Nonlinear Processes in Geophysical Fluid Dynamics

A tribute to the scientific work of Pedro Ripa

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HAMILTONIAN DESCRIPTION OF FLUID AND PLASMA SYSTEMS WITH CONTINUOUS SPECTRA

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In memory of Pedro Ripa 1946–2001

Abstract. We show how to transform a large class of infinite degree-of-freedom Hamiltonian systems into normal form. The energy-Casimir method that is widely used for ascertaining stability in Hamiltonian fluid and plasma systems is only the first step. A complete description involves changing to coordinates in which the energy is diagonal. This amounts to a transformation to action-angle variables. Because fluid and plasma systems typically have a continuous eigenspectrum, this transformation is nontrivial. It will be shown that a family of integral transforms, which is a generalization of the Hilbert transform, yields action-angle variables for a large class of fluid and plasma systems.

Key words: Hamiltonian fluid and plasma dynamics, normal forms, stability

1. Introduction

The goal of this paper is to show how to transform a class of infinite degree-of-freedom Hamiltonian systems, *i.e.* Hamiltonian field theories, into action-angle form. The class of systems is distinguished by two important features: first, it is Hamiltonian in the noncanonical sense of possessing a Lie-Poisson bracket description of the dynamics (*e.g.* Shepherd, 1990; Morrison, 1998; Marsden & Ratiu, 1999) and second, the class as a whole possesses a particular kind of continuous spectra that is akin to that discovered by Van Kampen (1955) in plasma physics. These two features present roadblocks to the usual construction of the transformation to action-angle form. Because of the noncanonical form one must first *canonize*, *i.e.* find a set of canonical variables. Because of the continuous spectrum the transformation constructed to *diagonalize* the Hamiltonian is novel and possesses some intricacies.

Finite linear Hamiltonian systems are simplified by transforming them into normal form (Birkhoff, 2002; Williamson, 1936; Moser, 1958), a form that is determined mostly by their eigenspectra. Action-angle form is the normal form that occurs when the system is stable, *i.e.* when the spectrum is neutral and nondegenerate. For finite systems the normal form for stable systems is given variously by one of the following:

$$H=\sum_{n=1}^{N}\frac{\omega_n}{2}\left(p_n^2+q_n^2\right)=i\sum_{n=1}^{N}\omega_n\,Q_n\,P_n=\sum_{n=1}^{N}\omega_n\,J_n\,, \tag{1}$$

where (q_n,p_n) and (Q_n,P_n) are canonical coordinates, and the last expression of (1) is the action-angle form, with J_n denoting the action variable. The frequency associated with the degree of freedom labeled by n is given by $\omega_n:=\sigma_n|\omega_n|$ with $\sigma_n\in\{-1,1\}$. The quantity σ_n is the signature that determines whether or not a stable oscillation possesses positive or negative energy and plays an important role in the bifurcation theory described by the Krein-Moser theorem (Kreĭn, 1950; Kreĭn & Jakubovič, 1980; Moser, 1958).

Infinite systems have the capacity for rich spectra composed of discrete, continuous, and residual components (*e.g.* Kato, 1966; Reed & Simon, 1980; Riesz & Nagy, 1955). If we restrict to the case of only a continuous spectrum we would expect the analog of the last term of (1) to be

$$H=\int\omega(u)\,\mathcal{J}(u)\,du\,, \tag{2}$$

where $\mathcal{J}(u)$ is a field action variable and the discrete sum over n is replaced by the integration over a continuum label u. Describing how to effect this transformation for the class of infinite dimensional Hamiltonian systems is the main result of this paper. This is a substantial task and we only attempt to sketch the basic ideas. (Greater detail for the cases of Vlasov-Poisson and shear flow dynamics can be found in Morrison & Pfirsch (1992), Morrison & Shadwick (1994), and Balmforth & Morrison (2002), with the most rigor given in Morrison (2000).)

The canonization and diagonalization procedure we describe here complements the stability arguments of Rayleigh (1896) and the energy-Casimir type of Kruskal & Oberman (1958), Arnol'd (1965), Ripa (1983), and others. (See *e.g.* Holm *et al.*, 1985; Morrison & Eliezer, 1986; Morrison, 1998 for more details.) The essence of the energy-Casimir method is to ascertain stability, as Dirichlet did for finite Hamiltonian systems, by using essentially the Hamiltonian as a Lyapunov fuction. Our diagonalization procedure completes the stability problem by finding the transformation to normal mode coordinates. Alternatively, one can view the procedure in an operator theory context. From this point of view we transform the operator that embodies the linear dynamics into a multiplication operator (Reed & Simon, 1980), akin to the procedure for diagonalizing matrices.

