Y. Aurégan A. Maurel V. Pagneux
J.-F. Pinton (Eds.)

Sound–Flow Interactions

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Editors

Yves Aurégan
Vincent Pagneux
Université du Maine, UMR CNRS 6613
Laboratoire d'Acoustique de l'Université du Maine
Av. O. Messiaen
72017 Les Mans, Cedex 9, France

Jean-Francois Pinton
Ecole normale supérieure de Lyon,
UMR CNRS 1325
Laboratoire de Physique
46 allée d'Italie
69007 Lyon, France

Agnes Maurel
ESPCI, UMR CNRS 7587
Lab. Ondes et Acoustique
10 rue Vauquelin
75005 Paris, France

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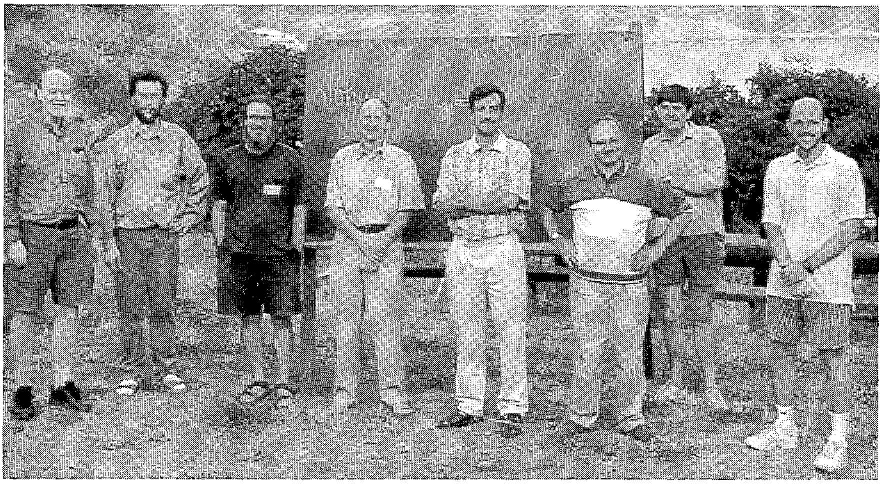
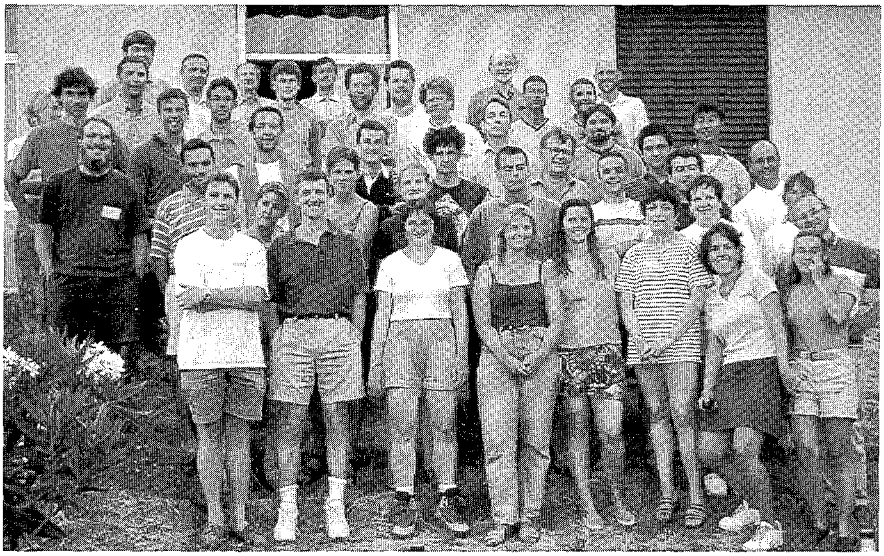


Some of the lecturers of the Cargèse School, from left to right: M.S. Howe, A. Hirschberg, P. Morrison, W. Lauterborn, V. Ostashev, A. Fabrikant, N. Peake, T. Colonius *(Photo C. Schram)*



