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Hamiltonian formulation and coherent structures in electrostatic turbulence

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Abstract

A Hamiltonian formulation is constructed for a finite ion Larmor radius fluid model describing ion temperature-gradient driven and drift Kelvin–Helmholtz modes. The Hamiltonian formulation reveals the existence of three invariants obeying detailed conservation properties, corresponding roughly to generalized potential vorticity, internal energy and ion momentum parallel to the magnetic field. These three invariants are added to the energy to form a variational principle that describes coherent structures (CSs), such as monopolar and dipolar vortices or modons. It is suggested that the invariants are responsible for the coherence and longevity of CSs and for their robustness during binary collisions.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The Hamiltonian formalism constitutes an effective framework for investigating the dynamics of fluid models [1]. In particular, it provides techniques for finding conserved quantities, obtaining first integrals of the equilibrium equations and constructing variational principles describing the stability of equilibria [1, 2] including propagating nonlinear coherent structures (CSs) [3]. It can also be used to guide the derivation of fluid closures by specifying the subset of higher order terms that need to be retained in order to preserve desired conservation properties [4]. More recently, the Hamiltonian formalism has been used to derive equations governing the generation of zonal flow and long wavelength CSs under the effect of stochastic forcing by short wavelength modes [5].

A possible objection to the application of the Hamiltonian formalism to models of turbulent transport is that energy conservation is generally violated in the open systems of interest in turbulent transport studies. In slab geometry, for example, energy is generally supplied to the system through one boundary and removed through the other. Such sources and sinks of energy, however, are known and controlled by the modeller and should be distinguished from unphysical sources arising from faulty dynamical equations. Models aiming to describe turbulent dynamics should satisfy energy conservation for closed boundary

conditions in the absence of known volumetric sources and dissipation terms. In particular, energy should be conserved in *local* interactions such as the collision between two vortices. The purpose of the Hamiltonian formulation is, thus, to shed light on the local properties of the dynamics that are independent of the drive.

A particularly appealing feature of the Hamiltonian formalism is that it readily provides the first integrals of the equations governing the properties of CSs. CS are two-dimensional soliton-like waves that usually consist of independent or paired vortex tubes propagating in the direction perpendicular to both the magnetic field and the equilibrium density gradient [6–15]. The paired vortices, called modons, are the simplest solutions that are free of damping by wake-field excitation. Figure 1 illustrates the role of nonlinearity in counteracting wave dispersion by comparing the evolution of a modon using the linearized and nonlinear dynamical equations. Modons have further been shown to have remarkable resilience, surviving collisions with other modons when the interaction time is shorter than the eddy turnover time [16–21]. For longer interaction times, it is common for one of the two modons to be split into independent vortices. The independent vortices experience damping through wake-field radiation, but in general this damping is weak and the vortices are quite long-lived [20].

The properties of CS are consistent with the common observation in simulations and experiments of patterns of flow or density perturbations that enjoy a lifetime substantially exceeding the correlation time for the turbulence. Such patterns are thought to play an important role in turbulent transport [19–23]. In particular, they give rise to intermittency and non-Gaussian statistics, and they determine the asymptotic behaviour of the turbulent spectra [23]. They appear to be a generic feature in simulations of turbulent transport and have been clearly identified in observations of edge turbulence [24–27], where their effect on the erosion of plasma-facing components is a source of concern. In the confinement region, CS are responsible for avalanches and have been observed as radially extended coherent signals in the electron cyclotron emission [28].

In this paper we present a Hamiltonian formulation of the equations governing the dynamics of the ion temperature gradient (ITG) instability with finite ion Larmor radius (FLR). We use, as a starting point, the model of Kim, Horton and Hamaguchi (henceforth KHH) describing electrostatic turbulence driven by the gradient of the ion temperature in slab geometry [29]. This model ensures energy conservation by including the divergence of the polarization drift in the heat equation. The closure scheme introduced by KHH was later extended by Zeiler *et al* [30] in their electromagnetic edge turbulence model, and is also used in the BOUT code developed by Xu *et al* [31]. We present Hamiltonian formulations for two different versions of the basic ITG model. The first corresponds to a traditional, fully adiabatic response for the electrons and the second to a more accurate, parallel adiabatic response where the electron density is insensitive to perturbations that are constant on a flux surface. We find that this second model is Hamiltonian only if the product $\Gamma = \gamma\tau$ of the adiabatic index γ with the ratio τ of ion and electron temperature is taken to be zero. Krommes and Kolesnikov [5] have recently proposed an alternative Hamiltonian model based on the two-field version of the gyrofluid equations of Dorland and Hammett [32]. Apart from the different treatment of FLR effects, our model differs from theirs in our inclusion of the effects of parallel flow and background drifts.

We derive a complete family of Casimir invariants for our two models and use these invariants to construct a variational principle describing the equilibrium and stability of propagating CS. Our solution extends previous descriptions of CS for ITG models [11, 14] by retaining both FLR and parallel flow effects. We show that for the parallel adiabatic model, however, the solubility conditions for two-dimensional CSs are violated when the product Γ is nonzero.

