On acoustic wave generation in uniform shear flow

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(Dated: October 2004)

Abstract

The linear dynamics of acoustic waves and vortices in uniform shear flow is studied. For flows with very low shear rates, the dynamics of perturbations is adiabatic and can be described by the WKB approximation. However, for flows with moderate and high shear rates the WKB approximation is not appropriate, and alternative analysis shows that two important phenomena occur: acoustic wave over-reflection and wave generation by vortices. The later phenomenon is a known linear mechanisms for sound generation in shear flows, a mechanism that is related to the continuous spectrum that arises in linear shear flow dynamics. A detailed analytical study of these phenomena is performed and the main quantitative and qualitative characteristics of the radiated acoustic field are obtained and analyzed.

PACS numbers: 43.20.El, 43.28.Ra, 47.27.Pa

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I. INTRODUCTION

Lighthill's acoustic analogy [1] is a well-known approach for understanding the generation of sound by turbulent shear flows. In this approach the flow is assumed to be known and the sound field is calculated as a small by-product of the flow. Lighthill's analogy has achieved considerable success in explaining the most prominent features of experimentally observed jet sound fields. However, more precise experiments (e.g. [2]) show other more subtle features that cannot be explained by Lighthill's analogy. To solve this problem various extensions of the analogy have been developed [3–8].

All of the above extensions of Lighthill's analogy involve, in one form or another, a nonlinear wave operator that must be linearized before meaningful calculations can be carried out. For low Mach number flows, researhers have typically followed Lilley (see e.g. [4, 7]) and assumed that all linear terms in the governing set of equations describe the sound propagation and all nonlinear terms represent sources of sound. Resent developments in this direction have led to the formulation of the generalized acoustic analogy [7], an approach in which the following procedure is adopted: (i) the flow variables are divided into their mean and fluctuating parts; (ii) the mean flow is subtracted out of the equation; (iii) all the linear terms are collected together on one side of equations with the nonlinear terms on the other side; and finally (iv) the nonlinear terms are treated as a known source of sound.

On the other hand, linear mechanisms of sound generation related to both large scale coherent structures in the flow [9] and perturbations related to shear flow dynamics [10–12] are also well-known [6, 7]. These linear mechanisms are especially effective in flows with relatively high Mach numbers. In Ref. [9] a comprehensive analysis of the sound generation by large scale structures has been performed in order to explain observed experimental results [13]. In Ref. [10] both analytical and experimental results indicate that linear terms can be effective sources of sound if the shear rate is sufficiently high. Also, this same linear mechanism, which is associated with the continuous spectrum in shear flow dynamics [15–17], has been shown to be an effective means of generating different kinds of wave modes in some astrophysical problems (see e.g. [14] and references therein).

The aim of the present paper is the quantitative study of the linear mechanism of wave generation by vortices. In particular, we study the linear evolution of hydrodynamic perturbations with two simplifying assumptions: firstly, in order to simplify the calculations we consider ony two spatial dimensions. It is straightforward to extend the analysis to three-dimensions. Secondly, we assume constant shear mean flow, an assumption that allows us to use the so-called shearing sheet solution [20].

We shown that for any shear in the flow, strong or weak, there exists a finite interval of time during which the evolution of perturbations is non-adiabatic. Outside this interval the evolution is adiabatic and therefore can be described by the WKB approximation. Thus we have a familiar asymptotic problem like that for obtaining connection formulae in turning point problems. Our analysis indicates that non-adiabatic behavior appears in the form of two phenomena: the over-reflection of acoustic waves and the generation of acoustic waves by vortex mode perturbations.

The equation that governs the evolution of acoustic waves in the linear shear flow profile is formally analogous to the one-dimensional Schrödinger equation with a parabolic potential barrier and also to the problem of propagation of electromagnetic waves in inhomogeneous media [21]. Thus, well-known results of existing asymptotic analysis can be adapted for our problem of acoustic wave evolution in uniform shear flow.

The second non-adiabatic process, the generation of acoustic waves by vortex mode perturbations, corresponds to the linear coupling of different modes in inhomogeneous media. This coupling is sometimes called mode conversion. There exist well-developed asymptotic methods for analysis of linear coupling of two wave modes in slowly varying media. These methods were first developed in the context of non-adiabatic transitions in two level quantum systems [22]. Later, the same asymptotic methods were successfully applied to various other problems including the interaction of plasma waves with inhomogeneous background magnetic field and/or density (see [23] and references therein), and the evolution of magnetohydrodynamic waves in shear flows [24]. However, these asymptotic methods fail for the study of linear coupling between vortical perturbations and acoustic waves. The main reason for this stems from the fact that the vortex mode has zero frequency and therefore variables that describe its evolution do not contain rapidly oscillating multipliers. Consequently, we have found it necessary to develop a different asymptotic technique for the analysis of the process. The technique allows us to obtain the quantitative and physical characteristics of the phenomenon. The technique is quite general and can be used for analysis of problems more complicated than that treated here.

The paper is organized as follows. The general formalism, including the derivation of the governing equations and the analysis of the adiabatic evolution of linear perturbations, is presented in Sec. II. The non-adiabatic evolution of acoustic waves is studied in Sec. III, and a second non-adiabatic process, the generation of acoustic waves by vortices, is studied in Sec. IV. Finally, in Sec. V we conclude with a summary of our results.

II. GENERAL FORMALISM

Consider the sound generation problem in parallel shear flow. We assume a background equilibrium velocity (or mean flow) of the form $\mathbf{V}_0 = (V_0(y), 0, 0)$, a uniform background pressure p_0, and uniform background density ρ_0, and then employ the standard technique Lighthill's acoustic analogy. Expanding all the variables $\Psi \equiv (\mathbf{V}, \rho, p)$ as a sum of background and fluctuating parts, $\Psi = \Psi_0 + \psi$, using the thermodynamic relation $p' = {c_s}^2 \rho'$, and combining the continuity and Euler equations one obtains Lilley's equation [25]:

$$\hat{L} \frac{p'}{\rho_0 c_s^2} = NLS, \tag{1}$$

where c_s is the sound speed,

$$\hat{L} \equiv \frac{D}{Dt} \left(\frac{D^2}{Dt^2} - c_s^2 \nabla^2 \right) + 2 c_s^2 \frac{dV_0}{dy} \frac{\partial^2}{\partial x \partial y}, \tag{2}$$

is the "third order linear wave operator" [25], $D/Dt \equiv \partial/\partial t + V_0 \partial/\partial x$ is the convective derivative, and the nonlinear source term is

$$NLS \equiv \frac{D}{Dt} \frac{\partial^2 (v_i' v_j')}{\partial x_i \partial x_j} - \frac{dV_0}{dy} \frac{\partial^2 (v_2' v_j')}{\partial x \partial x_j}, \tag{3}$$

where the indices i and j denote components and repeated indices are to be summed. The expression (3) is interpreted as a quadrupole source distribution, where the first and second terms are referred to as self-noise and shear-noise, respectively.