Some of the participants of the Cargèse School *(Photo C. Schram)*

Singular Eigenfunctions and an Integral Transform for Shear Flow

Philip J. Morrison

Department of Physics and Institute for Fusion Studies
The University of Texas, Austin, TX 78712, USA

Abstract. Euler's equation linearized about a shear flow equilibrium is solved by means of a novel invertible integral transform that is a generalization of the Hilbert transform. The integral transform provides a means for describing the dynamics of the continuous spectrum that is well-known to occur in this and other systems. The results are interpreted briefly in the context of infinite dimensional Hamiltonian systems theory, which serves as a unifying principle.

1 Introduction

This paper is about the classic problem of inviscid incompressible shear flow in a channel. The main contribution is an integral transform that is of general utility for describing physical systems with a continuous spectrum. The transform was previously developed with collaborators in a series of papers pertaining to the Vlasov–Poisson [14,12,15] and shear flow [1–3] systems. The reader is referred to these works for greater detail.

In Sect. 2 we describe Euler's equations in the channel geometry and discuss the equilibria of interest. The integral transform is applicable to the linear dynamics about these equilibria. We describe the normal mode approach and its associated difficulties because of the critical level singularity. Singular eigenfunctions are constructed. In Sect. 3 we introduce the transform, its inverse, and some identities, which are then used to obtain a solution of our problem. In Sect. 4 we conclude by offering some comments about the underlying Hamiltonian structure of our problem and why it serves as a unifying principle.

2 Shear Flow

The two-dimensional scalar vorticity equation can be obtained from the incompressible Navier–Stokes equations by neglecting viscosity and restricting to two spatial dimensions, say (x, y). This allows the introduction of a streamfunction, ψ, in terms of which the velocity field and the scalar vorticity can be written as $\mathbf{v} := (-\partial\psi/\partial y, \partial\psi/\partial x)$ and $\omega = \nabla^2\psi$, respectively. The resulting equation of motion for the single dynamical variable, ω, can be written as

$$\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla \omega = 0 \,. \qquad (1)$$

There are several notations in use for the term $\mathbf{v} \cdot \nabla\omega$:

$$\mathbf{v} \cdot \nabla\omega = [\psi, \omega] = J(\psi, \omega) = \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}. \tag{2}$$

We will use the notation $[\psi, \omega]$.

For shear flow in a channel, the domain D is an infinitely long channel of finite width, $(x, y) \in \mathbb{R} \times [-1, 1]$. The usual boundary condition of no normal flow at the boundaries located at $y = \pm 1$ will be assumed. This amounts to $\psi(x, y = \pm 1) =$ constant. We state no restrictions on the streamwise flow at $x = \pm\infty$. The only remaining requirement for well-posedness is a suitable initial condition $\omega(x, y, t = 0) = \mathring{\omega}(x, y)$. Global existence of solutions for this system was proven long ago. (See, for example, [6], a highly cited reference in this regard.)

By straightforward formal manipulation it can be shown that (1) conserves the energy functional

$$H[\omega] = \frac{1}{2}\int_D |\nabla\psi|^2 \,\mathrm{d}y\mathrm{d}x\,, \tag{3}$$

a quantity that will be seen to be the Hamiltonian functional for the infinite degree-of-freedom Hamiltonian description given in Sect. 4.

2.1 Equilibrium and Linearization

Equilibrium solutions of (1) satisfy

$$[\omega_e, \psi_e] = 0\,, \qquad \nabla^2\psi_e = \omega_e. \tag{4}$$

The solutions of (4) corresponding to shear flow in a channel depend only on y, and thus $\omega_e(y) = -U'(y) = \psi_e''(y)$, where prime denotes $\mathrm{d}/\mathrm{d}y$. However, the equilibrium velocity profile $U(y)$ is in general an arbitrary function of y. For convenience we assume it is monotonic for $y \in [-1, 1]$. Most importantly, we assume that the linear dynamics about U supports no instability. This can be assured by Rayleigh's criterion or by more sophisticated means (see [5,2]). We rule out instability and stipulate that the only spectrum of our problem is the continuous spectrum.