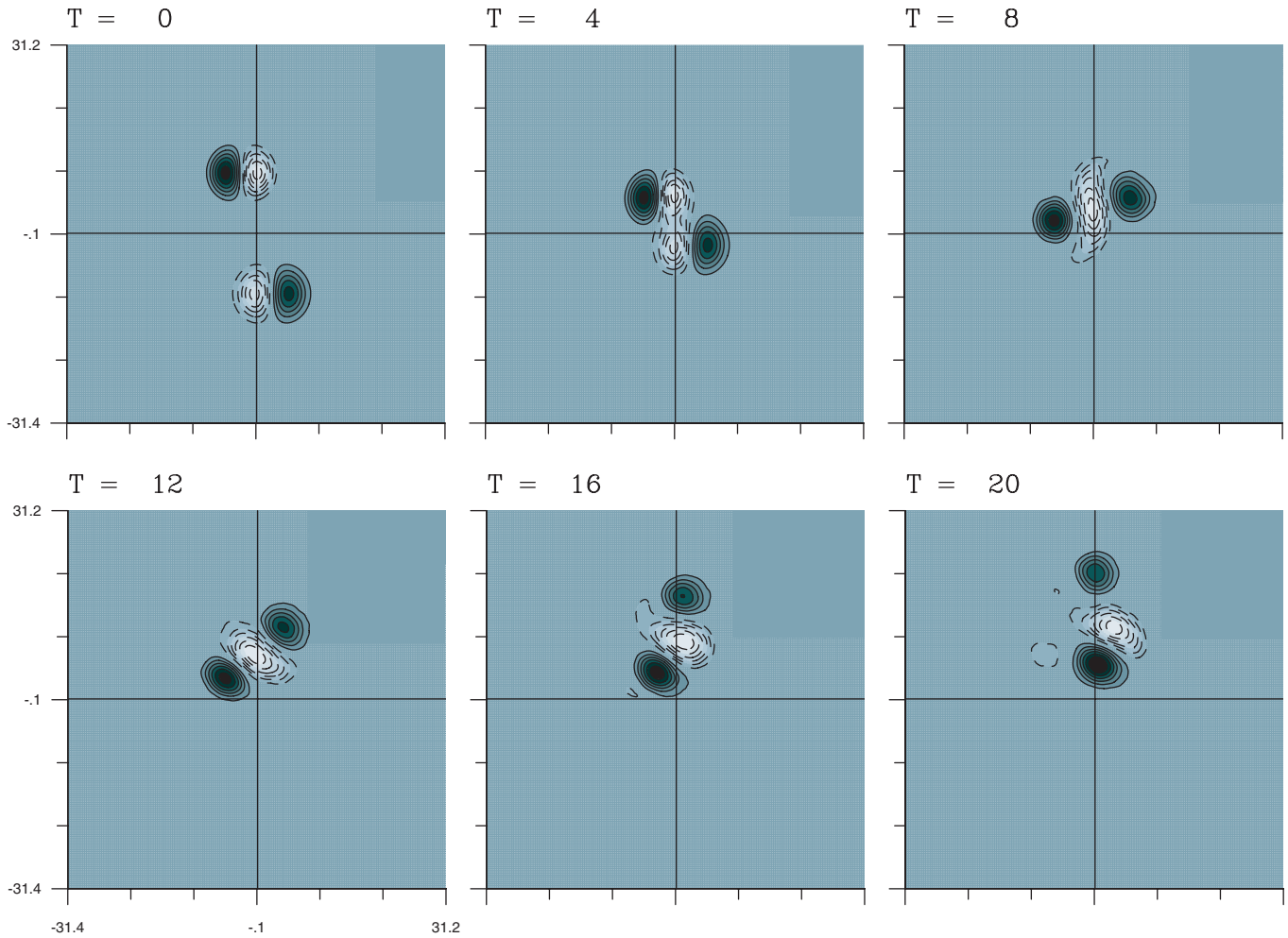

Figure 1. Comparison of the evolution of a dipolar structure as predicted by the linearized and the nonlinear equations, showing the coherence of the nonlinear solution over times long compared to the dispersion time.

This paper is organized as follows. In section 2 we present the KHH model and review the Hamiltonian formalism. In section 3 we construct the Hamiltonian formulation and derive the conserved quantities for the version of the model that uses the fully adiabatic electron response. We next use the conserved quantities to construct a variational principle describing the steady-state (equilibrium) solutions of the system, and we describe the modon solutions of these equations. In section 4 we construct a Hamiltonian formulation for the version of the model that uses the parallel adiabatic electron response. We end by discussing our results in section 5.

2. Formulation

2.1. Fluid model for ITG dynamics

In the electrostatic limit the turbulent dynamics caused by the ITG-driven instability can be described in terms of the four fluid variables n, ϕ, p and v representing the fluctuations in the density, the electrostatic potential, the pressure and the ion velocity parallel to the magnetic field. Following KHH, we normalize these variables to the background density, the electron temperature, the equilibrium pressure and the cold-ion sound speed $c_s = \sqrt{T_e/m_i}$, respectively. The evolution of n, p and v is governed by the ion continuity equation, the adiabatic heat equation and the parallel component of the ion momentum conservation equation:

$$\frac{dn}{dt} - \nabla_\perp \cdot \frac{d}{dt}\nabla_\perp(\phi + p) - \frac{\partial \phi}{\partial y} + \nabla_\parallel v = 0, \tag{1}$$

$$\frac{d}{dt}(p - \Gamma n) - (K - \Gamma)\frac{\partial \phi}{\partial y} = 0, \tag{2}$$

$$\frac{dv}{dt} + \nabla_\parallel(\phi + p) = 0, \tag{3}$$

where $\Gamma = \frac{5}{3}\tau$, $\tau = T_i/T_e$, $K = \tau(1 + \eta_i)$ and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}_E \cdot \nabla,$$

with $\boldsymbol{v}_E = \hat{\boldsymbol{z}} \times \nabla\phi$. All lengths in the plane perpendicular to the magnetic field are normalized to $\rho_s = c_s/\omega_{ci}$, where ω_{ci} is the ion Larmor frequency and all lengths along the magnetic field are normalized to the density gradient scale length L_n. The time t is normalized to L_n/c_s. We express the gradient in the direction of the magnetic field, $\nabla_\parallel$, in terms of the magnetic flux $\psi = x^2/2L_s$, where L_s is the magnetic shear length, according to

$$\nabla_\parallel f = \frac{\partial f}{\partial z} + (\hat{\boldsymbol{z}} \times \nabla\psi) \cdot \nabla v. \tag{4}$$

For the sake of clarity we have omitted the dissipation terms from (1)–(3). These terms are essential for a complete description of the turbulent dynamics, but they play no part in the Hamiltonian formulation and can easily be restored *a posteriori*.