It is clear that the linearization of Lilley's equation, $\hat{L} p' = 0$, is equivalent to the compressible Rayleigh equation. Therefore, in general this linear equation describes not only acoustic wave propagation effects, but also shear flow dyamics such as that associated with both large scale turbulent structures [9] and "fine-grained" [6] turbulent motions related to the continuous spectrum of Rayleigh's equation [15–17]. If $V_0 \ll c_s$, then the coupling between these solutions is negligibly small [6], i.e. linear mechanisms of the sound generation are inefficient and the acoustic output is fully determined by the nonlinear source term in

Eq. (1). However, this is not the case when the mean flow Mach number is on the order of unity.

In order to study the linear generation of sound by vorticity, we make two simplifying assumptions: we restrict the problem to two-dimensions and we assume a constant shear background equilibrium flow, $V_0 = Ay$. The first assumption is made to simplify our calculations, but the generalization to three-dimensions is straightforward. The second assumption is more significant. When the sound generation mechanism of interest is related to small scale motions the constant shear assumption might seem reasonable, and there is some numerical evidence to this effect in the related problem of flow in a Keplerian accretion disk [26]. In any event, this assumption results in a great simplification of the problem. Most importantly, the shear flow dynamics decouples from the compressible degrees for freedom and so this part of the problem can be solved by the methods of Refs. [15–17]. The degree to which this assumption provides generic results will not be addressed here.

With the above assumptions the linearized compressible Euler equations are given by the following:

$$(\partial_t + Ay\partial_x)\rho' + \rho_0(\partial_x v_x' + \partial_y v_y') = 0, \tag{4}$$

$$(\partial_t + Ay\partial_x)v_x' + Av_y' = -\partial_x p'/\rho_0, \tag{5}$$

$$(\partial_t + Ay\partial_x)v_y' = -\partial_y p'/\rho_0, \tag{6}$$

where we use the shorthand $\partial_y = \partial/\partial y$, etc. According to the compressible Euler equations, the potential vorticity $Q \equiv \Omega/\rho$, where Ω is the scalar vorticity, is advected, and we expect a remnant of this in the linear theory. Indeed, Eqs. (4), (5), and (6) imply

$$(\partial_t + Ay\partial_x)q'(x, y, t) = 0 \tag{7}$$

where

$$q' \equiv \partial_x v_y' - \partial_y v_x' + A\rho'/\rho_0. \tag{8}$$

Equation (7) is essentially Rayleigh's equation with a linear background shear flow. This equation was studied in [15–17], where it is shown that it possesses a continuous eigenspectrum (e.g. Ref. [18]) with a complete set of nonsquare integrable eigenfunctions. Indeed, the general solution of (7) has the form $q'(x, y, t) = q_0'(x - At, y)$, and if the initial condition is chosen to have the form $q_0'(x, y) = \breve{q}\exp(ik_x x + ik_y y)$, then the solution is given by

$$q'(x, y, t) = \breve{q}(\mathbf{k})\exp\left[ik_x x + i\kappa_y(t)y\right], \tag{9}$$

where $\kappa_y(t) = k_y - k_x At$. Following Van Kampen [19], the solution can be expressed as a sum over normal modes as follows:

$$q'(x,y,t) = \exp(ik_x x) \int dc\, \delta(c - Ay) \exp\left[-ik_x ct\right] \breve{q}(\mathbf{k}) \exp\left[ik_y c/A\right], \tag{10}$$

where the amplitude that gives our chosen initial condition is $\breve{q}(\mathbf{k}) \exp\left[ik_y c/A\right]$ and the set of nonsquare integrable singular eigenfunctions is represented by the Dirac delta function $\delta(c - Ay)$, with c being the eigenvalue that serves as a continuum eigenfunction label.

That we can explicitly solve (7) for the vorticity-like degrees of freedom allows us to lower the order of the system. In particular, we can change variables and consider the system that governs the variables ρ', $\nabla \cdot \mathbf{v}'$, and q'. Upon inserting the solution of (7) into the equations for ρ' and $\nabla \cdot \mathbf{v}'$, we would see that the q' serves as a time dependent drive in these equations.

Instead of following this path, we note that the solution of (9) suggests that we expand all the perturbed quantities as:

$$\psi'(\mathbf{x},t) = \breve{\psi}(\mathbf{k},t) \exp\left[ik_x x + i\kappa_y(t) y\right], \tag{11}$$

where again $\kappa_y(t) = k_y - k_x At$. This kind of solution was used in the shearing sheet approximation of Ref. [20], but here we will see that we obtain an exact reduction. The quantities k_x and k_y are the wave numbers of the spatial fourier harmonics (SFHs) at the initial moment of time. From Eq. (11) it follows that κ_y, an "effective" wavenumber, varies in time, and this can be interpreted as a "drift" of the SFH in k-space. This circumstance arises because perturbations cannot have a simple plane wave form because of the effect of the shearing background on wave crests [27].

Using $p' = c_s^2 \rho'$ and introducing the notations:

$$\begin{aligned} &\rho \equiv i\breve{\rho}/\rho_0, \quad u \equiv \breve{v}_x/c_s, \quad v \equiv \breve{v}_y/c_s, \quad S \equiv A/(k_x c_s), \\ &\tau \equiv c_s k_x t, \quad \beta(\tau) \equiv k_y/k_x - S\tau, \end{aligned} \tag{12}$$

we obtain the following set of ordinary differential equations for SFHs of the perturbations:

$$\dot{\rho} = u + \beta(\tau) v, \tag{13}$$

$$\dot{u} = -Sv - \rho, \tag{14}$$

$$\dot{v} = -\beta(\tau)\rho. \tag{15}$$

Here and henceforth an overdot denotes a τ derivative, and without loss of generality we assume $k_x > 0$.

From (7) it is clear that $\dot{\breve{q}} = 0$. Using the scalings of Eqs. (12) and the definition (8) we obtain $\breve{q} = ik_x c_s q/\rho_0$, where

$$q \equiv v - \beta u - S\rho. \tag{16}$$

Using Eqs. (13)-(14) one can easily verify [20, 29] that q is indeed a constant of motion for this dynamics. Here we have identified this invariant as the linearized potential vorticity.

Below we use another quantity,

$$E(\mathbf{k}, \tau) \equiv \frac{1}{2}(|u|^2 + |v|^2 + |\rho|^2), \tag{17}$$

as a quadratic energy-like measure of the excitation of the various degrees of freedom. This quantity is not conserved by the linear dynamics, but we will refer to it as the energy density. The actual conserved energy for shear flows is discussed in Refs. [17, 28].

Using (16), Eqs. (13)-(15) can be reduced to a single second order differential equation

$$\ddot{u} + \omega^2(\tau)u = -\beta(\tau)q, \tag{18}$$

where $\omega^2(\tau) = 1+\beta^2(\tau)$. The general solution of this equation, the main subject of our study, is the sum of the special solution and the general solution of the corresponding homogeneous equation,

$$\ddot{u} + \omega^2(\tau)u = 0. \tag{19}$$

Consequently if $u_{1,2}$ are any linear independent solutions of Eq. (19), with Wronskian W, then the general solution of Eq. (18) can be written as

$$\begin{aligned} u(\tau) &= C_1 u_1(\tau) + C_2 u_2(\tau) \\ &+ \frac{q}{W}\int_{\tau_0}^{\tau} \beta(\tau_1)[u_1(\tau)u_2(\tau_1) - u_1(\tau_1)u_2(\tau)]d\tau_1, \end{aligned} \tag{20}$$

where τ_0 is arbitrary. We assume $\tau_0 = -\infty$, for a reason discussed below. The three constants $C_{1,2}$ and q are defined by the initial conditions of the problem.