In Sec. 2 we describe our class of infinite-dimensional systems and give some examples. In Sec. 3 we see that the unifying principle of our class is the common Hamiltonian form. Also in this section we describe conservation laws with a particular emphasis on the momentum. Section 4 describes the nontrivial determination of equilibria and the equations for the nearby linear dynamics. The eigenvalue problem is described in Sec. 5, where the origin of a class of integral transforms, which are generalizations of the Hilbert transform, is also described. In Sec. 6 the canonization and diagonalization of the class of linear Hamiltonian systems is discussed. Finally in Sec. 7 we discuss future work.

2. A Class of Infinite-Dimensional Systems

2.1. SCALAR 2 + 1 MEAN FIELD THEORIES

The class of field theories we consider possesses a single independent variable $\zeta(q,p,t)$, which is a density-like variable that depends on the independent variables $z := (q,p)$ as well as time. Associated with the class of field theories are two phase spaces: the field phase space, which is the function space in which the density ζ resides, and the particle phase space of independent variables z. There is cause for confusion here because we introduce action-angle variables associated with both of these phase spaces: the field action-angle variables of (2), our main goal, and en route to this goal, a set of particle action-angle variables that below we denote by (θ, J). We write $\zeta\colon \mathcal{Z} \times \mathbb{R} \to \mathbb{R}$ where $\mathcal{Z}$ denotes the particle phase space, which we take to be $D_1 \times D_2$, $\Pi \times D_2$, or Π^2, where Π is the one-torus chosen depending on which of p and q is periodic, and $D_{1,2}$ are (not necessarily proper) subsets of $\mathbb{R}$. We will not be specific about the topology of $\mathcal{Z}$.

We suppose the density satisfies an equation of motion of the following form:

$$\frac{\partial \zeta}{\partial t} + [\zeta, \mathcal{E}] = 0\,, \tag{3}$$

where the particle Poisson bracket is defined by the usual expression $[f,g] = f_q g_p - g_q f_p$, where $f_q := \partial f/\partial q$ etc., and the quantity $\mathcal{E}$ is an energy-like quantity that we call the particle energy.

If (3) were a Liouville equation, then $\mathcal{E}$ would be a given function of z, and we would have a linear theory. However, we are concerned about mean field theories which are nonlinear partial integrodifferential equations. Such equations arise, for example, by truncation of BBGKY-like hierarchies, which results in a particular, generally global, functional dependence of the particle energy on the density. We determine our general class of systems by obtaining the particle energy in terms of the field energy given below.

2.2. FIELD ENERGY

The field energy is the integrated energy corresponding to a particular density ζ. We write the field energy in terms of energies corresponding to one-particle, H_1, two-particle, H_2, ... interactions, where

$$\begin{aligned} H_1[\zeta] &= \int_{\mathcal{Z}} h_1(z)\,\zeta(z)\,d^2z\,, \\ H_2[\zeta] &= \frac{1}{2}\int_{\mathcal{Z}}\int_{\mathcal{Z}} \zeta(z)\,h_2(z,z')\,\zeta(z')\,d^2z\,d^2z'\,, \end{aligned} \tag{4}$$

and the generalizations to H_3, H_4, ... are obvious. The quantities h_1 and h_2, the interaction kernels, are left unspecified. But, we suppose the two-particle interaction possesses the symmetry $h_2(z,z') = h_2(z',z)$.

The field energy of our class of systems is then given by $H[\zeta] = H_1 + H_2 + \ldots$. Henceforth, we will only consider field energies with the first two terms, $H[\zeta] = H_1 + H_2$.

The particle energy is obtained from the field energy by functional differentiation

$$\mathcal{E} := \frac{\delta H}{\delta \zeta} = h_1 + \int_{\mathcal{Z}} h_2(z,z')\,\zeta(z')\,d^2z'\,, \tag{5}$$

where the functional derivative is defined as usual by $\delta H = \int_{\mathcal{Z}} \delta\zeta\,\delta H/\delta\zeta\,d^2z$.

2.3. EXAMPLES

2.3.1. *Vlasov-Poisson*

In the case of Vlasov-Poisson we set $z = (x,p)$, which physically corresponds to a one degree-of-freedom particle phase space, and we set $\zeta = f(x,p,t)$, which is the phase space density that is chosen to give zero net charge. The one and two-particle interaction kernels are given respectively by the following:

$$h_1(z) = \frac{p^2}{2m}\,, \qquad h_2(z,z') = c|x - x'|\,, \tag{6}$$

where c is a constant. Thus the field energy is

$$H[\zeta] = \int_{\mathbb{R}^2} \frac{p^2}{2m}\, f\,dxdp + \frac{1}{8\pi}\int_{\mathbb{R}} E^2\,dx\,. \tag{7}$$

Upon taking the functional derivative, we obtain the usual Vlasov-Poisson particle energy, $\mathcal{E} := \delta H/\delta f = p^2/2m + e\phi[f](x)$.

