q_k} = \hat{G}^\dagger \left[\frac{\delta F}{\delta Q_k} \right], \qquad \frac{\delta B}{\delta p_k} = G \left[\frac{\delta B}{\delta P_k} \right], \tag{3.14}$$

where $\hat{G}^\dagger$ is the adjoint of $\hat{G}$. Upon inserting (3.14) into (3.9), using the adjoint property, and recognizing that $\hat{G}$ is the inverse of G we obtain

$$\begin{aligned}\{A,B\}_L &= \int_{D_k}\left(\frac{\delta A}{\delta q_k}\frac{\delta B}{\delta p_k} - \frac{\delta B}{\delta q_k}\frac{\delta A}{\delta p_k}\,dy\,dk\right)dy\,dk \\ &= \int_{D_k}\left(\frac{\delta A}{\delta Q_k}\frac{\delta B}{\delta P_k} - \frac{\delta B}{\delta Q_k}\frac{\delta A}{\delta Q_k}\right)dy\,dk. \end{aligned} \tag{3.15}$$

This verifies that (3.13) defines a canonical transformation.

Now we will insert the transformations of (3.13) into the Hamiltonian of (3.10) and show that this gives a diagonal form. Rewriting (3.10) as

$$H_L = \int_{D_L} (-ikUq_kp_k + ikU''\psi_kp_k)\,dy\,dk$$

and inserting $p_k = \hat{G}^\dagger[P_k]$ and $q_k = G[Q_k]$ yields

$$H_L = -i\int_{D_L} kP_k\big(\hat{G}[UG[Q_k]] - \hat{G}[U''\psi_k]\big)\,dy_c\,dk, \tag{3.16}$$

which upon making use of (2.15) and (2.16) gives

$$H_L = -i\int_{D_k} kUP_kQ_k\,dy_c\,dk = i\int_{D_k}\nu_kP_kQ_k\,dy_c\,dk. \tag{3.17}$$

Equation (3.17) is a kind of Hamiltonian normal form; it is a Hamiltonian that completely decouples each dimension of the system. However, it is not yet of the form of (1.1), our initial goal. So, we now take this last step to achieve our goal.

It is well-known lore of Hamiltonian mechanics that if one restricts to real canonical transformations, then for stable (purely oscillating) systems one cannot decouple each dimension; the best one can do is decouple each degree of freedom. Above we introduced complex analysis in order to avoid messy calculations; we could have defined the integral transform in such a way that only real variables were used. If we had done so, we would have arrived at the normal form of (1.1). It is a simple matter to recover this real variable Hamiltonian normal form by performing the following transformation:

$$Q_k(y_c,t) = \sqrt{J_k(y_c,t)}\,e^{i\sigma_k\theta_k(y_c,t)}, \qquad P_k(y_c,t) = -i\sqrt{J_k(y_c,t)}\,e^{-i\sigma_k\theta_k(y_c,t)},$$

where we have introduced a definition of signature, $\sigma_k(y_c) := \mathrm{sgn}(U''/U)$, an important quantity that will be discussed in the next subsection. In terms of the canonical variables (θ_k, J_k) the Hamiltonian becomes

$$H_L = \int_{D_k}\sigma_k(y_c)\nu(y_c)J_k(y_c,t)\,dy_c\,dk, \tag{3.18}$$

where $\nu_k(y_c) := |kU(y_c)|$ and the Poisson bracket becomes

$$\{F, G\}_L = \int_{D_k} \left(\frac{\delta F}{\delta \theta_k} \frac{\delta G}{\delta J_k} - \frac{\delta G}{\delta \theta_k} \frac{\delta F}{\delta J_k} \right) dy_c \, dk.$$

Thus we have achieved our goal of obtaining a continuum generalization of the oscillator form given in (1.1).

3.3 Signature and negative energy modes

In the course of arriving at the normal form of (3.18), the continuum version of (1.1), we defined the signature $\sigma_k(y_c) := \mathrm{sgn}(U''/U)$. This signature is the sign of the energy associated with a (stable) continuum eigenfunction. When σ_k is negative, the continuum eigenmode is a negative energy mode. A mode can have negative energy when the equilibrium state about which we expand, in order to obtain the linear theory, is not a minimum energy state. Vacuum states or states in thermodynamic equilibrium are minimum energy states, while an equilibrium state composed of a flow with shear is not. Consequently, for some shear flow equilibria there can be singular modes with $\mathrm{sgn}(U''/U) < 0$, i.e. there can exist negative energy modes whose presence lowers the energy of the system to below that of the equilibrium alone. In a system with a shear flow equilibrium, one can easily imagine that a perturbation that slows down the flow will remove kinetic energy and thus lower the energy of the system.