Further analysis of Eqs. (19) and (18) becomes straightforward if $\omega^2(\tau)$ is a slowly varying function of τ,

$$|\dot{\omega}(\tau)| \ll \omega^2(\tau), \tag{21}$$

which with the definition of $\omega^2(\tau)$ takes the form

$$S|\beta(\tau)| \ll [1+\beta^2(\tau)]^{3/2}. \tag{22}$$

Note, that the condition (22) holds for arbitrary β if $S \ll 1$. If condition (22) is satisfied, then the solutions of Eq. (19) can be approximate by the WKB solutions

$$\tilde{u}_{1,2} = \frac{1}{\sqrt{\omega(\tau)}} e^{\pm i \int \omega(\tau) d\tau}, \tag{23}$$

which can be identified as shear modified acoustic waves with positive and negative phase velocities along the x-axis.

Under condition (22), the approximate solution of the inhomogeneous equation (18) is well-known [29, 30],

$$u_{inh} \equiv q\tilde{u}_3 = q\sum_{m=0}^{\infty} S^m y_m(\tau), \tag{24}$$

with

$$y_0(\tau) = -\frac{\beta(\tau)}{\omega^2(\tau)}, \qquad y_{2n-1}(\tau) = 0, \tag{25}$$

$$y_{2n}(\tau) = -\frac{1}{\omega^2(\tau)} \frac{\partial^2 y_{2n-2}}{\partial \beta^2}, \tag{26}$$

and describes the adiabatic evolution of vortical perturbations [29]. As was mentioned above, τ_0 is arbitrary in Eq. (20). Our choice $\tau_0 = -\infty$ assures that the special solution of Eq. (19) in the limit $S \to 0$ corresponds to what can be termed purely vortical (non-oscillating) perturbations. This conclusion as well as some other mathematical details of the problem are discussed in Appendix A.

So, with the condition (22), the general solution of Eq. (18) can be approximated as

$$u \approx C_1\tilde{u}_1 + C_2\tilde{u}_2 + q\tilde{u}_3, \tag{27}$$

which can be interpreted as a linear superposition of the independent evolution of the different modes in the flow. The constants $C_{1,2}$ and q can be treated as the intensities of the corresponding types of perturbations. If u is found, then $\dot{u}$ is known, and v and ρ can be readily determined from Eqs. (14) and (16). Combining these equations with Eq. (17) and taking into account condition (22) shows that the energy density of acoustic waves satisfies a standard relation for adiabatic evolution:

$$E_{a1,2} \approx \omega(\tau)|C_{1,2}|^2. \tag{28}$$

Whereas for vortical perturbations,

$$E_w \approx \frac{|q|^2}{\omega^2(\tau) + S^2}. \tag{29}$$

From Eqs. (28) and (29) it follows that when $\tau \to \infty$, the energy density of acoustic waves increases linearly with τ [29], while that of vortical perturbations behaves as $1/\tau^2$ [15].

If $S \ll 1$, condition (22) holds for arbitrary $\beta(\tau)$. This means that the $C_{1,2}$ remain nearly constant (comments on the accuracy of the conservation of WKB amplitudes will be given below) during the evolution. A comprehensive study of the adiabatic (WKB) evolution of acoustic waves as well as vortical perturbations in uniform shear flow was presented in [29]. As noted in the Intoduction, the main purpose of the present paper is to focus on the non-adiabatic evolution of the perturbations, i.e. we study the dynamics of perturbations with moderate and high normalized shear parameter S.

When $S \sim 1$ condition (22) fails in the neighborhood of $\beta(\tau) = 0$. However, it remains valid for $|\beta(\tau)| \gg \sqrt{S}$. Thus, the problem is of a standard form for asymptotic analysis[30, 31]: assuming at $\tau = 0$, $\beta(0) \equiv k_y/k_x \gg \sqrt{S}$ and the intensities of the acoustic and vortical perturbations are $C_{1,2}$ and q, respectively, the problem is to determine the intensities after passing through the region of non-adiabatic evolution where $\beta(\tau) \ll -\sqrt{S}$, i.e. where $\tau > 2k_y/k_x\sqrt{S}$. We denote the intensities after traversing the non-adiabatic region by $D_{1,2}$ and J. Because q is an exact invariant

$$J \equiv q. \tag{30}$$

However, the quantities $C_{1,2}$, which are adiabatic invariants, are not exact invariants and thus change upon traversing the non-adiabatic region.

In the next section the non-adiabatic evolution of the acoustic waves ($q \equiv 0$) will be discussed, and in Sec. 4 the phenomenon of acoustic wave generation by vortices will be treated and analytical expression for the intensity of the generated acoustic waves will be obtained.

III. OVER-REFLECTION OF ACOUSTIC WAVES

In this section we consider non-adiabatic evolution of acoustic waves in uniform shear flow ($q \equiv 0$). The dynamics of perturbations is governed by Eq. (19):

$$\ddot{u} + \omega^2(\tau)u = 0. \tag{31}$$

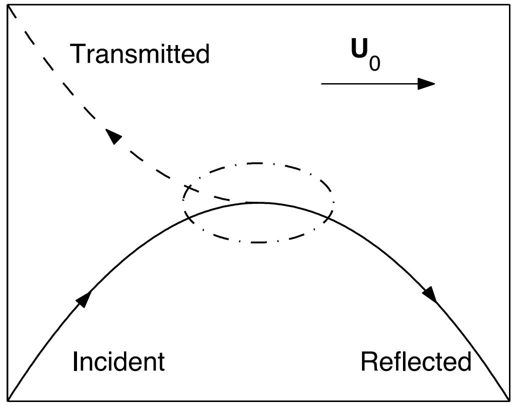

FIG. 1: Schematic illustration of acoustic wave reflection and transmission in the shear flow.

Assume that initially (at $\tau = 0$), $\beta(0) = k_y/k_x \gg \sqrt{S}$ only the wave with positive phase velocity along y-axis exists (without loss of generality we also assume that both, k_x and k_y are positive):

$$u_i = \frac{C_1}{\sqrt{\omega(\tau)}} e^{-i\int\omega(\tau)d\tau}. \tag{32}$$

In the general case when $\beta(\tau) \ll -\sqrt{S}$ there exist both WKB solutions (23). According to the notations in the theory of acoustic wave propagation in non-uniform flows [32, 33] the WKB solution with positive phase velocity along y-axis is treated as reflected wave (u_r) and another solution as transmitted wave (u_t, see Fig. 1):

$$u_r = \frac{D_1}{\sqrt{\omega(\tau)}} e^{-i\int\omega(\tau)d\tau}. \tag{33}$$

$$u_t = \frac{D_2}{\sqrt{\omega(\tau)}} e^{i\int\omega(\tau)d\tau}. \tag{34}$$

The reflection and transmission coefficients can be defined in the usual manner [30, 31, 36]:

$$R = \frac{|D_1|}{|C_1|}, \qquad T = \frac{|D_2|}{|C_1|}. \tag{35}$$

Normally the non-adiabatic evolution changes not only WKB amplitudes, but also the phases of WKB solutions:

$$D_1 = e^{i\phi_1} R C_1, \qquad D_2 = e^{i\phi_2} T C_1. \tag{36}$$

Combining Eq. (31) and its complex conjugated one it can be easily shown that if $\omega^2(\tau)$ is real function of τ, then an arbitrary solution u_x of Eq. (31) satisfies the condition:

$$u^*\dot{u} - u\dot{u}^* = constant, \tag{37}$$

where asterisk denotes complex conjugation.