Setting $\psi = \psi_e + \delta\psi$ and $\omega = \omega_e + \delta\omega$ and expanding (1) to first order, gives

$$\frac{\partial\delta\omega}{\partial t} + U\frac{\partial\delta\omega}{\partial x} - U''\frac{\partial\delta\psi}{\partial x} = 0\,, \tag{5}$$

with $\nabla^2\delta\psi = \delta\omega$. This linear integro-differential equation is our main object of study. It requires the boundary condition $\delta\psi(x, y = \pm 1) = 0$ and a suitable initial condition. (For mathematical results pertaining to this system see [17,20].)

The linear dynamics of (5) conserves the following functional:

$$H_L[\delta\omega] = \frac{1}{2}\int_D \left[\frac{U}{U''}(\delta\omega)^2 - \delta\omega\delta\psi\right] \mathrm{d}y\mathrm{d}x\,, \tag{6}$$

which is the total amount of energy contained in a perturbation away from the equilibrium state. In Sec. 4 we will see that this functional is the Hamiltonian for the linear dynamics.

For convenience we introduce the Fourier modal representation,

$$\delta\psi = \psi_k(y,t)\exp(ikx)\,, \qquad \delta\omega = \omega_k(y,t)\exp(ikx) \tag{7}$$

where k is the streamwise wavenumber, which could be summed over if desired, but in the sequel we will fix k and consider the problem with only a single mode. Equation (5)becomes

$$\frac{\partial\omega_k}{\partial t} + ikU\omega_k - ikU''\psi_k = 0 \tag{8}$$

with

$$\psi_k(y,t) = \int_{-1}^{1} \mathcal{K}_k(y,y')\,\omega_k(y',t)\,\mathrm{d}y'\,, \tag{9}$$

where

$$\mathcal{K}_k(y,y') = \begin{cases} \sinh[k(y-1)]\sinh[k(y'+1)]/k\sinh[2k] & \text{for } y > y', \\ \sinh[k(y'-1)]\sinh[k(y+1)]/k\sinh[2k] & \text{for } y \le y'. \end{cases} \tag{10}$$

The quantity ψ_k inherits its boundary conditions from $\delta\psi$, namely, $\psi_k(y = \pm 1, t) = 0$.

Because the channel is infinite in x, $k \in \mathbb{R}$. In acoustic as well as in quantum mechanics there is a continuous spectrum associated with the domain being infinite in extent. We emphasize that this is not the continuous spectrum that causes the major difficulty in shear flow and other problems of continuous media. This other continuous spectrum arises from a singularity in the equation, a singularity with a location dependent upon the eigenvalue. We discuss this difficulty below in Sec. 2.2, where k will be assumed fixed.

2.2 Singular Eigenmodes and the Continuous Spectrum

Now we search for eigenvalues of (8) in the usual way. More precisely, we are interested in the spectrum of a linear operator $\mathcal{L}_k$, which is defined by rewriting (8) as follows:

$$\frac{\partial\omega_k}{\partial t} = \mathcal{L}_k\,\omega_k\,. \tag{11}$$

Assuming

$$\omega_k(y,t) = \hat{\omega}_k(y,c)\exp(-ikct)\,, \qquad \psi_k(y,t) = \hat{\psi}_k(y,c)\exp(-ikct)\,, \tag{12}$$

we seek the eigenfunction $\hat{\omega}_k(y, c)$ associated with an eigenvalue that we choose to write in terms of the phase velocity c. With (12), (8) and (9) become

$$(U - c)\,\hat{\omega}_k - U''\hat{\psi}_k = 0\,, \tag{13}$$

and

$$\hat{\psi}_k(y, c) = \int_{-1}^{1} \mathcal{K}_k(y, y')\,\hat{\omega}_k(y', c)\,\mathrm{d}y'\,, \tag{14}$$