The above system of equations must be closed by a constitutive equation describing the response of the electron density to electrostatic perturbations. We will consider two models for the electron response. The first model generalizes the adiabatic model used in KHH, $n = \phi$, so as to ensure Galilean invariance,

$$n = \phi - ux, \tag{5}$$

where u is a constant background velocity in the y-direction. We will refer to this model as the *fully adiabatic model*. We will call the second, more realistic model the *parallel adiabatic model*. This second model is defined by

$$n = \tilde{\phi} := \phi - \bar{\phi}, \tag{6}$$

where the over-bar represents the flux-surface average,

$$\bar{\phi} := \oint\!\!\oint \frac{\mathrm{d}y\,\mathrm{d}z}{L_y L_z}\,\phi.$$

Equation (6) is obtained by observing that the parallel component of the electron momentum equation,

$$\nabla_{\parallel}(n + \phi) = 0,$$

requires that the electron density satisfy the Boltzmann relation only along the field. Thus,

$$n = \phi + f(\psi), \tag{7}$$

where $f(\psi)$ is an integration constant. One may determine this integration constant from the flux-surface average of the electron continuity equation,

$$\partial_t \bar{n} = -\overline{[\phi, n]} = \partial_x(\overline{n\partial_y \phi}) = 0,$$

where the second equality follows by integration by parts and the third is a consequence of equation (7). Thus, $\bar{n}$ is invariant, and equation (6) follows from the choice $\bar{n} = 0$.

The parallel adiabatic model avoids unphysical fluctuations in the averaged density, fluctuations that are implied by the fully adiabatic model [32]. The two models are illustrated in figures 2 and 3 showing a collision between two dipolar vortices governed by the Hasegawa–Mima equation [33] (fully adiabatic model, figure 2) and a modified version of the Hasegawa–Mima equation using the parallel adiabatic response (figure 3). These figures show that the parallel adiabatic model leads to the generation of zonal flows during the collision. Our analysis leads to the conclusion, however, that for finite ITG this model cannot be expressed as a Hamiltonian system (as described in section 2.2) except for vanishing specific heat index, $\Gamma = 0$. The fully adiabatic response model thus has the advantage of being able to address questions concerning the effects of Γ. It is also of historical interest in view of its prevalence in early studies of electrostatic turbulence and its isomorphism with the equivalent barotropic vorticity equation describing Rossby waves [33, 34]. Note that the linear properties of both models are identical. The stability of ITG and Kelvin–Helmholtz eigenmodes in equilibria with sheared flows is described in [35], and the convective amplification of wavepackets is described in [36].

2.2. The Hamiltonian formalism

The Hamiltonian formulation of fluid models is reviewed in [1]. Here we give only a brief description of a method for constructing Hamiltonian formulations.

The primary goal is to find a Hamiltonian H and a Poisson bracket $\{\cdot, \cdot\}$ such that the equations of motion can be written in the form

$$\dot{\xi}^j = \{\xi^j, H\}, \tag{8}$$

where the dot represents differentiation with respect to time and ξ^j represents the suitably chosen dynamical variables indexed by j. The Poisson bracket must be bilinear, antisymmetric and must satisfy the Jacobi identity,

$$\{a, \{b, c\}\} + \text{circular permutations} = 0.$$

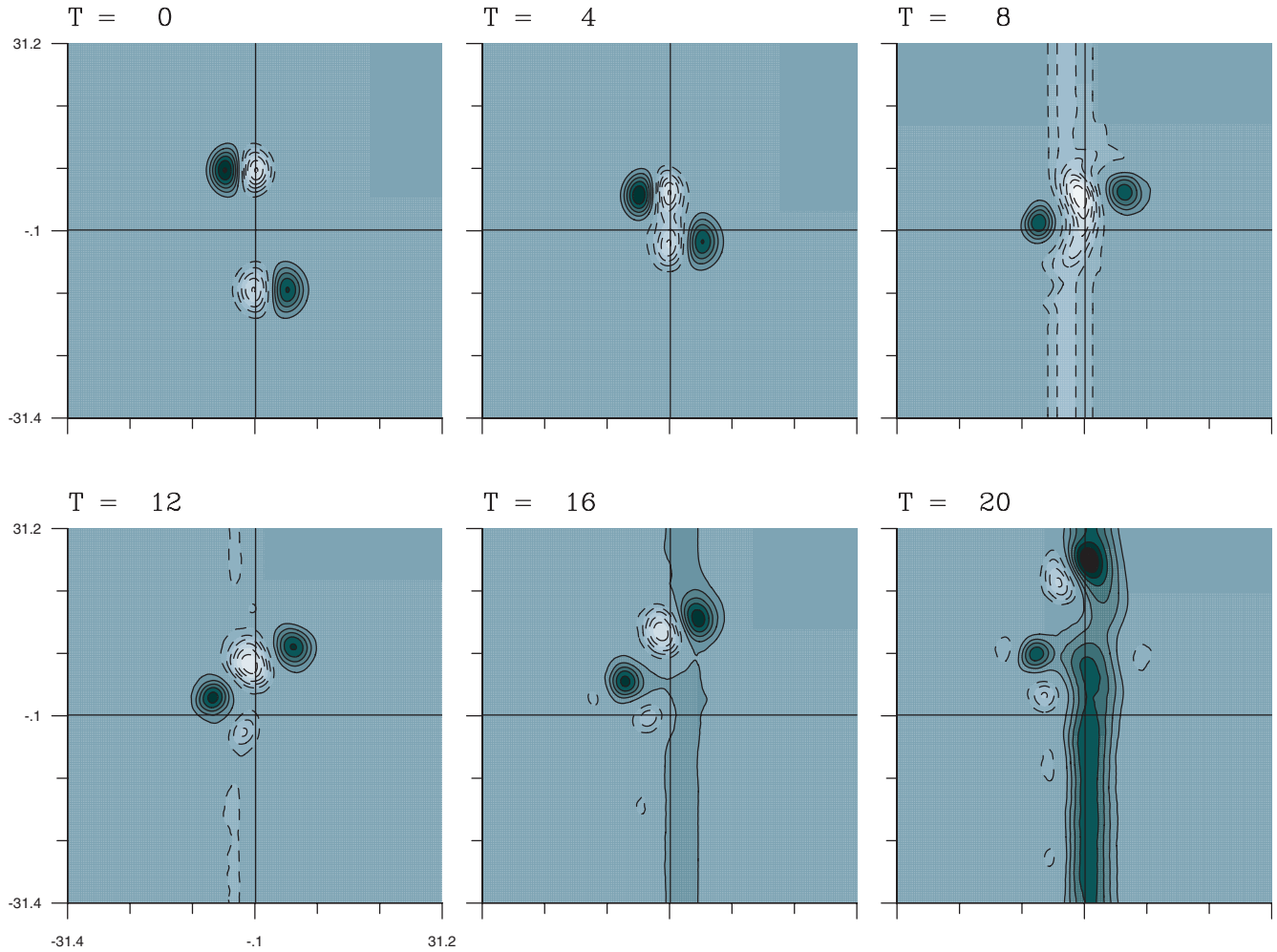

Figure 2. Collision between two dipolar vortices governed by the fully adiabatic model.