It is well known [34] that this equation represents the conservation of wave action during the evolution. Upon substituting Eqs. (32)–(36) into (37) one obtains

$$R^2 - T^2 = 1. \tag{38}$$

So, the amplitude of reflected wave is always larger than (or at least equals to) the amplitude of incident wave. This means, that non-adiabatic evolution of acoustic waves ($T \neq 0$) is always accompanied by the phenomenon of over-reflection, which was discovered by Miles [33], where the problem of acoustic wave reflection from the surface of tangential discontinuity of velocity was studied. From the derivation of Eq. (38) it is clear, that the same conclusion holds for non-adiabatic evolution of arbitrary wave mode that is governed by Eq. (31) with arbitrary real function $\omega^2(\tau)$.

For determination of R (and therefore T according to Eq. (38)) as well as $\phi_{1,2}$ there are well developed approximate asymptotic methods (sometimes called extended WKB or phase integral method) that works for wide classes of $\omega^2(\tau)$. These methods, based on pioneering papers of Jeffreys [35], are presented in textbooks of quantum mechanics [36] and asymptotic analysis [30, 31].

However we do not need to use above mentioned approximate asymptotic methods due to the fact, that in the case of $\omega^2(\tau) = 1 + \beta^2(\tau)$, exact analytical solutions of Eq. (31) can be presented by parabolic cylinder functions [12, 37]:

$$\bar{u}_1 = E\left(-\frac{1}{2S}, \sqrt{\frac{2}{S}}\beta(\tau)\right), \tag{39}$$

$$\bar{u}_2 = E^*\left(-\frac{1}{2S}, \sqrt{\frac{2}{S}}\beta(\tau)\right). \tag{40}$$

It has to be emphasized that the existence of the exact solution of Eq. (31) in general case do not guarantee neither simplification of R, T and $\phi_{1,2}$ calculation procedures nor improvement of accuracy of calculated results [21]. But in the specific case of parabolic cylinder functions, their asymptotic properties make possible to obtain the exact asymptotic solution of the problem. Since further mathematical analysis is similar to one presented in [21], we note only main steps skipping details of calculations.

Taking into account the asymptotic expansion of parabolic cylinder function [37]:

$$E(a,\eta) \approx \sqrt{\frac{2}{\eta}} \exp\left[i\left(\frac{\eta^2}{4} - a\ln\eta + \Phi + \frac{\pi}{4}\right)\right] \quad (41)$$

$$\text{for} \qquad \eta \gg |a|, \quad (42)$$

where $\Phi = \arg\Gamma\left(\frac{1}{2} + ia\right)$, and using connection formula [37]:

$$iE(a,-\eta) = e^{\pi a}E(a,\eta) - \sqrt{1+e^{2\pi a}}E^*(a,\eta), \quad (43)$$

one can conclude that for $\beta(\tau) \gg S^{1/2}$ exact solution (39) coincide with incident wave (32) accurate to the constant multiplier. Whereas for $\beta(\tau) \ll -S^{1/2}$, exact solutions (39) and (40) agree with reflected (33) and transmitted (34) waves respectively. Taking this in mind and combining Eqs. (32)-(36) and (41)-(43) one can readily obtain:

$$R = \sqrt{1+e^{-\pi/S}}, \quad (44)$$

$$T = e^{-\pi/2S}. \quad (45)$$

$$\phi_1 = \frac{\ln(2S)}{2S} - \arg\left[\Gamma\left(\frac{1}{2} + \frac{i}{2S}\right)\right], \quad (46)$$

$$\phi_2 = \frac{\pi}{2}. \quad (47)$$

It is important to note that Eqs. (44)-(47) represent exact asymptotic solution of the problem, i.e., they are valid for arbitrary value of normalized shear parameter S.

Calculating the phase integral [30, 31]:

$$\delta = \frac{1}{S}\left|\int_{-i}^{i}\sqrt{1+\beta^2}d\beta\right| = \frac{\pi}{2S}, \quad (48)$$

and noting that $T_{ph} = e^{-\delta}$ in the framework of the method, we see that the method of phase integrals gives the same results for T and R as in Eqs. (45) and (44). However the validity of the method is bounded by condition $S \ll 1$.

It is also well known [30, 31] that if incident wave has negative phase velocity with respect to x-axis:

$$\bar{u}_i = \frac{C_2}{\sqrt{\omega(\tau)}}e^{i\int\omega(\tau)d\tau}, \quad (49)$$

then the expressions for reflection and transmission coefficients remain the same as in Eq. (44)-(45), whereas the expressions (46)-(47) for $\phi_{1,2}$ change the sign.

It can be seen from Eq. (45), that if $S \ll 1$, then the intensity of transmitted wave is exponentially small with respect to the large parameter $1/S$. On its turn, this means that, if $S \ll 1$, then the adiabatic invariants $C_{1,2}$ have exponential (with respect to $1/S$) accuracy of conservation [30, 31, 36].

It follows from Eq. (45), that the intensity of transmitted wave never exceeds intensity of incident wave ($T < 1$), and therefore $R < \sqrt{2}$.

The last problem that we discuss in this section is estimation of the time scale of transmitted wave generation process. It follows from the condition of validity of Asymptotic expansion (41), that the time scale $\Delta\tau_b$ of the "birth" [38] of transmitted wave is of order:

$$\Delta\tau_b \sim S^{-1/2}. \tag{50}$$

Note that this estimation is valid for arbitrary S. Returning to the dimensional variables per Eq. (12) this equation takes the form:

$$\Delta t_b \sim \frac{S^{-1/2}}{c_s k_x} = (A c_s k_x)^{-1/2}. \tag{51}$$

Taking into account that the period of acoustic wave $P \sim 1/c_s k_x$, and the time scale of the linear drift in the k-space $\Delta t_d \sim 1/A$, one can conclude that in the limit $S \ll 1$:

$$P \ll \Delta t_b \ll \Delta t_d. \tag{52}$$

This means that the birth of the transmitted wave is slow process in comparison with a wave period, but it is the fast process in comparison with the linear drift in the k-space.

We have to note that the estimation similar to (50) was first obtained by Berry [38] for a wide class of $\omega^2(\tau)$ satisfying the condition (21), which in our specific case reduces to the condition $S \ll 1$.