2.3.2. *2D Euler*

In the case of the two-dimensional Euler fluid equations, we set $z = (x, y)$, which physically corresponds to a two-dimensional configuration space, and we set $\zeta = \zeta(x, y, t)$, which is the scalar vorticity. The one- and two-particle interaction kernels are given respectively by the following

$$h_1(z) \equiv 0\,, \qquad h_2(z, z') = c\ln[(x - x')^2 + (y - y')^2]\,, \tag{8}$$

where c is a constant. Thus the field energy is given by

$$H[\zeta] = \int_{\mathbb{R}^2} \frac{v^2}{2}\, dxdy\,, \tag{9}$$

where the velocity is related to the scalar vorticity by $\zeta = \hat{z} \cdot \nabla \times \mathbf{v}$, and functional differentiation gives $\mathcal{E} := \delta H/\delta\zeta = \psi[\zeta](x)$. For this case the particle energy corresponds to the streamfunction, where $\triangle\psi = \zeta$.

2.3.3. *Other Examples*

Many other examples with physical content exist. For example, Jeans equation for stellar dynamics is obtained by changing the sign of the h_2 of the Vlasov equation and removing the zero net charge condition, and quasi-geostrophy is obtained by changing the relationship between the vorticity and the streamfunction. Interesting examples where the underlying characteristics correspond to integrable n-body problems are the Cologero-Moser system (Moser, 1975; Illner, 2000), and the apparently unstudied cases of Stäckel potential interaction, Smereka's product potential (Smereka, 1998), and the Toda-Vlasov equation.

3. Hamiltonian Form and Conservation Laws

3.1. MEAN FIELD HAMILTONIAN FORM

It is evident from the above that the energy, H, which will turn out to be the Hamiltonian, is quadratic in ζ. Usually quadratic Hamiltonians correspond to linear dynamics, as is the case for the simple oscillations described in Sec. 1. However, the Vlasov-Poisson equation, the Euler equations, and indeed every equation in the class described in Sec. 2, are quadratically nonlinear. The discrepancy lies in the fact that the variable ζ does not constitute a set of canonically conjugate field variables. Theories of the kind described in Sec. 2 possess a description in terms of noncanonical degenerate Poisson brackets, brackets that are sometimes referred to as Lie-Poisson brackets. These brackets are linear in the field variables and thus account for the nonlinearity missing in the Hamiltonians. Much has been written about Lie-Poisson brackets, so we will not dwell, but refer the reader interested in more detail to Morrison (1998) and Marsden & Ratiu (1999). It is this Hamiltonian form that is the unifying theme that defines our class of systems.

The noncanonical Lie-Poisson bracket of our class is given by

$$\{F,G\} = \int_{\mathcal{Z}} \zeta \left[\frac{\delta F}{\delta \zeta}, \frac{\delta G}{\delta \zeta}\right] d^2 z\,. \tag{10}$$

Observe that this bracket depends explicitly upon the variable ζ, unlike usual Poisson brackets that only depend on (functional) derivatives of the canonical variables. Like canonical Poisson brackets, the bracket of (10) is antisymmetric and satisfies the Jacobi identity. Using (10) the equations of motion are obtained in the form

$$\frac{\partial \zeta}{\partial t} = \{\zeta, H\} = -[\zeta, \frac{\delta H}{\delta \zeta}] = -[\zeta, \mathcal{E}]\,, \tag{11}$$

where $H = H_1 + H_2$ is defined by (4). This constitutes the Hamiltonian form.

Associated with this Hamiltonian form are natural constants of motion; viz. the energy or Hamiltonian $H[\zeta] = H_1 + H_2$ and an infinity of constants known as Casimir invariants,

$$C[\zeta] = \int_{\mathcal{Z}} \mathcal{C}(\zeta)\, d^2 z\,, \tag{12}$$

where $\mathcal{C}(\zeta)$ is an arbitray function. The Casimir invariants arise from degeneracies in the Poisson bracket and do not occur in canonical theories. An important additional invariant is the momentum $P[\zeta]$, a quantity that is Hamiltonian dependent. We discuss this invariant further below.

3.2. MOMENTUM

Momentum invariants generally arise from translation symmetries that in the present context might be determined by the form of h_2. This is how the strong version of Newton's third law is built into the n-body problem. We generalize this idea significantly as follows. We state that our system has a momentum invariant if there exists a canonical transformation

$$z = (q,p) \longleftrightarrow \bar{z} := (\chi, \pi)$$

such that in the new particle coordinates $\bar{z} := (\chi, \pi)$, the interactions h_1 and h_2 have the form

$$h_1 \circ z = \bar{h}_1(\pi)\,, \qquad h_2 \circ (z, z') = \bar{h}_2(\pi, \pi', |\chi - \chi'|) \tag{13}$$

upon composition with $z(\bar{z})$. We will refer to the coordinates (χ, π) as *momentum coordinates.*

If such a transformation exists, then the following *momentum* is conserved:

$$P[\zeta] = \int_{\mathcal{Z}} \pi(z)\,\zeta(z)\,d^2z\,. \tag{14}$$

This can be shown by differentiating (14), changing to momentum coordinates, and using the fact that h_2 depends on $|\chi - \chi'|$.