respectively. Equation (13) is essentially Rayleigh's equation, which is usually written in terms of $\hat{\psi}_k$ as follows:

$$(U - c)\,(\hat{\psi}_k'' - k^2\hat{\psi}_k) = U''\hat{\psi}_k\,. \tag{15}$$

The eigenvalue problem associated with (15) has a feature that makes it different from eigenvalue problems encountered in quantum mechanics or other fluid mechanics problems that give rise to dispersion relations. The feature is the presence of the singularity that occurs where the phase velocity matches the equilibrium velocity, i.e. at values of y such that $U = c$. This singularity differs from that of Legendre's equation or the equations for other special functions because the singularity depends upon the eigenvalue c. In fact there is a continuous correspondence between values of c and points in the channel y_c (where $U(y_c) = c$), and this is the origin of the continuous spectrum.

This singularity, which in fluid mechanics is called a critical layer or critical level, has a long history dating to Rayleigh [16], and has since been considered by many researchers. One approach is to give up the search for eigenmodes and to use the Laplace transform to solve the initial value problem [4,7,17]. Another is to change the physics by adding some form of viscosity or nonlinearity. We will stick to the eigenvalue problem and solve it by following a procedure introduced by Van Kampen in the context of plasma physics [22].

The difficulty arises in part because the eigenfunctions associated with singular problems like this one do not exist in a usual Hilbert space, i.e. are not square integrable. In fact one must resort to the theory of distributions or generalized functions (see e.g. [8]) in order to obtain the eigenfunctions. In the calculus of generalized functions, an equation of the form $xf(x) = 0$ has the solution $f(x) = C\delta(x)$, where C is a constant and $\delta(x)$ is Dirac's delta function. Van Kampen's procedure is to employ this device to obtain a homogeneous solution to (13) that is added to a particular solution that incorporates the Cauchy principal value. Applying his procedure to the present problem gives the solution

$$\hat{\omega}_k = \mathcal{P}\left(\frac{U''\hat{\psi}_k}{U - c}\right) + \mathcal{C}_k\delta(U - c)\,, \tag{16}$$

where $\mathcal{P}$ signifies the Cauchy principal value and $\mathcal{C}_k$ is (as yet) an arbitrary constant.

By writing the vorticity eigenfunction in the form of (16), we explicitly display the singular nature of the mode. Moreover, we see that the eigenvalue c does not depend upon the wave number k, i.e. there is no dispersion relation, and that any value of c in the range of U is acceptable. This suggests the continuous spectrum. (A proof of the continuous spectrum requires analyzing the complement of the point spectrum of $\mathcal{L}_k$ in a Banach space setting.)

Although (16) gives a general form for the eigenfunction, it remains indeterminant for two reasons: because we do not have a value for the constant $\mathcal{C}_k$ and even more so because the function $\hat{\psi}_k$ is unknown. The latter problem is addressed by inserting $\hat{\omega}_k$ as given by (16) into (14). This yields the following inhomogeneous integral equation:

$$\hat{\psi}_k(y, y_c) = \mathcal{P}\int_{-1}^{1} \mathcal{K}_k(y, y')\frac{U''(y')\hat{\psi}_k(y', y_c)}{U(y') - c}\, \mathrm{d}y' + \frac{\mathcal{C}_k}{|U_c'|}\mathcal{K}_k(y, y_c)\,, \tag{17}$$

where for convenience we have chosen to write $\hat{\psi}_k$ with the argument y_c instead of c (recall we have assumed U is monotonic so $c = U(y_c)$ establishes a unique correspondence) and $U_c' := U'(y_c)$. The kernel of this integral equation is singular at the critical layer, which adds to the difficulty, but we have yet to our second point of indeterminancy, the specification of $\mathcal{C}_k$