IV. GENERATION OF ACOUSTIC WAVES BY VORTICES

In this section we study another non-adiabatic process: generation of acoustic waves by vortices. This phenomenon was already intensively studied by different authors [10–12]. Our study is concentrated on the derivation of analytical results for main characteristics of the process, such as intensity of generated acoustic wave, time length of the generation process by concrete vortex SFH and the acoustic power of the considered mechanism in turbulent shear flow.

Introducing a new variable $z = -\beta(\tau)$, Eq. (18) can be presented as:

$$\frac{d^2u}{dz^2} + \frac{1}{S^2} f(z) u = \frac{q}{S^2} z. \tag{53}$$

with $f(z) = 1 + z^2$. Eq. (20) takes the form:

$$u(z) = C_1 u_1(z) + C_2 u_2(z) + q u_3(z), \tag{54}$$

where

$$u_3(z) = \frac{1}{2iS} \int_{-\infty}^{z} \zeta Q(z, \zeta) d\zeta, \tag{55}$$

$Q(z, \zeta) \equiv u_1(z) u_2(\zeta) - u_1(\zeta) u_2(z)$ and $u_{1,2}$ are solutions of corresponding homogeneous equation:

$$\frac{d^2u}{dz^2} + \frac{1}{S^2} f(z) u = 0, \tag{56}$$

chosen such that for $|z| \gg \sqrt{S}$ they converge to WKB solutions (23).

Let assume that initially (at $\tau = 0$), $\beta(0) = k_y/k_x \gg \sqrt{S}$, i.e., $z \ll -\sqrt{S}$, there exist only vortical perturbations: $C_{1,2} = 0$ and $q \neq 0$. The problem is to determine the intensities of acoustic waves for $z \gg \sqrt{S}$.

It is clear that the generation process takes place at some neighborhood of the moment $z = -\beta(\tau) = 0$. In Subsection D we consider this problem in detail and obtain that the time scale of acoustic wave generation process satisfies the condition (50).

For $z \gg \sqrt{S}$ we can rewrite Eq. (55) as:

$$u(z) = i G_1 q u_1 + i G_2 q u_2 - \frac{q}{2iS} \int_{z}^{\infty} \zeta Q(z, \zeta) d\zeta, \tag{57}$$

where:

$$G_{1,2} = \frac{\mp 1}{2S} \int_{-\infty}^{\infty} z u_{2,1} dz. \tag{58}$$

One can show that for arbitrary choice of $u_{1,2}$:

$$\int_{-\infty}^{-z} \zeta Q(z, \zeta) d\zeta = \int_{z}^{\infty} \zeta Q(z, \zeta) d\zeta. \tag{59}$$

Indeed, due to the fact that $f(z)$ is even function of z there always exist such a choice of linear independent solutions of Eq. (56), that one of them (u_{ev}) will be even function of z

and another one (u_{od}) an odd function [31]. Taking into account that any other solution of Eq. (56) can be presented as a linear combination of u_{ev} and u_{od} one can readily prove (57).

Then we conclude that the first two terms in Eq. (57) represent the generated acoustic waves with intensities $|G_{1,2}q|$. In analogy with Eq. (35), we interpret $|G_{1,2}|$ as acoustic wave generation coefficients.

The aim of our further mathematical analysis is derivation of approximate expressions for generation coefficients $G_{1,2}$. This problem can be divided into two steps. First we need to choose solutions of homogeneous equation (56) that are reasonable for our purposes and the second is the derivation of approximate expressions for $G_{1,2}$.

Neither approximate WKB solutions (23) nor exact solutions (39)-(40) are optimal for the evaluation of $G_{1,2}$. As it is well known accuracy of WKB solutions is not enough for derivation of correct results in this kind of problems [38, 39]. Usage of the exact solutions (39)-(40) do not allow exact evaluation of integrals in Eq. (58). This circumstance makes it necessary to use the series expansion of parabolic cylinder functions valid for arbitrary point on the real z-axis. There exists only one such expansion of parabolic cylinder function [37], but it also is useless for the evaluation of $G_{1,2}$.

Instead we use another approximate solutions of Eq. (56) – solutions in a form of formal series [30]. It is well known [30], that WKB solutions represent the first term of the solutions in a form of formal series. Accuracy of this solutions in the limit $S \to 0$ is $O(S^\infty)$. In the next subsection we obtain the solutions of Eq. (56) in the form of formal series, and in the Subsection B these solutions will be used for the evaluation of integrals in the expressions of generation coefficients (58).

A. Formal series

Let us turn back to Eq. (56):

$$\frac{d^2u}{dz^2} + \frac{1}{S^2}(1 + z^2)u = 0. \tag{60}$$

Making standard Liouville transform:

$$\xi \equiv \int_0^z \sqrt{f(\zeta)}\, d\zeta, \tag{61}$$

$$\bar{U} \equiv f^{1/4}u, \tag{62}$$

Eq. (60) can be rewritten as:

$$\frac{d^2\bar{U}}{d\xi^2} + \left(\frac{1}{S^2} - \psi\right)\bar{U} = 0, \tag{63}$$

where $\psi = -f^{-3/4}d^2(f^{-1/4})/dz^2$. Substituting the expansion:

$$\bar{U}_1 = e^{i\xi/S}\sum_{m=0}^{\infty} B_m(\xi)S^m \tag{64}$$

into Eq. (63) one can easily obtain the following recurrence relation for the expansion coefficients B_m:

$$B_m(\xi) = \frac{-1}{2i}\frac{dB_{m-1}}{d\xi} + \frac{1}{2i}\int \psi B_{m-1}d\xi, \tag{65}$$

or alternatively:

$$B_m(z) = \frac{-1}{2i}f^{-1/2}\frac{dB_{m-1}}{dz} + \frac{1}{2i}\int \Lambda B_{m-1}dz, \tag{66}$$

where:

$$\Lambda \equiv \frac{4ff'' - 5(f')^2}{32f^{5/2}} = \frac{2-3z^2}{8(1+z^2)^{5/2}}, \tag{67}$$

and prime denotes z derivative, and $B_0 \equiv 1$.

Second linear independent solution is:

$$\bar{U}_2 = e^{-i\xi/S}\sum_{m=0}^{\infty}(-1)^m B_m(\xi)S^m. \tag{68}$$

Taking into account Eqs. (61)-(62) one can readily obtain that the first term of the expansions (64) and (68) represent the standard WKB solutions (23).

It also has to be noted that the series in Eqs. (64) and (68) are convergent at least for such values of $z \equiv -\beta(\tau)$ and S that satisfy the condition (22).

Combining Eqs. (66) and (67) we obtain, that B_m can be presented as:

$$B_m(z) = \frac{b_m}{(1+z^2)^{3m/2}}z^{\frac{1-(-1)^m}{2}} + \frac{p_m(z)}{(1+z^2)^{3m/2-1}}, \tag{69}$$

where $p_m(z)$ is some polynomial, $b_0 = 1$ and $b_{m>0}$ satisfies following recurrence relations:

$$b_m = (-1)^m i b_{m-1}\left[\frac{3}{2}(m-1) + \frac{5}{24m}\right]. \tag{70}$$

As we see in the next subsection, only the first term in Eq. (69) contributes to the leading term of the generation coefficient in the limit $S \to 0$.