Building in momentum conservation in the manner above is not the most general way possible. Ultimately, momentum conservation should arise from Nöther's theorem in an action principle, which could then be reduced to obtain the class of systems presented here. However, the definition given is sufficient for our purposes. It includes that for the Vlasov system, where $P = \int pf dpdq$ and the two momenta for 2D Euler system, where $P_y = \int x\zeta dxdy$ and $P_x = -\int y\zeta dxdy$. In the case of 2D Euler, there exists a coordinate system in which h_2 possesses translation invariance in both spatial directions, and for this reason we get the two conserved momenta.

4. Equilibria and Linearization

We now effect the usual procedure of expanding about an equilibrium state by setting $\zeta = \zeta_e + \delta\zeta$ and retaining terms of first order in $\delta\zeta$. We will see that our class of Hamiltonian systems has the property that eigenvalue problems resulting from the linearization procedure possess continuous spectra.

4.1. EQUILIBRIA

Equilibria, ζ_e, satisfy

$$\frac{\partial \zeta_e}{\partial t} = 0 = \{\zeta_e, H\}\,, \tag{15}$$

which upon using (10) implies

$$[\zeta_e, \mathcal{E}_e] = 0\,, \tag{16}$$

where we add the subscript 'e' to the particle energy because it depends functionally on ζ_e. Equation (16) implies functional dependence of ζ_e on $\mathcal{E}_e$ or vice versa. More generally it implies the existence of a single variable, say J, such that $\zeta_e(J)$ and $\mathcal{E}_e(J)$.

Given $J(q,p)$ one can obtain a θ such that the pair (θ, J) constitutes a canonically conjugate system of particle coordinates. We call these coordinates *equilibrium coordinates*, which are in general distinct from the momentum coordinates (χ, π) defined above. Thus we have three sets of canonical particle coordinates at our disposal

$$(q,p) \longleftrightarrow (\theta, J) \longleftrightarrow (\chi, \pi)\,.$$

We assume the problem is stated in terms of (q,p) and that we know (χ,π), somehow, possibly because of the physics. We now describe in more detail how one might obtain (θ, J).

If we knew $\mathcal{E}_e(q,p)$ then we could attempt to use the usual procedure for obtaining action-angle variables for a one degree-of-freedom Hamiltonian system. But the question remains, how do we obtain $\mathcal{E}_e(q,p)$ from the equilibrium equation (16)? It is clear that (16) alone cannot determine an equilibrium because this equation allows the choice of a free function, *e.g.* any function $\zeta_e(\mathcal{E}_e)$ is a solution of (16). In fact there are two routes one can follow in hope of finding a solution: one can assume the function $\zeta_e(\mathcal{E}_e)$ and seek $\mathcal{E}_e = \mathcal{E}_e(z)$ or, conversely, one can assume $\mathcal{E}_e = \mathcal{E}_e(z)$ and seek $\zeta_e(\mathcal{E}_e)$. Once the free function is chosen, we seek to remove the ambiguity by using the expression for the particle energy.

Let us follow the first route and assume a form for the function $\zeta_e(\mathcal{E}_e)$. We then insert this function into the expression for the particle energy (5) to obtain

$$\mathcal{E}_e(z) = h_1(z) + \int_{\mathcal{Z}} h_2(z,z')\, \zeta_e(\mathcal{E}_e(z'))\, d^2z'\,. \tag{17}$$

This integral equation is the generalization of the equilibrium elliptic equations that are obtained in the cases of the Vlasov-Poisson and 2D Euler examples. The point here is that there may be no elliptic equation corresponding to the inverse of the integral operator with the kernel h_2. However, we are fortunate that (17) has the form of a Hammerstein integral equation (Hammerstein, 1930), a nonlinear integral equation about which much is known. In the sequel we will assume we have a solution of this equation, one that is sufficiently well-behaved for our purposes. There are very interesting analysis questions pertaining to (17), but we leave these to a future publication.

We conclude this subsection with a discussion of some special cases. If we have a momentum invariant, then we can rewrite (17) in terms of momentum coordinates as follows:

$$\mathcal{E}_e = \bar{h}_1(\pi) + \int_{\mathcal{Z}} \bar{h}_2(\pi, \pi', |\chi - \chi'|)\, \zeta_e(\mathcal{E}_e)\, d\chi' d\pi'\,.$$

Some advantage is achieved by this form because the difference kernel makes the problem amenable to Fourier transform techniques. Other simplifications occur for the cases below:

4.1.1. *Vlasov-like*

For this case, the momentum and equilibrium particle coordinates coincide. For Vlasov, $\pi = J$, $\bar{h}_2 = \bar{h}_2(\chi - \chi')$, and the convolution form leads to a solution. In addition to the Vlasov equation, defect dynamics (Balmforth *et al.*, 1996) falls into this category. In analogy to Vlasov theory, one can have homogeneous equilibria or the more complicated case of BGK-like equilibria.