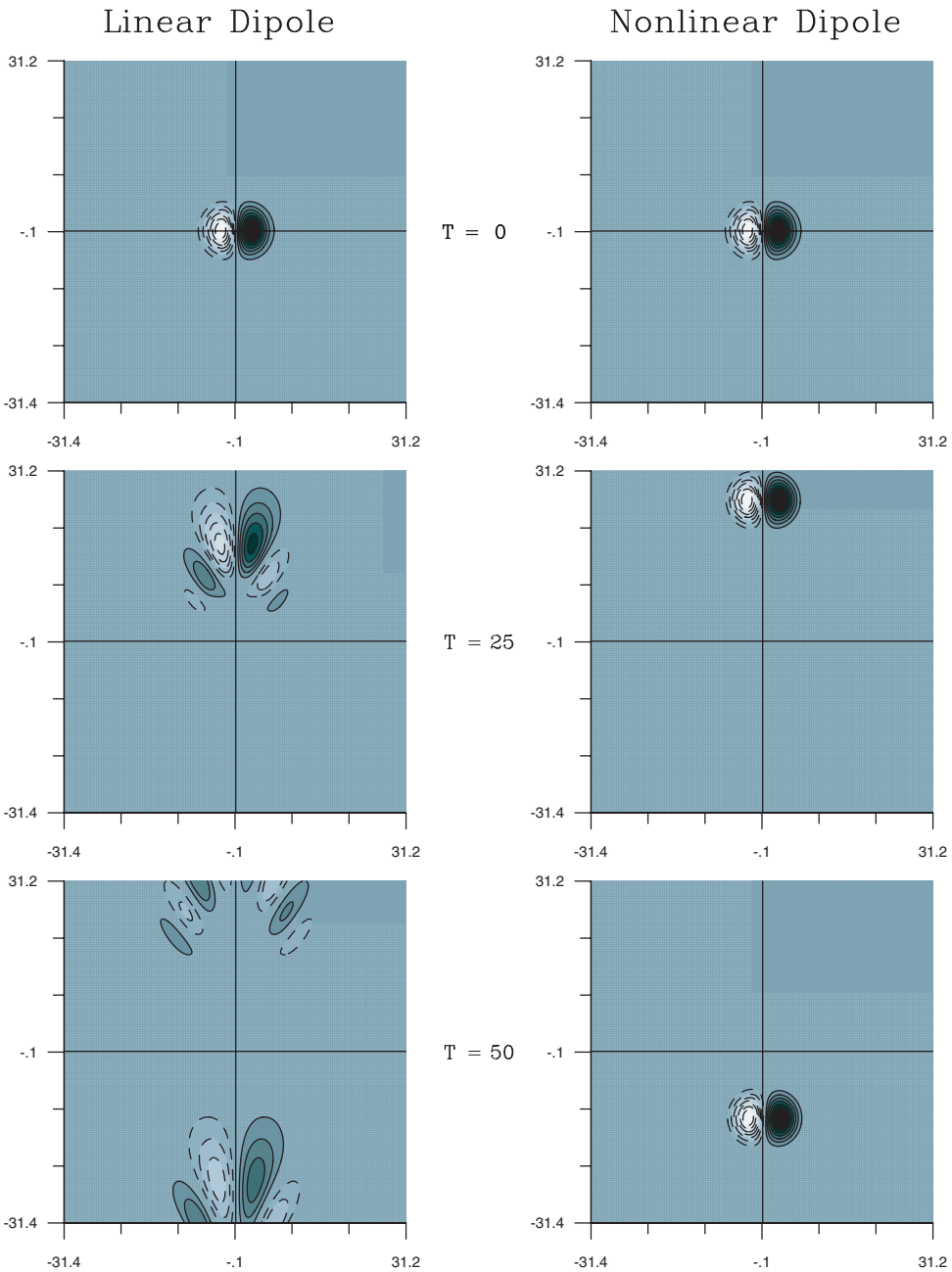

Figure 3. Collision between two dipolar vortices governed by the parallel adiabatic model.

It follows immediately from the antisymmetry of the bracket that the Hamiltonian, usually identified with the energy of the system, is conserved.

It is generally the case that there exist quantities C that vanish under the action of the Poisson bracket,

$$\{f, C\} = 0, \tag{9}$$

for any choice of f. Such quantities are clearly also conserved. They are called Casimir invariants, or simply Casimirs. Two well-known examples of Casimirs are the circulation for an inviscid fluid and the magnetic helicity in magnetohydrodynamics. A complete set of Casimirs for any given model can be constructed systematically by solving (9).