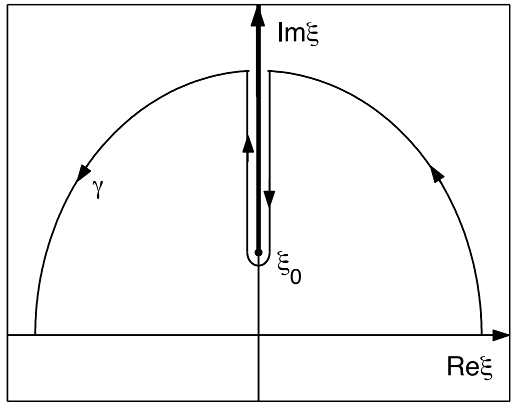

FIG. 2: Path γ in the complex ξ-plane.

B. the generation coefficient

In this subsection we derive the leading term of asymptotics of the generation coefficients (58):

$$G_{1,2} = \frac{\mp 1}{2S} \int_{-\infty}^{\infty} z u_{2,1} dz \tag{71}$$

in the limit $S \to 0$.

Using Eqs. (64) and (68), it is easy to show, that $-G_1 = G_2 \equiv G$. Changing the variable of integration to ξ and substituting Eq. (64), Eq. (71) can be rewritten as:

$$G = \frac{1}{2S} \int_{-\infty}^{\infty} \frac{z(\xi)}{f^{3/4}(\xi)} \left(\sum_{m=0}^{\infty} B_m(\xi) S^m \right) e^{i\xi/S} d\xi. \tag{72}$$

Consider the integrand in the upper half of the complex ξ-plane. First of all one has to note, that only irregularity of the integrand is branch point at $\xi_0 = i\delta/2$, or equivalently at $z(\xi_0) = i$, where δ is the standard phase integral defined by Eq. (48). It is well known [30], that to construct single valued analytical continuation of the integrand in the upper half of the complex ξ-plane, one needs to make branch cut, that starts at the branch point ξ_0 and ends either in another branch point or infinity. We choose to make branch cut that tends to infinity along positive direction of the imaginary ξ-axis (see Fig. 2).

Consider the integral (72) on the closed path γ (see Fig. 2) that consists of real ξ-axis, two quarters of circle and path ℓ that starts at $+i\infty$ along the left hand side of the branch cut, turns over ξ_0 and returns to $+i\infty$ along the right hand side of the branch cut. If one tends the radius of the circle to infinity then it is easy to see, that integral on the both quarters of the circle tends to zero exponentially. Therefore, according to Cauchy's theorem

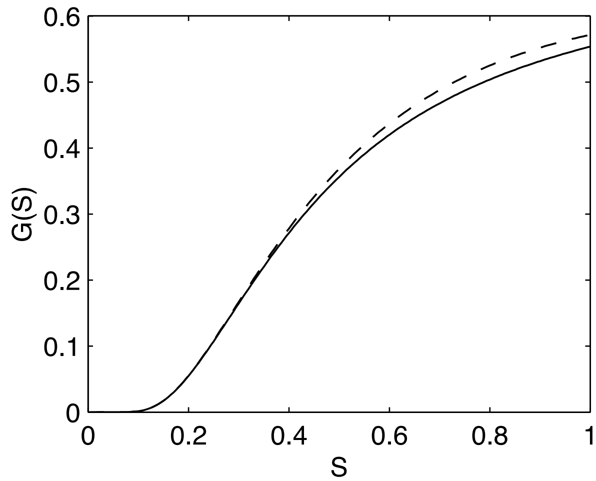

FIG. 3: Acoustic wave generation coefficient G vs normalized shear parameter S. Solid line represents generation coefficient obtained by numerical solution. Dotted line is the curve of analytical expression (76).

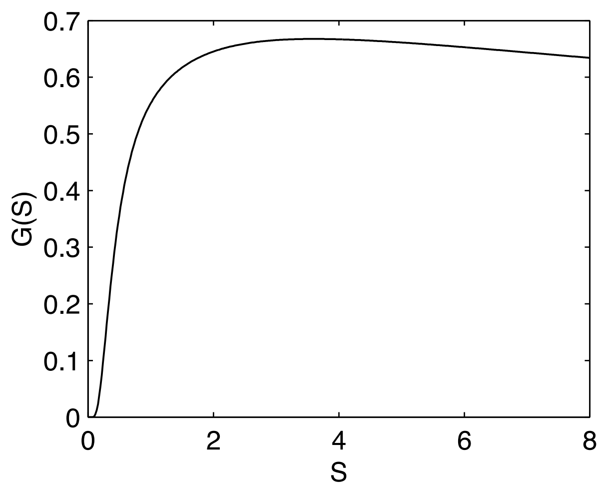

FIG. 4: Acoustic wave generation coefficient G vs normalized shear parameter S obtained by numerical solution of Eqs. (13)-(15)

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the integral along real ξ-axis equals to the integral along ℓ. Taking into account, that in the neighborhood of ξ_0, $z(\xi_0) = i$ and:

$$f^{3/4} = [3i(\xi - i\delta/2)]^{1/2}, \tag{73}$$

then the leading term of asymptotics in Eq. (72) along ℓ as well as the order of remainder term can be obtained using Watson's lemma for the loop integrals [30]. After straightforward calculations we obtain:

$$G = S^{-1/2} e^{-\pi/4S} \left[\frac{2\pi}{3^{1/2}} \sum_{m=0}^{\infty} \frac{|b_m|}{3^m \Gamma\left(m+\frac{1}{2}\right)} + O(S^{2/3}) \right]. \tag{74}$$

As it was mentioned above, the accuracy of WKB solutions is not enough for derivation of right expression of generation coefficient. Eq. (74) provides that all the terms of the solution in the form of formal series (64) contributes to the leading term of the asymptotics of generation coefficient (72).

Taking into account Eq. (70), it can be easily shown, that the number series in Eq. (74) is rapidly convergent. numerical evaluation of the series yields:

$$\frac{2\pi}{3^{1/2}} \sum_{m=0}^{\infty} \frac{|b_m|}{3^m \Gamma\left(m+\frac{1}{2}\right)} \approx 1.2533, \tag{75}$$

and therefore:

$$G \approx 1.2533 S^{-1/2} e^{-\pi/4S}. \tag{76}$$

Dependence of generation coefficient G on S for $S \leq 1$ is presented in Fig. 3. Solid line shows the generation coefficient obtained by numerical solution of Eqs. (13)-(15), where initial conditions (at $\beta(0) \ll 1$) for u, v and ρ are chosen by WKB solutions (23) and (24). Dotted line is the curve of analytical expression (76). Note, that even for $S = 1$ accuracy of (76) is better then four percent.

Dependence of generation coefficient G on S for $S \leq 8$, obtained by numerical solution of the set of Eqs. (13)-(15), is presented in Fig. 4. As it can be seen generation coefficient reaches its maximum at $S \approx 3$ and then slowly decreases.