4.1.2. *Euler-like*

For this case, $\bar{h}_1 \equiv 0$ and $\bar{h}_2 = \bar{h}_2(\pi - \pi', \chi - \chi')$ and we have two momenta and two difference directions available for Fourier transform techniques.

4.1.3. *Others*

There are many other cases that possess special properties. The integrable systems mentioned above; *i.e.*, the Cologero-Moser systems, systems with Stäckel potential interaction, and the Toda-Vlasov equation would be interesting to analyze.

4.2. LINEARIZATION

Now we suppose that the equilibrium problem has been solved and that we have found the equilibrium coordinates; *i.e.*, it is assumed that the transformation $(q,p) \longleftrightarrow (\theta, J)$ is in hand and the function $\zeta_e(J)$ is known. Moreover, it is assumed known that $\zeta_e(J)$ posseses only a continuous spectrum. Energy-Casimir or Rayleigh-like stability arguments, arguments based on the Green transform (Hille, 1976), or arguments based on the Nyquist method (Balmforth & Morrison, 1998) can be used to rule out stable or unstable discrete spectra.

Setting $\zeta = \zeta_e(J) + \delta\zeta(\theta, J, t)$ and expanding (11) to first order in $\delta\zeta$ gives

$$\frac{\partial \delta\zeta}{\partial t} + [\delta\zeta, \mathcal{E}_e] + [\zeta_e, \delta\mathcal{E}] = 0\,, \tag{18}$$

where $\Omega(J) := d\mathcal{E}_e/dJ$, $\delta\mathcal{E} = \int_{\mathcal{Z}} h_2(z,z')\,\delta\zeta(z')\,d^2z'$ written in terms of (θ, J), and because these equilibrium coordinates are canonical $[f,g] = f_\theta g_J - g_\theta f_J$. Thus, (18) is equivalent to

$$\frac{\partial \delta\zeta}{\partial t} + \Omega(J)\frac{\partial \delta\zeta}{\partial \theta} = \frac{d\zeta_e}{dJ}\frac{\partial \delta\mathcal{E}}{\partial \theta}\,. \tag{19}$$

Equations of the form of (19) generally possess a continuous spectrum, which we now turn to.

5. Spectrum and Integral Transform

5.1. SPECTRUM

To see the origin of the continuous spectrum we insert

$$\delta\zeta = \sum_k \zeta_k(J)\, e^{ik\theta - ik\omega t}$$

into (19) and obtain

$$\Big(\Omega(J) - \omega\Big)\zeta_k = \zeta_e'\,\mathcal{E}_k\,, \tag{20}$$

where $\zeta_e' := d\zeta_e/dJ$ and $\mathcal{E}_k$ is given by the following expression

$$\mathcal{E}_k(J) = \sum_{k'} \int \mathcal{H}_{k,k'}(J,J')\,\zeta_{k'}(J')\,dJ'\,, \tag{21}$$

which is obtained from $\delta\mathcal{E}$ by changing to equilibrium coordinates and Fourier expanding in the angles. (Note that $\mathcal{H}_{k,k'}$ depends upon h_2 and ζ_e.) Equation (20) is an eigenvalue problem for the eigenvalue ω.

The left hand side of (20) vanishes for values of J such that $\omega = \Omega(J)$, and this singularity is recognized in plasma physics and fluid mechanics as the origin of the continuous spectrum. In plasma physics it corresponds to wave-particle resonance, while in the fluid mechanics of shear flow it is called the critical layer (or line) and it corresponds to the matching of a background equilibrium shear velocity to the phase velocity of a perturbation.

Following Van Kampen (1955) we write a solution of (20) in the form:

$$\zeta_k = \lambda_k\,\delta(\Omega - \omega) + \mathcal{P}\,\frac{\zeta_e'\,\mathcal{E}_k}{\Omega - \omega}\,, \tag{22}$$

where δ is the Dirac distribution and $\mathcal{P}$ denotes Cauchy principal value. Equation (22) is of the form of a continuum eigenfunction corresponding to the continuous eigenspectrum labeled by ω. We assume $\Omega(J)$ is monotonic. Thus the eigenfunction labeled by ω can equally well be labeled by J_ω where $\omega = \Omega(J_\omega)$.