The Poisson bracket often takes the form of a Lie–Poisson bracket, which in two-dimensional systems can be expressed as

$$\{F, G\} = \langle W_k^{ij} \xi^k [F_{\xi^i}, G_{\xi^j}] \rangle, \tag{10}$$

where the W_k^{ij} are constant coefficients, summation over repeated indices is understood,

$$\langle f \rangle = \int_0^{L_x} \mathrm{d}x \oint \mathrm{d}y \oint \mathrm{d}z \, f(x, y, z) \tag{11}$$

is the volume integral, and

$$[f, g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y} \tag{12}$$

is called the inner Poisson bracket. Note that the inner bracket acts on fields (functions of space and time), whereas the 'outer' bracket $\{\cdot, \cdot\}$ acts on functionals (functions of fields). The arguments of the inner bracket are the functional derivatives of F and G defined by

$$\langle \eta F_{\xi^i} \rangle := \frac{\mathrm{d}}{\mathrm{d}\delta} F[\ldots, \xi^i + \delta\eta, \ldots]\Big|_{\delta=0}, \tag{13}$$

where η is an arbitrary function of the spatial coordinates and δ is a real coefficient. Since the inner bracket is antisymmetric, W must be symmetric in its upper indices to ensure antisymmetry of the Lie–Poisson bracket. Thiffeault and Morrison [37] have shown that the Lie–Poisson bracket defined by (10) satisfies the Jacobi identity when the product

$$W_k^{ij} W_i^{lm} \tag{14}$$

is symmetric in all three free upper indices. Equivalently, the Lie–Poisson bracket (10) satisfies the Jacobi identity when all the matrices $\boldsymbol{W}^{(i)}$ with elements $(W^i)_k^j$ commute.

Substitution of (10) into the equation of motion and integration by parts results in

$$\dot{\xi}_i = -W_k^{ij} [\xi^k, H_{\xi^j}]. \tag{15}$$

This suggests a pedestrian, but effective, procedure for finding the Poisson bracket of a Hamiltonian model. First, write the two-dimensional version of the equations of motion in a form resembling (15). Identify the coefficients W_k^{ij}, and verify the Jacobi identity. Second, extend the Poisson bracket to allow for three-dimensional perturbations [38]. We will carry out this procedure in the following sections.

3. Fully adiabatic model

3.1. Conserved energy

We may construct a conserved energy as follows. We begin by multiplying the equations of motion by $p+\phi$, p/Γ and v, respectively, summing the resulting equations, and integrating over all space. Assuming periodic boundary conditions in y and z, there follows after integrating by parts [29]

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \phi^2 + [\nabla_\perp(\phi + p)]^2 + \frac{p^2}{\Gamma} \right\rangle = \left\langle \frac{Kp}{\Gamma}\frac{\partial \phi}{\partial y} \right\rangle. \tag{16}$$

We have assumed in (16) that the surface term resulting from the integration by parts vanishes. This is true for either periodic (in x) or impermeable ($\phi(0) = \phi(L_x) = 0$) boundary conditions. The volume integral on the right-hand side of (16) represents the work done by the background pressure gradient (K) when a pressure perturbation p is convected across the gradient by the electric drift $v_{Ex} = -\partial_y \phi$.

In order to eliminate the right-hand side of (16) we multiply the pressure equation by x and integrate over the volume. We find

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x(p - \Gamma\phi)\rangle = -\left\langle p\frac{\partial \phi}{\partial y}\right\rangle + L_x(\overline{v_{Ex}\, p})_{x=L_x}. \tag{17}$$

Here the boundary term vanishes for impermeable walls, $v_{Ex} = 0$, at the $x = 0, L_x$ boundaries, but not for periodic boundary conditions. Clearly, if a fluid element is removed from one side of the simulation volume and reintroduced on the other side where the background pressure is higher, the energy in the system will change. Since our goal is to investigate the Hamiltonian form and CS of the system, we henceforth adopt impermeable boundary conditions for both electrons and ions: $\partial_y \phi = \partial_y p = 0$. It follows then from (16) and (17) that

$$E = \frac{1}{2}\left\langle \phi^2 + [\nabla_\perp(\phi + p)]^2 + \frac{(p + Kx)^2}{\Gamma} - 2Kx\phi\right\rangle \tag{18}$$

is conserved: $\dot{E} = \mathrm{d}E/\mathrm{d}t = 0$.

We may obtain a second conserved quantity by multiplying the vorticity equation by x and again integrating over all space:

$$\frac{\partial}{\partial t}\langle x\phi\rangle = \Upsilon, \tag{19}$$

where Υ is a boundary term that vanishes when $\partial_x(\phi + p)$ is constant on the $x = 0$, $x = L_x$ walls, corresponding to a fixed velocity parallel to the walls. Assuming this boundary condition to hold, it follows that

$$A = 2\langle x\phi\rangle \tag{20}$$

is a conserved quantity. Recalling that $\phi = n + ux$, we see that A is related to the cross-gradient position of the centre of mass.

3.2. Construction of the Poisson bracket

In order to take advantage of the properties of the Lie–Poisson bracket, described in section 2.2, we specialize at first to the two-dimensional case where $\partial/\partial z = 0$, for which

$$\nabla_\parallel v = [\psi, v].$$

We will see that the generalization to three dimensions is straightforward [38, 14].

We begin by noting that the heat transport equation is a simple convection equation,

$$\frac{\partial s}{\partial t} = [s, \phi], \tag{21}$$

where

$$s := p - \Gamma\phi + [K - (1 - u)\Gamma]x \tag{22}$$

is the linearized change in the entropy per unit mass with respect to a homogeneous reference state. We use equation (22) to eliminate p in terms of s in the ion continuity equation. We find

$$\frac{\partial \Omega}{\partial t} = [\Omega, \phi] - [\nabla_\perp \phi; \nabla_\perp s] + [\psi, v], \tag{23}$$

where ψ is the magnetic flux defined above equation (4) and

$$\Omega = \nabla_\perp^2[(1+\Gamma)\phi + s] - \phi - (1-u)x$$

is a generalized potential vorticity. We observe that except for the term

$$[\nabla_\perp\phi;\, \nabla_\perp s := [\partial_x\phi,\, \partial_x s] + [\partial_y\phi,\, \partial_y s], \tag{24}$$

the right-hand side of equation (23) has the form of a sum of inner Poisson brackets acting on the fields (Ω, s, v). This form is consistent with the general form of the Lie–Poisson bracket (10).