C. On the method of derivation of acoustic wave generation coefficient

It has to be emphasized that the presented method of derivation of acoustic wave generation coefficient (76) is quite general. It can be successfully applied to the problems that are governed by equations similar to(18):

$$\ddot{u} + \omega^2(\tau) u = \gamma(\tau), \tag{77}$$

with $\omega^2(\tau)$ and $\gamma(\tau)$ such that:

(a) $\omega^2(\tau)$ have only a pair of complex conjugated turning points in the complex τ-plane;

(b) $\omega^2(\tau)$ satisfies the condition (21) on the real τ-axis.

(c) Special solution of inhomogeneous equation (24)-(26) are convergent on the real τ-axis.

Indeed, conditions (b) and (c) provide that the solutions of corresponding homogeneous equation (19) in a form of formal series as well as the special solution (24)-(26) are convergent on the real τ-axis. If so, the generation coefficient defined similar to (71), after substitution of the solution in a form of formal series, represents a sum of converging integrals, that can be calculated by Watson's lemma or Cauchy's theorem.

D. Transition matrix

Performed analysis allows us to determine entirely all the components of so-called transition matrix [30, 31] which connects initial and final intensities of different kind of perturbations.

Assume $C_{1,2}$ and q are initial ($\beta(0) \equiv k_y/k_x \gg \sqrt{S}$) intensities of acoustic and vortical perturbations respectively and $D_{1,2}$ and J are intensities of the same perturbations for $\beta(\tau) \ll -\sqrt{S}$. Performed analysis allows us entirely determine all the components of the transition matrix:

$$\begin{pmatrix} D_1 \\ D_2 \\ J \end{pmatrix} = \begin{pmatrix} e^{i\phi_1}R & iT & -iG \\ -iT & e^{-i\phi_1}R & iG \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ q \end{pmatrix}, \tag{78}$$

where R, T, ϕ_1, ϕ_2 and G are defined by Eqs. (44), (45), (46), (47) and (74), respectively.

E. On the time length of acoustic wave generation process

In this subsection we discuss the time scale of acoustic wave generation process by the vortex mode SFH. If condition $S \ll 1$ fails, estimation for the time length of the process $\Delta\tau_g$, coincident with Eq. (51) automatically follows from Eq. (22):

$$\Delta\tau_g \sim S^{-1/2}. \tag{79}$$

Returning to the dimensional variables this equation takes the form:

$$\Delta t_g \sim \frac{S^{-1/2}}{c_s k_x} = (A c_s k_x)^{-1/2}. \tag{80}$$

The problem becomes much more complicated if $S \ll 1$. In this case the condition of adiabatic evolution (22) holds during the whole evolution (therefore amplitude of generated waves are exponentially small with respect to the large parameter $1/S$) and there seems no obvious way how one can estimate $\Delta\tau_g$. Detailed mathematical analysis of the problem (which is based on the ingenious method developed in [38]) presented in Appendix B shows that the estimation (79) holds also for $S \ll 1$. In the later case we can also conclude that the generated waves are propagating nearly along the main flow and in the opposite direction. Indeed, combining Eqs. (12) and (80) one obtains that at the end of the generation process:

$$|k_x| \gg |k_y| \sim S^{1/2}|k_x|. \tag{81}$$

F. Acoustic power

Due to the fact that considered mechanism is linear, the analysis of linear problem has been performed, when each SFH evolves independently. But obtained results allow us to estimate also main characteristics of turbulent shear flow generated acoustic field by the considered mechanism. Consider the case $S \ll 1$. Assume there exists stationary shear turbulence with spectral energy tensor $\Phi_{ij}(\mathbf{k})$. The shear of main velocity profile leads to permanent distortion of turbulent vortices. Therefore, the spectral energy density can be stationary only if the linear distortion caused by the shear is balanced by nonlinear interaction of turbulent vortices [40]. This circumstance leads to conclusion that the time scale t_{NL} of nonlinear interaction of energy containing vortices:

$$t_{NL} \sim \Delta t_d \sim \frac{1}{A} \gg \Delta t_g. \tag{82}$$

If this condition is satisfied, future analysis becomes straightforward. Taking into account that during some time interval $\Delta\tau$ acoustic waves are generated only by vortices that passes through generation area at $k_y = 0$, noting that for such a vortices according to Eq. (16) $q \approx v$ and using Eq. (23) for total acoustic energy ΔE_a generated during Δt we obtain

$$\Delta E_a \approx \int_{-\infty}^{\infty} dk_x \int_0^{Ak_x\Delta t} dk_y \frac{G^2}{\omega}\Phi_{22}(\mathbf{k}). \tag{83}$$

Therefore for total acoustic power we have

$$P_a \equiv \frac{\Delta E_a}{\Delta t} \approx \int_{-\infty}^{\infty} 2A|k_x|G^2\Phi_{22}(k_x,0)dk_x, \tag{84}$$

where the integrand represents the acoustic power spectrum.

As a concrete example of shear flow turbulence consider Tchen model [40]. In this case

$$\Phi_{22}(k_x, 0) = \frac{E_t(|k_x|)}{2\pi|k_x|}, . \tag{85}$$

with $E_t(k) = \alpha k^{-1}$, for $k^{min} < k < k^{max}$, and α is some constant. Substituting these equations into Eq. (84) and using Eq. (76), after integration we obtain

$$P_a \approx \frac{6A\alpha}{\pi^2} \exp\left(-\frac{\pi c_s k^{min}}{2A} \right), \tag{86}$$

G. Comparison of the wave generation and resonant transformation processes

Generation of acoustic waves by vortices represents the example of the linear coupling of different modes of perturbations in inhomogeneous media. As it was mentioned above, existing asymptotic methods for analysis of linear coupling of two wave modes in slowly varying media are useless for the study of linear coupling between vortical perturbations and acoustic waves due to the fact that the vortical perturbations has zero frequency and therefore variables describing the evolution of vortical perturbations [see e.g. Eqs. (24)-(26)] do not include rapidly oscillating multipliers. In this context it seems interesting to compare obtained results with well known [23, 24, 36] properties of linear coupling of different wave modes in inhomogeneous media.

First of all, let us shortly represent well known properties [23, 24, 31, 36] of linear coupling of different wave modes in slowly varying media. Under slowly varying we mean that the wavelength of the wave modes is much shorter than the spatial scale of inhomogeneity and/or the period of the wave modes is much shorter then the temporal scale of inhomogeneity, i.e.:

$$\bar{S} \sim \frac{\dot{\Omega}_{1,2}(\tau)}{\Omega^2_{1,2}(\tau)} \ll 1, \tag{87}$$

where $\Omega_{1,2}(\tau)$ are frequencies of the wave modes.

In this case effective coupling between wave modes occur only in some vicinities (so-called resonant area) of the resonant point τ_* where frequencies of the wave modes become equal. Transformation coefficient is usually defined as absolute value of corresponding term of the transition matrix, which similar to Eq. (78) connects WKB amplitudes of wave modes before and after passing through resonant area.