Because the problem is linear we are free to scale the amplitude of the eigenfunction, and the value of the constant $\mathcal{C}_k$ can be set by a normalization procedure. A convenient one is (cf. [9,10,18])

$$\Xi_k = \int_{-1}^{1} \hat{\omega}_k\, \mathrm{d}y\,, \tag{18}$$

where the modal amplitude Ξ_k can be taken to be any desired function of the eigenvalue c (or y_c). Inserting the (16) into (18) gives

$$\Xi_k = \frac{\mathcal{C}_k}{|U_c'|} + \mathcal{P}\int_{-1}^{1} \frac{U''(y')\hat{\psi}_k(y', y_c)}{U(y') - U(y_c)}\, \mathrm{d}y'\,. \tag{19}$$

Solving (19) for $\mathcal{C}_k$ and inserting the result into (17) gives finally

$$\hat{\psi}_k(y, y_c) = f_k(y) + \int_{-1}^{1} \mathcal{F}_k(y, y'; y_c)\, \hat{\psi}_k(y', y_c)\, \mathrm{d}y'\,, \tag{20}$$

where the inhomogeneous term $f_k(y) := \Xi_k \mathcal{K}_k(y, y_c)$ and the kernel is defined by

$$\mathcal{F}_k(y, y'; y_c) = \left[\frac{\mathcal{K}_k(y, y') - \mathcal{K}_k(y, y_c)}{U(y') - U(y_c)}\right] U''(y')\,. \tag{21}$$

This kernel is regular at the critical level, and thus the choice of normalization (18) has regularized the singular integral equation (17). Note that the quantity y_c appears only as a parameter.

The procedure above resulted in an eigenfunction, but unfortunately one that cannot be written down explicitly because it is given in terms of the solution to the (albeit regular) integral equation (20). The good news is this only needs to be done once for a given profile $U(y)$ in order to obtain the eigenfunctions. The solutions $\hat{\psi}_k(y, y_c)$ should be viewed as a special function associated to a given profile. Once $\hat{\psi}_k(y, y_c)$ is known a general solution to the initial value problem can be obtained by summation. In the next section we view this in a more general context by using the singular eigenfunctions to construct an integral transform that can in turn be used to effect a coordinate change that dramatically simplifies the problem.

3 The Integral Transform–Coordinate Change

Now we turn to the description of our integral transform solution. Rather than constructing solutions by multiplying the singular eigenfunction $\hat{\omega}_k$ by an amplitude that is selected according to the initial condition, multiplying the result by e^{-ikct}, and finally integrating over c (or y_c), we construct an integral transform that involves only the y-dependence and makes no assumptions about time. This approach is Hamiltonian in spirit in that one solves the problem by transforming to coordinates that make the problem simple. This procedure is, of course, what is usually done with conventional integral transforms and we will view this problem in that light.

3.1 Integral Transform Pair

A superposition of the singular eigenfuntions (16), each of which is labeled by y_c, with amplitudes given by $\Lambda_k(y_c, t)$ suggests the following transform:

$$G_k[\Lambda_k](y,t) := \int_{-1}^{1} \mathcal{G}_k(y, y_c)\Lambda_k(y_c, t)\,\mathrm{d}y_c \tag{22}$$

where the transform kernal is given by

$$\mathcal{G}_k(y, y_c) := A_k(y)\delta(y - y_c) + \mathcal{P}\frac{U''(y)\psi_k(y, y_c)}{U(y) - U(y_c)}\,, \tag{23}$$

with

$$A_k(y) := 1 - \mathcal{P}\int_{-1}^{1} \frac{U''(y')\psi_k(y', y)}{U(y') - U(y)}\,\mathrm{d}y'\,, \tag{24}$$

and $\psi_k(y, y_c)$ a solution to (20). The transform of (22) takes the function $\Lambda_k(y_c, t)$ into a function $\omega_k(y, t) := G_k[\Lambda_k](y, t)$.

Alternatively, we can rewrite (22) by inserting (23) into (22) to obtain the following equivalent form:

$$G_k[\Lambda_k](y,t) := A_k(y)\Lambda_k(y_c, t) + \mathcal{P}\int_{-1}^{1} \frac{U''(y)\psi_k(y, y_c)}{U(y) - U(y_c)}\Lambda_k(y_c, t)\,\mathrm{d}y_c\,, \tag{25}$$

whence we see that G_k is the sum of a piece that merely multiplies the function to be transformed by $A_k(y)$ with a piece that is a generalization of the Hilbert transform (see e.g. [21,19]).