The offending term may be eliminated with a similar term that arises when applying the $\nabla_\perp^2$ operator to the pressure equation,

$$\frac{\partial \nabla_\perp^2 s}{\partial t} = [\nabla_\perp^2 s, \phi] + [s, \nabla_\perp^2\phi] + 2[\nabla_\perp s; \nabla_\perp\phi] = 0. \tag{25}$$

There follows

$$\frac{\partial N}{\partial t} = [N, \phi] + \frac{1}{2}[s, \nabla_\perp^2\phi] + [v, \psi], \tag{26}$$

where

$$N = \phi - (1+\Gamma)\nabla_\perp^2\phi - \tfrac{1}{2}\nabla_\perp^2 s + (1-u)x, \tag{27}$$

is the density of guiding centres [32].

We next turn to the velocity equation. Eliminating the pressure in favour of s leads to

$$\frac{\partial v}{\partial t} = [v, \phi] - [\psi, (1+\Gamma)\phi + s]. \tag{28}$$

The last term involves a bracket of ψ and ϕ, neither of which is an independent dynamical variable. To remedy this, we change variables to $V = v - (1+\Gamma)\psi$. The equation for V,

$$\frac{\partial V}{\partial t} = [V, \phi] + [s, \psi], \tag{29}$$

has the desired form.

In order to complete the construction of the Lie–Poisson bracket for the dynamical equations (21), (26) and (29) we rearrange the various terms appearing in these equations so that each Poisson bracket acts on one of the fields (N, s, V) and on a functional derivative of the Hamiltonian, as in equation (15). We expect the Hamiltonian to be given by a linear combination of the energy E and the conserved quantity A,

$$H = E + \alpha A,$$

where α is a coefficient that we must determine. The functional derivatives of the Hamiltonian, denoted by H_{ξ^j}, are

$$H_N = (1+\Gamma)\phi + s + (\alpha - K + \Gamma)x, \tag{30}$$

$$H_s = N + \frac{1}{2}(1+\Gamma)\nabla_\perp^2\phi + \frac{s}{\Gamma}, \tag{31}$$

$$H_V = V + (1+\Gamma)\psi. \tag{32}$$

We must now rearrange the terms to make each inner Poisson bracket operate on a functional derivative of H and on one of the dynamical fields (N, s, V). This task is facilitated by noting that ϕ and $\nabla_\perp^2\phi$ can only enter through the functional derivatives H_N and H_s, respectively.

Eliminating ϕ from equation (21), we find that we must take $\alpha = K - \Gamma$ in order that equation (21) takes the form

$$\dot{s} = \frac{[s, H_N]}{1+\Gamma}, \tag{33}$$

consistent with equation (10). We next eliminate $\nabla_\perp^2\phi$ in favour of H_s in equation (26). We find

$$\dot{N} = \frac{[N, H_N] + [s, H_s] + [V, H_V]}{1 + \Gamma}. \tag{34}$$

Lastly, we express equation (29) as

$$\dot{V} = \frac{[V, H_N] + [s, H_V]}{1 + \Gamma}. \tag{35}$$

We may now determine the coefficients W_k^{ij} by comparison of equations (33)–(35) with the general form of the Lie–Poisson bracket given in equation (10). The corresponding Poisson bracket is

$$\{F, G\}_2 = (1 + \Gamma)^{-1}\langle N[F_N, G_N] + V([F_N, G_V] + [F_V, G_N]) + s([F_N, G_s] + [F_s, G_N] + [F_V, G_V])\rangle, \tag{36}$$

where the subscript 2 is included to remind us that this is a two-dimensional bracket since we have omitted the longitudinal derivatives $\partial/\partial z$ in the parallel gradient.

To verify that the above bracket satisfies the Jacobi identity we must show that the three matrices $(W^i)_k^j$, $i = 1, 2, 3$ commute. Aside from the common $(1 + \Gamma)^{-1}$ factor, these matrices are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{37}$$

One easily verifies that they commute.

To complete the construction of the Poisson bracket we now extend the bracket to three-dimensional perturbations. The only change this requires is the replacement of $[\psi, \cdot]$ by the full parallel gradient $\nabla_\parallel$ in equations (26) and (28). It is easy to show that this is realized by adding to the two-dimensional bracket the term

$$\{F, G\}_z = \left\langle F_N \frac{\partial G_V}{\partial z} - \frac{\partial F_V}{\partial z} G_N \right\rangle. \tag{38}$$

The complete three-dimensional bracket is thus

$$\{F, G\} = \{F, G\}_2 + \{F, G\}_z, \tag{39}$$

where the component brackets are given in equations (36) and (38). It is not difficult to verify that the Jacobi identity survives the addition of the longitudinal terms [38].

3.3. Casimir invariants

Recall that a Casimir invariant is a functional $C[N, s, V]$ that vanishes when inserted in the Poisson bracket,

$$\{F, C\} = 0$$

for arbitrary F. Using integration by parts and assuming that all the boundary terms vanish we may write this condition

$$\langle F_{\xi^i}[W_k^{ij}\xi^k, C_{\xi^j}]\rangle = 0.$$

In order that this is satisfied for any F the coefficients of each of the functional derivatives F_{ξ^i} must vanish:

$$[N, C_N] + [s, C_s] + [V, C_V] = 0, \tag{40}$$

$$[s, C_N] = 0, \tag{41}$$

$$[V, C_N] + [s, C_V] = 0. \tag{42}$$

The second of these equations implies that $C_N = f(s)$, where f is an arbitrary function of s. Functional integration yields

$$C(N, s, V) = \langle Nf(s) + g(s, V)\rangle.$$

Substituting this in equation (42), we find

$$[s, g_V - Vf'(s)] = 0,$$

or, equivalently,

$$g_V = Vf'(s) + q(s).$$

The remaining equation, equation (40), is automatically satisfied. The complete solution is

$$C(N, s, V) = \left\langle c(s) + Vq(s) + \frac{V^2}{2} f'(s) + Nf(s)\right\rangle. \tag{43}$$