Note, the eigenfunction of (22) is indeterminate because the parameter λ_k is unknown and because it is self-referential in that $\mathcal{E}_k$ depends on ζ_k. The parameter λ_k is determined by a normalization condition, *e.g.* $\int \zeta_k\,dJ = 1$, and the following equation for $\mathcal{E}_k$ is obtained by inserting (22) into (21):

$$\mathcal{E}_k(J,J_\omega) = \sum_{k'} \mathcal{H}_{k,k'}(J,J_\omega) + \sum_{k'} \int\, \mathcal{E}_{k'}(J',J_\omega)\,\mathcal{F}_{k,k'}(J,J',J_\omega)\,dJ'\,, \tag{23}$$

where the kernel

$$\mathcal{F}_{k,k'}(J,J',J_\omega) := \left[\frac{\mathcal{H}_{k,k'}(J,J') - \mathcal{H}_{k,k'}(J,J_\omega)}{\Omega(J') - \Omega(J_\omega)}\right] \zeta_e'(J')$$

is well-behaved enough to apply the Fredholm theory of integral equations. (See Morrison & Pfirsch (1992), Morrison (2000), and Balmforth & Morrison (2002) for more details in the context of the Vlasov and 2D Euler equations.)

More rigorous statements regarding the spectrum are made in a Banach space, $\mathcal{B}$, setting. In the remainder of this subsection we make a few comments in this regard and leave serious analysis for a possible future publication. To this end we write

$$\mathcal{L}_k\zeta_k := \Omega(J)\zeta_k - \zeta_e'\,\mathcal{E}_k[\zeta_k] = \omega\,\zeta_k\,, \tag{24}$$

where the linear operator $\mathcal{L}_k : \mathcal{B} \to \mathcal{B}$ is our concern. We partition the spectrum of $\mathcal{L}_k$ as follows: $\sigma = \sigma_p \cup \sigma_c \cup \sigma_r$. An eigenvalue ω is in the point spectrum, σ_p, if $\mathcal{L}_k - \omega\mathcal{I}$ is not one-one, where $\mathcal{I}$ is the identity operator. If ω is such that the range of $\mathcal{L}_k - \omega\mathcal{I}$ is not dense in the Banach space of interest, then ω is in the residual spectrum σ_R, and if ω is such that the inverse of $(\mathcal{L}_k - \omega\mathcal{I})$, defined on its range, is unbounded, then ω is in the continuous spectrum σ_c. We find this partition convenient because if σ_r is null, then the approximate or Weyl spectrum corresponds to $\sigma_p \cup \sigma_c$. (Note, there are other commonly used decompositions of the spectrum (*e.g.* Reed & Simon, 1980; Riesz & Nagy, 1955.)

In Sec. 4.2 we made some comments on how, for a given kernel, an equilibrium can be ensured to have no discrete modes, *i.e.*, ensured to have a null point spectrum. We assume this has been arranged, *i.e.* we know $\mathcal{L}_k - \omega\mathcal{I}$ is one-one.

The operator $\mathcal{L}_k$ is the sum of a multiplication operator and an integral operator that under mild conditions is bounded. It is well-known that a multiplication operator possesses a continuous spectrum and early theorems by Friedrichs and others (*e.g.* Kato, 1966) state conditions under which this continuous spectrum survives perturbation by the addition of an integral operator. For example, an operator that is composed of a bounded self-adjoint piece perturbed by the addition of a compact piece retains its continuous spectrum.

One interpretation of the diagonalization procedure we are attempting here is akin to the diagonalization of matrices by coordinate changes. Our goal is to transform the operator $\mathcal{L}_k$ into a pure multiplication operator by a coordinate change. This procedure is described in Reed & Simon (1980) for bounded self-adjoint operators, but it is not confined to such operators. A sum over eigenfunctions of the form of (22) suggests a form for the diagonalizing transform that converts (24) into a pure multiplication operator. We turn to this now.

5.2. INTEGRAL TRANSFORM

The general form of the integral transform that diagonalizes the continuous spectrum of our class of Hamiltonian systems is given by

$$G[g](J) = \epsilon(J)\, g(J) + \mathcal{P}\int \zeta_e'(J)\, \frac{\mathcal{E}(J, J_\omega)\, g(J_\omega)}{\Omega(J) - \Omega(J_\omega)}\, dJ_\omega\,, \qquad (25)$$

where g is the function that is being transformed, and the functions ϵ and $\mathcal{E}$ (given below) are determined by the functions h_1 and h_2 that define our system, as well as the equilibrium being studied, ζ_e. In this way the transform is tailored to the problem at hand. Here, for clarity, we have suppressed the dependence on k to display that the transformation is the sum of a multiplication operator, a multiplication of g by a function ϵ, and an integral part that involves the Cauchy principal value, a generalization of the Hilbert transform with a kernal given by $\mathcal{E}\zeta_e'$.

To simplify matters we will assume the momentum and energy coordinates coincide so that $\bar{h} = \bar{h}(J, J', \theta - \theta')$. Consequently (21) and (23) become

$$\mathcal{E}_k(J) = \int h_k(J, J')\, \zeta_k(J')\, dJ'\,, \qquad (26)$$

and

$$\mathcal{E}_k(J, J_\omega) = h_k(J, J_\omega) + \int \zeta_e(J')\, \mathcal{E}_k(J', J_\omega)\, \frac{[h_k(J, J') - h_k(J, J_\omega)]}{\Omega(J') - \Omega(J_\omega)}\, dJ'\,, \qquad (27)$$

respectively. For this choice of interaction $\mathcal{H}_{k,k'}(J, J') = \delta_{k,k'}\, h_k(J, J')$.