Linear transformation of wave modes has the following properties:

(i) If the condition (87) is satisfied, then the total energy of perturbations conserves during the transformation process. This means that transformed wave is generated at the expense of the energy of incident wave;

(ii) The coefficient of the wave transformation from one into another equals to the transformation coefficient of the reverse process:

$$T_{12} = T_{21}, \tag{88}$$

(iii) for a pair of complex conjugated first order resonant points [the resonant point is called of order n if $(\Omega_1^2 - \Omega_2^2) \sim (\tau - \tau_*)^{n/2}$ in the neighborhood of τ_*] in the complex τ-plane the transformation coefficient is:

$$T_{12} = \exp\left(-\delta_{12}\right)\left[1 + O(\bar{S}^{1/2})\right], \tag{89}$$

where δ_{12} is corresponding phase integral:

$$\delta_{12} = \left| \Im \int_{\tau_0}^{\tau_*} [\Omega_1(\tau) - \Omega_2(\tau)] d\tau \right|, \tag{90}$$

τ_0 is arbitrary point on the real τ-axis and τ_* is either of the resonant points.

Taking into account Eq. (29) and noting that q is absolute (not adiabatic) invariant of the system, one can conclude, that generated acoustic waves gain the energy from the background flow, in contrary to the linear transformation of wave modes.

Another important discrepancy between the processes is strongly delineated asymmetry of interaction of perturbation modes: in our concrete problem acoustic waves can be generated by vortices whereas reverse process is impossible.

The third difference follows from the comparison of Eqs. (76) and (89). Note that in our dimensionless notations the frequency of acoustic waves is $\Omega_a(\tau) = \pm[1 + \beta^2(\tau)]^{1/2}$ and the frequency of vortical mode is $\Omega_w = 0$, calculation of the phase integral yields:

$$\delta_{aw} = \left| \Im \int_0^i [\Omega_a(\tau) - \Omega_w(\tau)] d\tau \right| = \frac{\pi}{4S}. \tag{91}$$

Then we see that the exponential multipliers in Eqs. (76) and (89) are of the same origin: both of them are caused by difference between frequencies of interacting modes. The principal difference between the expressions (76) and (89) is multiplier $S^{-1/2}$ in Eq. (76).

V. SUMMARY

Evolution of two dimensional hydrodynamic perturbations was studied in uniform shear flow. It is shown that for any value of the shear rate there exists a finite interval of time, when the evolution of perturbations is non-adiabatic. Outside this interval the evolution is adiabatic and fully described by WKB approximation. This circumstance allows to formulate asymptotic problem in the usual manner. It is shown that non-adiabatic behavior of perturbations consist of two phenomena: over-reflection of acoustic waves and generation of acoustic waves by vortex mode perturbations. The later phenomenon is known as one of the linear mechanisms of sound generation in shear flows that is related to continuous spectrum perturbations. Detailed analytical study of the problem is performed and main quantitative and qualitative characteristics of the radiated acoustic field are obtained and analyzed.

Acknowledgments

The research described in this publication was made possible in part by Award No. 3315 of the Georgian Research and Development Foundation (GRDF) and the U.S. Civilian Research and Development Foundation for the Independent States of the Former Soviet Union (CRDF). The research was supported in part by ISTC grant No. G 553. One of us (PJM) was supported by the US Department of Energy Contract No. DE-FG03-96ER-54346.

Appendix A

Let first consider the special solution of Eq. (18):

$$u_{inh} = \frac{q}{W}\int_{\tau_0}^{\tau} \beta(\bar\tau)[u_1(\bar\tau)u_2(\tau) - u_1(\tau)u_2(\bar\tau)]d\bar\tau, \tag{A1}$$

with $\tau_0 = -\infty$. Substituting WKB solutions (23) into this equation and multiply integrating by parts allow us to obtain the special solution in the form of series:

$$u_{inh} = q\sum_{m=0}^{\infty} S^m \bar{y}_m(\tau). \tag{A2}$$

Simple calculations show that the first term of this series coincides with the first term of Eq. (24):

$$\bar{y}_0(\tau) = \bar{y}_0(\tau) = -\frac{\beta(\tau)}{\omega^2(\tau)}, \tag{A3}$$

whereas for $m > 1$:

$$\bar{y}_m(\tau) \neq y_m(\tau). \tag{A4}$$

The reason of this circumstance is clear: WKB solutions are approximate solutions of Eq. (18) and their accuracy $O(S)$ allows to obtain correct expression only for the first term of the special solution (24), i.e., leading term of u_{inh} in the limit $S \to 0$.

The same technique allows to show that the special solution of Eq. (18) that do not contain oscillating terms corresponds to the choice $\tau_0 = -\infty$. Indeed, repeating the same steps as for the derivation of Eq. (A2) one can readily check that for any finite τ_0 the leading term of the special solution in the limit $S \to 0$ contains rapidly oscillating terms that are proportional to WKB solutions (23).

Appendix B

It is obvious that in case $S \ll 1$ acoustic waves are generated in some vicinity of the point $z \equiv -\beta(\tau) = 0$. Let us turn back to Eq. (72):

$$G = \frac{1}{2S} \sum_{m=0}^{\infty} S^m \int_{-\infty}^{\infty} \frac{z(\xi)}{f^{3/4}(\xi)} B_m(\xi) e^{i\xi/S} d\xi. \tag{B1}$$

Consider mth term of the sum:

$$G_m = \frac{S^{m-1}}{2} \int_{-\infty}^{\infty} \frac{z(\xi)}{f^{3/4}(\xi)} B_m(\xi) e^{i\xi/S} d\xi. \tag{B2}$$

Introducing a new variable [38]:

$$F = \frac{1}{S} \int_{i}^{z} f^{1/2}(\zeta) d\zeta \equiv \frac{1}{S} \left(\xi - \frac{i\delta}{2} \right) \tag{B3}$$

and taking into account Eqs. (69), (70) and (73), one can readily obtain:

$$G_m \sim e^{-\delta/2S} b_m \int_{-\infty - i\delta/2S}^{\infty - i\delta/2S} \frac{m!}{F^{m+1/2}} e^{iF} dF. \tag{B4}$$

Using Stirling's formula and introducing polar coordinates in the complex F-plane:

$$F = -i|F|e^{i\theta}, \tag{B5}$$

and noting that in the vicinity of $\xi = 0$:

$$dF \sim |F| d\theta, \tag{B6}$$

then Eq. (B4) can be rewritten as:

$$G_m \sim e^{-\delta/2S} b_m \int d\theta \left(\frac{m}{|F|}\right)^{m+\frac{1}{2}} \times$$
$$\exp[|F|cos\theta - m - i(|F|sin\theta - m - 1/2)]. \quad \text{(B7)}$$

If m is close to F:

$$|m - |F|| \ll |F|^{1/2}, \quad \text{(B8)}$$

Eq. (B7) simplifies and reduces to:

$$G_m \sim e^{-\delta/2S} b_m \int \exp(-F\theta^2/2) d\theta. \quad \text{(B9)}$$

From this equation it follows that G_m is generated in $\Delta\theta \sim S^{1/2}$ vicinity of the point $\theta_0 = 0$.

In fact the same result remains valid for arbitrary m. Indeed, one can readily check that any component of the series in Eq. (B1) can be reduced to the form (B4) with m satisfying condition (B8) by multiple integration by parts.

Taking into account that:

$$\Delta\theta \sim \Delta z \sim \Delta\tau_g S, \quad \text{(B10)}$$

we obtain:

$$\Delta\tau_g \sim S^{-1/2}. \quad \text{(B11)}$$

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