Although our definition of signature for the continuum modes above and its predecessor in Vlasov theory (Morrison & Pfirsch 1992) are novel, signature is a well-known concept in finite degree-of-freedom Hamiltonian systems theory. Also, it is well-known in infinite dimensional systems that describe the beam–plasma instability (see e.g. Davidson 1972 and many references therein), and it has previously appeared in the context of fluids (Cairns 1979). However, these developments describe modes that are not continuum modes, but normal discrete modes. Let us now turn to a discussion of the ramifications of signature.

As noted in the Introduction, negative energy modes and signature arise in the context of bifurcation theory and stability theory. That the signature of modes can have repercussions on possible bifurcations was shown for Hamiltonian systems of finite dimension by Kreĭn (1950) and Moser (1958) (see Kreĭn & Jakubovič 1980 for a collection of original papers). These authors established that the Hamiltonian Hopf bifurcation is only possible when pairs of modes of opposite signature collide on the real (stable) axis of the spectral plane. For discrete modes in the infinite dimensional plasma physics context, this is a well-known result, and in a fluid dynamical context, MacKay and Saffman (1986) have previously considered the role of signature in bifurcations of water waves. For infinite-dimensional systems with a continuous spectrum there is an analogue of Kreĭn's theorem, which states among other things that instabilities must emerge from places where positive and negative continua meet, but a discussion of this is beyond our present scope.

Signature also has implications regarding dissipative and finite-amplitude instability. Finite-dimensional systems that are linearly stable but possess negative energy modes are generically destabilized by the addition of dissipation. This is the so-called Thompson–Tait theorem (Poincaré, 1885; Lamb, 1907; Thompson & Tait, 1888; Greene & Coppi, 1965). Linearly stable systems which have both positive and negative energy modes (in all frames of reference) can be destabilized by the inclusion of nonlinear mode interaction (Cherry, 1925; Davidson, 1972; Weiland & Wilhelmsson, 1977; Kueny & Morrison, 1995; Morrison, 1998). Thus, we expect that equilibrium shear flow profiles with negative-energy continuous spectra can be destabilized by the addition of dissipation (the generalization of the Thompson–Tait theorem), and that linearly stable flows that violate Fjørtoft's criterion for stability may be nonlinearly unstable.

The importance of signature leads one to ask whether or not signature is invariant under coordinate changes. For time-independent transformations that involve only the dependent variables, Sylvester's theorem (Whittaker, 1937) guarantees that this is the case. However, for time-dependent transformations that have an explicit time dependence, energy is not a covariant quantity and signature can change. An important example, which is relevant here, is afforded by Galilean transformations to frames moving at a constant velocity in the streamwise direction. This is a time-dependent canonical transformation on the Lagrangian variable level.

Normally, equilibria that are equivalent up to Galilean transformations are obtained as extremal points of the Hamiltonian plus a constant multiple of the conserved, total momentum. In noncanonical variables this leads us to extremize

$$F_u[\omega] := F + uP_x, \tag{3.19}$$

where u is the Galilean boost velocity, and P_x the total streamwise momentum, as given in (1.6). Equation (3.19) provides us with a new Hamiltonian for linear theory in a moving or boosted frame:

$$H_L^u = \frac{1}{2}\int_D \left(|\nabla\delta\psi|^2 + \frac{(U-u)}{U''}\delta\omega^2 \right) dy\, dx. \tag{3.20}$$

Therefore, the second term of the integrand of (3.20), for any value of y, has a sign that depends upon u. Although one can change the sign of an individual mode by choosing a frame appropriately, one cannot, for general U, find a frame in which all of the modes possess the same sign. It is because of this and the fact that dissipation is also frame dependent, that the discussion above regarding the significance of signature for bifurcation and stability has meaning. Now we further discuss stability.

3.4 Rayleigh-like criteria as energy stability criteria

The Hamiltonian H_L^u of (3.20) is the physical perturbation energy in the boosted frame. Such energy expressions are very useful for obtaining stability criteria. If the energy is such that all perturbations increase or decrease the total energy of the system, then the corresponding configuration is stable. This is a basic means for ascertaining stability in Hamiltonian systems, which is sometimes called Dirichlet's theorem, and is in fact the essence of the energy-Casimir method of Kruskal and Oberman (1958) and Arnol'd (1965, 1966) (see also Holm *et al.* 1985 and Morrison 1998). In infinite-dimensional Hamiltonian systems, a positive or negative definite form for the perturbation energy guarantees linear (and is suggestive of nonlinear) stability.