Many properties of transforms of the form of (25) can be proven by techniques similar to those used in Hilbert transform theory (*e.g.* Stein & Weiss (1971) and other works on Calderón-Zygmund theory). Under mild conditions it can be shown that G is a bounded linear operator on suitable Banach spaces (L_p and $C^{0,\alpha}$). With more difficulty, the inverse can be constructed, and it is of the same general form as (25). In addition there exist identities which can be used to show that the transform diagonalizes the Hamiltonian. The justification of these statements follows closely those for the Vlasov (Morrison, 2000) and shear flow (Balmforth & Morrison, 2002) cases. Below we list the results, but present their justification elsewhere.

5.2.1. *Transform and its Inverse*

Multiplying (24) by an amplitude, $g_k(J_\omega)$, and integrating over J_ω motivates the following form for the transform as a sum over eigenfunctions:

$$G_k[g_k](J,t) := \epsilon_k^{(r)}(J)\, g_k(J,t) + \mathcal{P}\int \frac{\zeta_e'(J)\mathcal{E}_k(J,J_\omega)}{\Omega(J)-\Omega(J_\omega)}\, g_k(J_\omega,t)\, dJ_\omega\,. \qquad (28)$$

Observe we have explicitly displayed the t dependence to reinforce the idea that this is a coordinate change. Now, specifically, we define

$$\epsilon_k^{(r)}(J_\omega) := 1 - \mathcal{P}\int \frac{\zeta_e'(J)\mathcal{E}_k(J,J_\omega)}{\Omega(J)-\Omega(J_\omega)}\, dJ \qquad (29)$$

and $\mathcal{E}_k(J,J_\omega)$ is the solution to (23).

The inverse of (28), under our assumed conditions of monotonicity and no discrete spectrum, is given by the following:

$$\widehat{G}_k[f_k](J_\omega,t) := \frac{1}{|\epsilon_k(J_\omega)|^2}\left[\epsilon_k^{(r)}(J_\omega) f_k(J_\omega,t) + \mathcal{P}\int \frac{\zeta_e'(J_\omega)\mathcal{E}_k(J,J_\omega)}{\Omega(J)-\Omega(J_\omega)} f_k(J,t)\, dJ\right], \qquad (30)$$

where $|\epsilon_k(J)|^2 := (\epsilon_k^{(r)})^2 + (\epsilon_k^{(i)})^2$ and

$$\epsilon_k^{(i)}(J_\omega) := \pi\mathcal{E}_k(J_\omega,J_\omega)\frac{\zeta_e'(J_\omega)}{\Omega'(J_\omega)}\,. \qquad (31)$$

That $\widehat{G}$ is the inverse of G is is most simply shown by making use of the Poincaré-Bertrand theorem on the interchange of the order of integration for singular integrals.

5.2.2. *Transform Identities*

We record two identities for (30) that will be needed below.

$$\widehat{G}_k[\Omega\,\zeta_k](J_\omega) = \Omega(J_\omega)\widehat{G}_k[\zeta_k](J_\omega) + \frac{\zeta_e'(J_\omega)}{|\epsilon_k|^2(J_\omega)}\mathcal{P}\int \zeta_k(J,t)\,\mathcal{E}_k(J,J_\omega)\, dJ\,, \qquad (32)$$

and

$$\widehat{G}_k[\zeta_e'\mathcal{E}_k](J_\omega) = \frac{\zeta_e'(J_\omega)}{|\epsilon_k|^2(J_\omega)}\mathcal{P}\int \zeta_k(J)\,\mathcal{E}_k(J,J_\omega)\, dJ\,. \qquad (33)$$

These are shown by techniques similar to those used for verifying the inverse.

6. Linear Canonization and Diagonalization

6.1. LINEAR HAMILTONIAN FORM

In the energy-Casimir method (Holm *et al.*, 1985; Morrison & Eliezer, 1986; Morrison, 1998) one chooses the Casimir invariant C of (12) such that the vanishing of the first variation of the quantity $F := H + C$ gives the equilibrium of interest, and then (modulo some technicalities) one examines the second variation $\delta^2 F$ for positive definiteness in order to prove stability. Physically, $\delta^2 F$ corresponds to the energy content of a perturbation away from equilibrium, and it serves as the Hamiltonian for the linear dynamics. From (4) and (12), it is seen to be given by

$$\delta^2 F = \delta^2 H + \frac{1}{2}\int \mathcal{C}''(\zeta_e)\,(\delta\zeta)^2\, d\theta dJ = \delta^2 H - \frac{1}{2}\int \frac{\mathcal{E}_e'(J)}{\zeta_e'(J)}\,(\delta\zeta)^2\, d\theta dJ\,. \quad (34)$$