The inverse of (22), subject to the conditions we have placed on U (monotonicity and no discrete spectrum), is given by the following:

$$\widehat{G}_k[\omega_k](y_c, t) := \int_{-1}^{1} \widehat{\mathcal{G}}_k(y_c, y)\omega_k(y,t)\, \mathrm{d}y\,, \tag{26}$$

where

$$\widehat{\mathcal{G}}_k(y_c, y) = \frac{1}{A_k^2(y_c) + B_k^2(y_c)} \left[A_k(y_c)\delta(y - y_c) + \mathcal{P}\frac{U''(y_c)\psi_k(y, y_c)}{U(y) - U(y_c)} \right], \tag{27}$$

and where

$$B_k(y) := -\pi\psi_k(y_c, y_c)\frac{U''(y)}{U'(y)}\,. \tag{28}$$

Note that on the right hand sides of (22) and (26), the functions $\Lambda_k(y_c, t)$ and $\omega_k(y,t)$ are functions of y_c and y, respectively, and the explicit time dependence appears only as a parameter. We include this time dependence to emphasize that the integral transforms are coordinate changes; the actual time dependence will be determined by the equation of motion.

If $\widehat{G}_k$ is to be the inverse of the transform G_k, then $\widehat{G}_k[G_k[\Lambda_k]] \equiv \Lambda_k$, which follows if

$$\int_{-1}^{1} \widehat{\mathcal{G}}_k(y_c', y)\mathcal{G}_k(y, y_c)\, \mathrm{d}y = \delta(y_c - y_c')\,. \tag{29}$$

This inverse relation can also be viewed as a relation that verifies the completeness of the continuous spectrum. Similarly, there exists a reciprocal completeness relation,

$$\int_{-1}^{1} \widehat{\mathcal{G}}_k(y_c, y')\mathcal{G}_k(y, y_c)\, \mathrm{d}y_c = \delta(y - y')\,. \tag{30}$$

Just as the Fourier and other integral transforms posses useful identities, there are a variety of identities possessed by G_k and $\widehat{G}_k$. We record two such identities that will be used below:

$$\begin{aligned}\widehat{G}_k[U\omega_k](y_c, t) &= U(y_c)\widehat{G}_k[\omega_k](y_c, t) \\ &\quad + \frac{U''(y_c)}{A_k^2(y_c) + B_k^2(y_c)}\mathcal{P}\int_{-1}^{1} \omega_k(y,t)\psi_k(y, y_c)\, \mathrm{d}y\,,\end{aligned} \tag{31}$$

and

$$\widehat{G}_k[U''\psi_k](y_c, t) = \frac{U''(y_c)}{A_k^2(y_c) + B_k^2(y_c)}\mathcal{P}\int_{-1}^{1} \omega_k(y,t)\psi_k(y, y_c)\, \mathrm{d}y\,, \tag{32}$$

where (32) is valid if ω_k is related to ψ_k according to (14).

The above and other related transform formulas are verified in [1,3] and their counterparts in the Vlasov-Poisson context are proved rigorously in [12].

3.2 Solution

Now we are in a position to use the transform pair to solve (8). Let $\Lambda_k(y,t) := \widehat{G}_k[\omega_k]$ and then $\omega_k(y,t) = G_k[\Lambda_k]$. Transforming (8) with $\widehat{G}_k$ gives

$$\frac{\partial \Lambda_k}{\partial t} + ikc\Lambda_k = 0\,, \tag{33}$$

where use has been made of both (31) and (32), with the obvious cancellation, and we have set $c = U(y_c)$. This simple form is obtained because the transform has been designed especially for (8); it is a coordinate change in which the equation becomes trivial to solve. The solution to (33) is clearly $\Lambda_k = \mathring{\Lambda}_k e^{-ikct}$, which upon transforming back gives

$$\omega_k(y,t) = G\left[\widehat{G}[\mathring{\omega}_k]e^{-ikct}\right]. \tag{34}$$

Here we have written $\mathring{\Lambda}_k$ in terms of $\mathring{\omega}_k(y) := \omega_k(y, t=0)$ by means of $\mathring{\Lambda}_k = \widehat{G}[\mathring{\omega}_k]$.