We may interpret the above result by choosing alternately each one of the free functions f, c, q to be a delta function. We obtain, in this way, three families of detailed conservation laws. For $c = \delta(s - \hat{s})$, we find that

$$C^{(c)}(\hat{s}) := \oint \frac{\mathrm{d}\ell}{|\nabla s|} \tag{44}$$

is conserved, where $\mathrm{d}\ell$ is the element of length along the curve formed by the intersection of the surface $s = \hat{s}$ and the plane $z = 0$. This shows that the volume inside any tube of constant s is conserved, as expected for a field convected by the incompressible electric drift. Taking next $q(s) = \delta(s - \hat{s})$, we find that

$$C^{(q)}(\hat{s}) := \oint \frac{\mathrm{d}\ell}{|\nabla s|} V \tag{45}$$

is conserved, corresponding to the conservation of parallel ion momentum in each s-tube. Lastly for $f = \delta(s - \hat{s})$ we find the third family of conserved quantities,

$$C^{(f)}(\hat{s}) := \oint \frac{\mathrm{d}\ell}{|\nabla s|} N - \frac{\mathrm{d}}{\mathrm{d}s} \oint \frac{\mathrm{d}\ell}{|\nabla s|} V^2. \tag{46}$$

This last Casimir generalizes the conservation of potential vorticity given by Ertel's theorem.

It is interesting to compare the Casimirs found above for the KHH model to those given by Krommes and Kolesnikov (K2) [5] for the gyrofluid model [32]. The gyrofluid model has the bracket [5]

$$\{F, G\} = \langle N[F_N, G_N] + T([F_N, G_T] + [F_T, G_N]) + (N + T)[F_T, G_T]\rangle. \tag{47}$$

It has been pointed out by K2 that symmetry of the $(W^i)^j_k$ matrices implies that the quantity $\xi^i\xi^i$ is a Casimir invariant, which they interpret as the enstrophy:

$$Z := \left\langle \frac{N^2 + T^2}{2} \right\rangle. \tag{48}$$

For the KHH model, in contrast, the $(W^i)^j_k$ matrices are asymmetric so that there is no conserved enstrophy. We next show that the enstrophy of K2 is a particular member of a more general family of Casimirs for the gyrofluid equations.

The bracket (47) can be transformed into one where the two fields N and T are decoupled, called a *direct product* form. This is achieved by a linear change of the dependent variables (a general theory of such coordinate changes is presented in [37]). The required transformation is

$$\bar{N} = N + \gamma T, \qquad \bar{T} = N - \gamma^{-1} T, \tag{49}$$

where $\gamma := (1 + \sqrt{5})/2$ is the golden mean. This expression is derived by calculating the coefficients of a general linear transformation to make the resulting bracket fit the direct product form. Letting $F(N, T) = \hat{F}(\hat{N}, \hat{T})$, there follows the chain rule expressions

$$F_N = \hat{F}_{\hat{N}} + \hat{F}_{\hat{T}}, \qquad F_T = \gamma \hat{F}_{\hat{N}} - \gamma^{-1} \hat{F}_{\hat{T}}. \tag{50}$$

Using these in (47) and inserting the inverse of (49) to replace N and T by the linear expressions involving $\hat{N}$ and $\hat{T}$ gives

$$\{\hat{F}, \hat{G}\} = \langle c_1 \hat{N}[\hat{F}_{\hat{N}}, \hat{G}_{\hat{N}}] + c_2 \hat{T}[F_{\hat{T}}, G_{\hat{T}}] \rangle, \tag{51}$$

where $c_1 = 1 + \gamma^2$ and $c_2 = 1 + \gamma^{-2}$ are constants that can be scaled out. The bracket (51) has the Casimirs

$$C^a = \langle a(\hat{N}) \rangle, \qquad C^b = \langle b(\hat{T}) \rangle, \tag{52}$$

where a and b are arbitrary functions. Thus, in terms of the original variables, the Casimirs are

$$C^a = \langle a(N + \gamma T) \rangle, \qquad C^b = \langle b(N - \gamma^{-1} T) \rangle. \tag{53}$$

The Casimir of (48) is clearly a special case, composed of a sum of $C^{(a)}$ and $C^{(b)}$, where the functions a and b are quadratic.

3.4. Coherent structures

The calculation and characterization of propagating CS is an important application of the Hamiltonian formalism. Such CS may arise as a result of the saturation of a primary instability [39], they may be driven by small scale fluctuations [22,40], or they may be formed through inverse cascade in two-dimensional turbulence [40]. The standard approach is to look for solutions of the equations of motion of the form $\xi^j = \xi^j(x, y - ut)$ where u is the propagation velocity of the CS. Equivalently, we may transform to a frame moving with the perturbation and look for equilibria by setting the partial time derivatives to zero in the moving frame. For realistic multi-field models, however, the resulting equations can be formidable, especially in the presence of FLR effects.

In Hamiltonian systems, the task of solving the equilibrium equations can be greatly facilitated by utilizing the Casimirs. To see this, we use the definition of the Casimir functional, equation (9), to write the equations of motion (8) as

$$\dot{\xi}^j = \{\xi^j, F\}, \tag{54}$$

where $F = H - C$ and C is the complete Casimir given in equation (43). It follows that the extrema of the functional F are solutions of the equilibrium equations, $\{\xi^j, F\} = 0$ (see [1,2]). That is, the set of solutions of $\delta F = 0$ or

$$F_{\xi^i} = 0, \qquad i = 1, 2, 3 \tag{55}$$

automatically satisfies

$$\dot{\xi}^j = \{\xi^j, H\} = 0, \qquad j = 1, 2, 3. \tag{56}$$

Comparison of equations (55) and (56) reveals one of the key advantages of the Hamiltonian approach to equilibrium calculations: since the Poisson bracket is a derivation operator, the variational principle (55) amounts to a first integral of the equilibrium equations represented by (56). We emphasize that different choices for the Casimir yield different equilibria. We will see that there is a correspondence between the choice of Casimir and the choice of profile functions for the equilibrium.