In the current context, Dirichlet's theorem states that linear stability is assured by the definiteness of either H_L or H_L^u. The latter expression is more powerful for ascertaining stability because it contains u, which is an arbitrary constant parameter that can be used to advantage. Definiteness of H_L^u for some u implies stability in the frame boosted by u, and because stability is frame independent this implies stability in all frames. We will use this idea to obtain the Rayleigh and Fjørtoft criteria. (Arnol'd (1965) suggests this frame shift idea but does not make it explicit.)

If there is no inflection point, then U'' has the same sign for all $y \in [-1, 1]$. Therefore, one can select u so that

$$(U - u)U'' > 0 \tag{3.21}$$

throughout the flow. Since the first term of (3.20) is obviously positive, the energy H_L^u can be made positive definite by an appropriate choice of frame. Thus we see that Rayleigh's criterion (the nonexistence of inflection points implies stability) is in fact an energy stability criterion (a positive definite energy functional implies stability). Hence, we see there is a deep physical reason for the stability underlying the ad hoc manipulations of Rayleigh's criterion that were performed in Section 1.2.

When there are no inflection points there are no discrete modes, and therefore the coordinate change of (3.13) is well-defined and we can then introduce into H_L^u the transformation to action-angle variables. This gives the diagonal Hamiltonian of (3.18) with the frequency shifted, i.e. with $\sigma_k \nu_k$ replaced by

$$\sigma_k \nu_k^u = k(U - u)\,\mathrm{sgn}(U''). \tag{3.22}$$

From this expression it is clear that the signature is frame dependent and that there is a frame in which all the modes have positive signature.

Fjørtoft's sufficient condition for stability, which as we saw in Section 1.2 applies when the equilibrium profile has an inflection point, also follows directly from (3.20). This is seen by taking u to be the flow speed at $y = y_I$. Clearly, if $U''/(U - U_I) > 0$, then the energy $H_L^{U_I}$ is positive definite and we have stability. Therefore, like Rayleigh's theorem, Fjørtoft's condition is simply a statement of Dirichlet's energy theorem.

Note that when the system is stable by Fjørtoft's condition, then we can use the transformation to action-angle variables and write the Hamiltonian H_L in diagonal form.

Both of the energy criteria above are sufficient conditions for stability that are based on positive definiteness of the energy. However, Dirichlet's theorem gives sufficient conditions for stability when the energy is either positive or negative definite. Thus, given an equilibrium, it is natural to ask whether or not H_L is negative definite. For this to be the case the first term of (1.11) must dominate the second. This idea is the essence of what is sometimes called Arnol'd's second theorem (Arnol'd 1965, 1966), which uses Poincaré's inequality to formally estimate the first term in comparison to the second and to thereby show that the energy is negative definite. When the system is stable by this criterion, we can again use the transformation to action-angle variables and write the Hamiltonian H_L in the diagonal form. The resulting diagonal form has a negative integrand.

Now we treat one last case, which is possibly the most interesting. It is possible to have one or more inflection points and for the system to be stable. This was discussed in Balmforth & Morrison (1999) where necessary and sufficient conditions for stability were obtained by using the Nyquist method. When there is more than one inflection point, there is no frame in which the energy of (3.20) is sign definite. This is true whether or not the system is stable. However, when the system is stable, the transformation to action-angle variables that we described in Section 3.2 exists, but the resulting Hamiltonian of (3.18), with frequencies shifted according to (3.22), cannot be made sign definite by a choice of u.

When the Hamiltonian is indefinite and the system is in fact unstable, or when there exist discrete neutral modes embedded in the continuous spectrum, then the transformation of Section 3.2 fails. The integral transforms G_k and $\hat{G}_k$ are no longer well-defined inverses of each other. Thus, one should not attempt to effect the transformation and then ascertain stability. When a system is unstable or has embedded modes, the appropriate transformation is obtained by adding in the discrete modes. The resulting transformation does not lead to the normal form of (3.18), but to a different normal form, one that is appropriate for either the unstable or embedded-mode Hamiltonian systems. We will record this in a future paper.

4 Conclusions

In this paper we have described the shear flow problem and have reviewed associated material that was needed for our purposes. We introduced an integral transform, a generalization of the Hilbert transform, that amounts to a coordinate change to variables that make the linear shear flow dynamics trivial. We have briefly reviewed the noncanonical Hamiltonian structure of Euler's equation and we have used it to interpret the integral transform in the Hamiltonian context; i.e. we showed how to use the integral transform to obtain the mixed variable generating functional that transforms the system to action-angle variables. We defined signature for continuum eigenmodes, which enabled us to identify negative energy modes. The role of signature in bifurcation and stability theory was discussed. It was shown how the Rayleigh and Fjørtoft criteria for stability are in essence energy criteria for stability.