The solution (34) is exact, but not entirely explicit because it requires $\psi_k(y, y_c)$, which in turn requires the solution of the integral equation (20). This is a fairly easy numerical exercise because (20) is a regular Fredholm integral equation. We reiterate that the fact that this calculation only needs to be done once for each flow profile, $U(y)$. After $\psi_k(y, y_c)$ is obtained, (34) gives the solution for all initial conditions.

4 Hamiltonian Interpretation

It has been know for a long time that the equations for continuous media without dissipation effects are an infinite dimensional Hamiltonian system (see e.g. [11] for review). An advantage of this Hamiltonian structure is that systems that possess it have phenomena in common. Thus when approaching a new problem one can appeal to a large body of lore. For example, it is known that all stable finite degree-of-freedom Hamiltonian systems can be reduced, by a coordinate change, to a system of uncoupled oscillators. That is, the Hamiltonian can be transformed into the following:

$$H(p,q) = \sum_k \frac{\nu_k}{2}\left(q_k^2 + p_k^2\right) = \sum_k \nu_k\, J_k\,, \tag{35}$$

where the second equality corresponds to action-angle variables. The integral transform of Sec. 3 is in fact the essence of a canonical transformation that does this for the infinite dimensional fluid system, which because of the continuous spectrum takes the form

$$H_L = \int\!\!\int \nu_k(y_c) J_k(y_c, t)\, \mathrm{d}y_c \mathrm{d}k\,. \tag{36}$$

In this concluding section, we briefly describe how this transpires, and refer the reader to [3] for details.

There are various ways of describing the Hamiltonian structure of fluids; the one we adopt is that of [13] in term of the noncanonical Poisson bracket. In the present context, the upshot of this is that (1) can be written as follows:

$$\frac{\partial \omega}{\partial t} = \{\omega, H\} = [\omega, \psi] \,, \tag{37}$$

where the noncanonical Poisson bracket is given by

$$\{A, B\} = \int_D \omega \left[\frac{\delta A}{\delta \omega}, \frac{\delta B}{\delta \omega}\right] \mathrm{d}y\mathrm{d}x \,, \tag{38}$$

and the Hamiltonian functional H is given by (3). (Note $\delta A/\delta\omega$ is the functional or variational derivative.)

The linear dynamics obtains its Hamiltonian structure from that of the full dynamics given in (37) by linearizing. This procedure involves linearizing the bracket as well as the Hamiltonian. Upon inserting $\omega = \omega_e + \delta\omega$ into the bracket and retaining the lowest order piece, we obtain

$$\{A, B\}_L = \int_D \omega_e \left[\frac{\delta A}{\delta\delta \omega}, \frac{\delta B}{\delta\delta \omega}\right] \mathrm{d}y\mathrm{d}x \,, \tag{39}$$

where $\delta A/\delta\delta\omega$ is the functional derivative with respect to the linear dynamical variable, $\delta\omega$. The linear dynamics of (5) can be written as

$$\frac{\partial \delta\omega}{\partial t} = \{\delta\omega, H_L\}_L \,. \tag{40}$$

where H_L, the Hamiltonian for the linear dynamics, is given by (6).

Our desire is to transform (6) into (36), but to do this requires first writing the Hamiltonian description (40) in terms of canonically conjugate variables. Once this is achieved the transformation to action-angle variables can be effected. Performing these calculations is beyond the scope of this paper, but we mention in closing that this can be done by constructing a mixed variable generating functional that incorporates the integral transform of Sec. 3. The details of this along with other Hamiltonian insights are given in [3].

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