Because $\delta^2 F$ is the Hamiltonian for the linear dynamics we rename it H_L. It, together with the linear Poisson bracket,

$$\{F,G\}_L = \int \zeta_e(J)\left[\frac{\delta F}{\delta\delta\zeta}, \frac{\delta G}{\delta\delta\zeta}\right]\, d\theta dJ\,,$$

generates the linear dynamics as follows:

$$\frac{\partial\delta\zeta}{\partial t} = \{\delta\zeta, H_L\}_L\,.$$

Upon expanding the angle-like dependence, θ, of ζ_k in a Fourier sum, we obtain the following expressions for the linear Hamiltonian and Poisson bracket, respectively:

$$H_L = \frac{1}{2}\sum_{k,k'}\int\!\!\int \zeta_k(J)\,\mathcal{H}_{k,k'}(J,J')\,\zeta_{k'}(J')\,dJdJ' - \frac{1}{2}\sum_k \int \frac{\mathcal{E}_e'(J)}{\zeta_e'(J)}\,\zeta_{-k}\zeta_k\,dJ \quad (35)$$

and

$$\{F,G\}_L = \sum_{k=1}^{\infty} ik\int \zeta_e'\left(\frac{\delta F}{\delta\zeta_k}\frac{\delta G}{\delta\zeta_{-k}} - \frac{\delta G}{\delta\zeta_k}\frac{\delta F}{\delta\zeta_{-k}}\right)dJ\,. \quad (36)$$

Note, because the sum now extends only over positive integers, ζ_k and ζ_{-k} are to be viewed as independent variables. Also, note we have not made any Fourier expansion in the time variable.

6.2. CANONIZATION AND DIAGONALIZATION

If it were not for the presence of $ik\zeta_e'$ in the bracket of (36), ζ_k would be the canonical conjugate of ζ_{-k}. If we define

$$q_k(J,t) := \zeta_k(J,t) \qquad \text{and} \qquad p_k(J,t) = \frac{\zeta_{-k}(J,t)}{ik\zeta_e'}\,, \tag{37}$$

then (36) becomes

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int \left(\frac{\delta F}{\delta q_k}\frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k}\frac{\delta F}{\delta p_k} \right) dJ\,,$$

i.e. we have canonized the bracket.

Diagonalization is a more difficult procedure, but formally we can define the mixed variable generating functional

$$\mathcal{F}[q,P] = \sum_{k=1}^{\infty} \int P_k(J)\,\widehat{G}[q_k](J)\,dJ$$

where $\widehat{G}$ is defined by (30). This type-2 mixed variable generating functional effects the canonical transformation from the old field coordinates (q_k, p_k) to the new field coordinates (Q_k, P_k) according to

$$p_k(J) = \frac{\delta \mathcal{F}[q,P]}{\delta q_k(J)} = \widehat{G}^{\dagger}[P_k](J) \quad \text{and} \quad Q_k(J) = \frac{\delta \mathcal{F}[q,P]}{\delta P_k(J)} = \widehat{G}[q_k](J)\,. \tag{38}$$

Making use of (37) and (26) we write the linear Hamiltonian (35) in the form

$$H_L = \sum_{k=1}^{\infty} ik \int p_k\left[\zeta_e'\mathcal{E}_k - q_k\mathcal{E}_e'\right] dJ\,, \tag{39}$$

into which we insert p_k and q_k from (38), to obtain

$$H_L = \sum_{k=1}^{\infty} ik \int P_k \left(\widehat{G}[\zeta_e'\mathcal{E}_k] - \widehat{G}[\mathcal{E}_k' G[Q_k]] \right) dJ\,, \tag{40}$$

Using (32) and (33) the new Hamiltonian takes the form

$$H_L = -\sum_{k=1}^{\infty} \int ik\,\Omega(J) Q_k(J)\,P_k(J) dJ\,. \tag{41}$$

Demonstrating the above achieves our final goal, because transforming from (41) to (2) is elementary.

7. Conclusions

It is evident that there are many avenues for future work. In conclusion we list some of them.

- Investigate the consequences of the signature of the continuous spectrum; *i.e.* prove a Krein-Moser theorem in a Banach space setting.
- Investigate the theory of adiabatic invariants in this infinite dimensional Hamiltonian context by adding explicit time dependence to the Hamiltonian. Some results for finite systems in the context of atmospheric models appear in Wirosoetisno & Shepherd (2000).
- Obtain the analog of Birkhoff's nonlinear normal form theory for our class of infinite dimensional Hamiltonian systems with continuous spectra. Some results in this direction appear in Yudichak (2001).
- Obtain our class of infinite dimensional Hamiltonian systems by reduction (Morrison, 1998; Marsden & Ratiu, 1999) of a canonical system with symmetry.
- Investigate the role played by (25) in attempts to understand the function phase space tangent bundle geometry.

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