One of the reasons for studying Hamiltonian systems is because of their generality; one discovers properties for one system that turn out to be true for a large class of systems. It is evident from what we have presented here and from previous work on the Vlasov equation, that the subject matter of this paper is far reaching. The techniques we have developed apply to a large class of Hamiltonian systems that possess linear theories with continuous spectra. There is much more we could say, and there are many future directions in which to proceed. Below we briefly describe the content of some ongoing research.

We have used the integral transform as a numerical tool for examining the long time behaviour of linear shear flow, and we have weakened the requirements on the profiles U. Namely, the monotonicity requirement and the requirement that profiles only support continuous spectra can be relaxed. The level of mathematical rigor given in Morrison (2000) can be achieved for the shear flow problem, although with greater difficulty.

The Hamiltonian interpretation presented here suggests many additional projects, such as a rigorous discussion of Kreĭn's theorem and a theory of adiabatic invariants for systems with the continuous spectrum. We leave these and many other topics to possible future publications.

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References

Arnol'd, V. 1965. Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid. *Dokl. Akad. Nauk SSSR* **162**, 975–978. Eng. transl.: *Soviet Math. Dokl* **6**, 773–777 (1965).

Arnol'd, V. 1966. On an a priori estimate in the theory of hydrodynamic stability. *Izv. Vyssh. Uchebn. Zaved. Matematika* **54(5)**, 3–5. Eng. transl.: *Am. Math. Soc. Transl. Ser. 2* **79**, 267–269 (1969).

Arnol'd, V. 1978. *Mathematical Methods of Classical Mechanics*. Springer-Verlag.

Balmforth, N.J. & Morrison, P.J. 1995a. Normal modes and continuous spectra. *Ann. New York Acad. of Sci.* **773**, 80–94.

Balmforth, N.J. & Morrison, P.J. 1995b. Singular eigenfunctions for shearing fluids I. *Institute for Fusion Studies, University of Texas, Austin*, Report No. **692**. Available as NTIS document DE95008952INZ. See `http://www.ntis.gov`.

Balmforth, N.J. & Morrison, P.J. 1999. A necessary and sufficient instability condition for inviscid shear flow. *Stud. in Appl. Math.* **102**, 309–344.

Barston, E.M. 1964. Electrostatic oscillations in inhomogeneous cold plasmas. *Annals Phys.* **29**, 282–303.

Cairns, R.A. 1979. The role of negative energy waves in some instabilities of parallel flows. *J. Fluid Mech.* **92**, 1–14.

Case, K.M. 1960. Stability of inviscid plane Couette flow. *Phys. Fluids* **3**, 143–148.

Cherry, T.M. 1925. Some examples of trajectories defined by differential equations of a generalized dynamical type. *Trans. Cam. Phil. Soc.* **23**, 199–200.

Davidson, R.C. 1972. *Methods in Nonlinear Plasma Theory*. Academic Press.

Greene, J.M. & Coppi, B. 1965. Dissipative instability in mirror containment systems. *Phys. Fluids* **8**, 1745–1747.

Eliassen, A., Høiland, E. & Riis, E. 1953. Two-dimensional perturbation of a flow with constant shear of a stratified fluid. *Institute for Weather and Climate Res. Oslo, Sweden* Pub. No. **1**.

Engevik, L. 1966. On a stability problem in hydrodynamics. *Dept. Applied Math. Bergen, Norway* Pub. No. **11**.

Fjørtoft, R. 1950. Application of integral theorems in deriving criteria of stability of laminar flow and for the baroclinic circular vortex. *Geofys. Publ.* **17**, 1–52.

Gakhov, F.D. 1990. *Boundary Value Problems*. Dover.

Holm, D.D., Marsden, J.E., Ratiu, T.S., & Weinstein, A. 1985. Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.* **82**, 1–116.

Hille, E. 1976. *Ordinary Differential Equations in the Complex Domain*. Dover.

Kamp, L.P.J. 1991. Integral-equation approach to the instability of two-dimensional sheared flow of inviscid fluid in a rotating system with variable Coriolis parameter. *J. Phys. A: Math. Gen.* **24**, 2029–2052.

Kowalski, K.L. & Feldman, D. 1961 Transition matrix for nucleon–nucleon scattering I. *J. Math. Phys.* **2**, 499–511.

Kreĭn, M.G. 1950. A generalization of some investigations on linear differential equations with periodic coefficients. *Dokl. Akad. Nauk SSSR A* **73**, 445–448.

Kreĭn M.G. & Jakubović, V.A. 1980. *Four Papers on Ordinary Differential Equations*. American Mathematical Society.

Kruskal, M.D. & Oberman, C. 1958. On the stability of plasma in static equilibrium. *Phys. Fluids* **1**, 275–280.

Kueny, C.S. & Morrison, P.J. 1995. Nonlinear instability and chaos in plasma wave–wave interactions. *Phys. Plasmas* **2**, 1926–1940.

Lamb, H. 1907. On kinetic stability. *Proc. Roy. Soc.* **80**, 168–177.

MacKay, R.S. & Saffman, P.C. 1986. Stability of water waves. *Proc. R. Soc. Lond. A* **406**, 115–125.

Morrison, P.J. & Greene, J.M. 1980. Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics. *Phys. Rev. Lett.* **45**, 790–793; **48**, 569 (1982).

Morrison, P.J. 1982. Poisson brackets for fluids and plasmas. In *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems*, M. Tabor and Y. Treve (eds.), American Institute of Physics conference proceedings, no. 88.

Morrison, P.J. 1998. Hamiltonian description of the ideal fluid. *Rev. Mod. Phys.* **70**, 467–521.

Morrison, P.J. 2000. Hamiltonian description of Vlasov dynamics: action-angle variables for the continuous spectrum. *Trans. Theory and Stat. Phys.* **29**, 397–414.

Morrison, P.J. & Pfirsch, D. 1989. Free energy expressions for Vlasov–Maxwell equilibria. *Phys. Rev. A* **40**, 3898–3910.

Morrison, P.J. & Pfirsch, D. 1990. The free energy of Maxwell–Vlasov equilibria. *Phys. Fluids B* **2**, 1105–1113.

Morrison, P. J. & Pfirsch, D. 1992. Dielectric energy versus plasma energy, and action-angle variables for the Vlasov equation. *Phys. Fluids B* **4**, 3038–3057.

Morrison, P.J. & Shadwick, B. 1994. Canonization and diagonalization of an infinite dimensional noncanonical Hamiltonian system: linear Vlasov theory. *Acta Phys. Pol.* **85**, 759–769.

Moser, J.K. 1958. New aspects in the theory of stability of Hamiltonian systems. *Comm. Pure Appl. Math.* **11**, 81–114.

Poincaré, H. 1885. Sur I'equilibre d'une masse fluide animée d'un mouvement de rotation. *Acta Math.* **VII**, 259–380.

Rayleigh, J.W.S. 1880. On the stability, or instability, of certain fluid motions. *Proc. Lond. Math. Soc.* **9**, 57–70.

Rayleigh, J.W.S. 1896. *Theory of Sound*, Art. 369, Macmillan.

Rice, O.K. 1929. Perturbations in molecules and the theory of predissociation and diffuse spectra. *Phys. Rev.* **33**, 748–759.

Rosencrans, S.I. & Sattinger, D.H. 1966. On the spectrum of an operator occurring in the theory of hydrodynamic stability. *J. Math. and Phys.* **45**, 289–300.

Shadwick, B. 1995. *On the Hamiltonian structure of the linearized Maxwell–Vlasov system*, Ph.D. thesis, The University of Texas at Austin, Austin, TX.

Sattinger, D.H. 1966. A singular eigenfunction expansion in anisotropic transport theory. *J. Math. Anal. and App.* **15**, 497–511.

Sedláček, Z. 1971. Electrostatic oscillations in cold inhomogeneous plasma. II: Integral equation approach. *J. Plasma Phys.* **6**, 187–199.

Stein, E.M. & Weiss, G. 1971. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press.

Thiffeault, J.-L. & Morrison, P.J. 2000. Classification of Casimir invariants of Lie–Poisson brackets. *Physica D* **136**, 205–244.

Thompson, W. & Tait, P.G. 1988. *Treatise on Natural Philosophy*, part 1, p. 388. Cambridge University Press.

Titchmarsh, E.C. 1937. Introduction to the theory of Fourier integrals. Clarendon Press.

Van Kampen, N.G. 1951. Contribution to the quantum theory of light scattering. *Kgl. Danske Videnskab. Selskab, Mat-Fys. Medd.* **26**, no. 15, 1–77.

Van Kampen, N.G. 1955. On the theory of stationary waves in plasmas. *Physica* **21**, 949–963.

Vanneste, J. 1996. Rossby wave interactions in a shear flow with critical levels. *J. Fluid Mech.* **323**, 317–338.

Weiland, J. & Wilhelmsson, H. 1977. *Coherent nonlinear interaction of waves in plasmas*. Pergamon Press.

Whittaker, E.T. 1937. *Analytical dynamics*. Cambridge University Press.