We note that the variational functional F can also be used to investigate the stability properties of a given equilibrium [1, 2]. This is based on the observation that since F is

conserved, convexity of F implies that displacements from the equilibrium are bounded [1,2]. Examination of F can thus yield sufficient conditions for stability. It is generally the case, however, that the construction of the Casimir functional is restricted in part to two-dimensional systems (in magnetized fluids, for example, the construction of the Casimir describing magnetic flux conservation depends on the existence of good flux surfaces). Since the most unstable perturbations break the symmetry of the equilibrium, stability investigations using F are of limited value in three-dimensional systems. An alternative approach that sometimes allows this limitation to be side-stepped is to restrict the stability analysis to perturbations that preserve the Casimirs, the so-called dynamically accessible perturbations [1]. This approach is related to the well-known energy principle of magnetohydrodynamics. We will not investigate stability here, and will instead refer the interested reader to [3,41] which discuss the stability of isolated model CSs, and to [42] which shows how instability of a periodic array of convection cells can lead to the generation of zonal flows.

Applying the variational principle given by equation (55) to the problem of finding CS leads to the following three equilibrium equations

$$F_N = (1+\Gamma)\phi + s - f(s) = 0, \tag{57}$$

$$F_s = (1 - f'(s))N + \tfrac{1}{2}(1+\Gamma)\nabla_\perp^2\phi - \hat{c}'(s) - q'(s)V - \tfrac{1}{2}V^2 f''(s) = 0, \tag{58}$$

$$F_V = V + (1+\Gamma)\psi - q(s) - f'(s)V = 0, \tag{59}$$

where $\hat{c}(s) = c(s) - s/\Gamma$. We see that the equilibria are specified by the choice of the three functions $f(s)$, $\hat{c}(s)$ and $q(s)$. This corresponds to the freedom to determine the density, pressure and vorticity profiles of the equilibrium state.

The choice of the three profile functions depends on the problem at hand. In the limit where the time scales are long compared to the characteristic time of diffusive relaxation for the structure of interest, the unknown functions and the profiles they specify are determined by solving the transport equations. These transport equations can be obtained by expressing the solubility conditions for the equilibrium equations in the presence of the dissipation terms [43]. In the opposite limit of very rapid evolution, in contrast, the unknown functions may be obtained from the condition that the Casimirs must be conserved [44]. A third class of solutions corresponds to soliton-like structures, called modons [6, 7, 9]. The modon solutions were discovered by Larichev and Reznik in the context of geophysical fluid dynamics and were subsequently introduced to plasma physics by Meiss and Horton [9]. They are obtained by seeking CS that correspond to disturbances that are localized in space and are such that the relationship between the vorticity and stream functions inside the convection cells is linear. We will describe these solutions in greater detail after completing the analysis of the general problem.

In the general case, we may reduce the equilibrium equations to a single equation for s by using equations (57) and (59) to eliminate the fields ϕ and V from equation (58). There follows

$$f'(f'+1)\nabla_\perp^2 s + \left[\left(f' + \frac{1}{2}\right)(\nabla_\perp s)^2 - \frac{1}{2}\left(\frac{q + (1+\Gamma)\psi}{f'+1}\right)^2\right] f''$$
$$-(f'+1)x - \frac{(f'+1)(f+s)}{1+\Gamma} + \hat{c}' - \left(\frac{q+(1+\Gamma)\psi}{f'+1}\right) q' = 0.$$

This equation may be simplified by changing variables so as to eliminate the squared gradient term. To this end we note that

$$\nabla_\perp^2 \chi(s) = \chi'(s)\nabla_\perp^2 s + \chi''(s)(\nabla_\perp s)^2.$$

We may thus eliminate the square gradient term by replacing s by the field χ determined by

$$\frac{\chi''}{\chi'} = \frac{(f' + 1/2)f''}{f'(f' + 1)}.$$

Integration yields

$$\chi = \int \mathrm{d}s \sqrt{f'(f' + 1)}, \tag{60}$$

where the argument of the square root represents the damping decrement for electron drift waves. The square root is thus well defined whenever the propagation speed of the CS lies outside the band of frequency where drift waves propagate. If the equation (60) can be solved for the field s, this field can be completely eliminated in favour of χ. The resulting equation is

$$\nabla_\perp^2 \chi = Q(\chi, x), \tag{61}$$

where

$$Q(\chi, x) = -\left[\frac{1}{2}\left(\frac{q + (1 + \Gamma)\psi}{f' + 1}\right)^2 f'' + (f' + 1)x + \frac{(f' + 1)(f + s)}{1 + \Gamma}\right.$$
$$\left. -\hat{c}' + \left(\frac{q + (1 + \Gamma)\psi}{f' + 1}\right) q'\right] [f'(f' + 1)]^{-1/2}.$$

In the absence of simplifying assumptions concerning the profiles, analytical solutions of the above equation can only be obtained in two limits. For structures much smaller than an ion gyroradius ($\nabla_\perp \gg 1$), the left-hand side of (61) dominates to lowest order and this results in a linear dependence of χ on x. In this limit the solution reduces to that predicted by linear theory. In the opposite limit where the CS are much larger than the ion gyroradius ($\nabla_\perp \ll 1$), the right-hand side of (61) dominates and all the fields are functions of x to leading order. This leads to the long wavelength class of solutions that are identified with zonal flows.

3.5. Modon solutions

Modons constitute a particular family of solutions of equations (57)–(59) corresponding to localized disturbances. The assumption of localization allows the profile functions f, $\hat{c}$ and q to be almost completely determined. The following description of the modon solutions follows closely that of Meiss and Horton [9].

In the unperturbed reference state, the fields take the form $N = x, \phi = ux, s = (K - \Gamma)x$ and $V = vx - (1 + \Gamma)\psi(x)$. Imposing the condition that the modon fields asymptote to their unperturbed values for $x,\ y \to \infty$ yields

$$f(s) = \left(1 + \frac{1 + \Gamma}{K - \Gamma} u\right) s, \tag{62}$$

$$q(s) = \frac{1 - f'}{K - \Gamma} vs + (1 + \Gamma)\Psi(s) f', \tag{63}$$

$$\hat{c}(s) = \frac{1 - f'}{K - \Gamma} s - \left(\frac{vs}{K - \Gamma} - (1 + \Gamma)\Psi(s)\right) q'(s), \tag{64}$$

where

$$\Psi(s) = \psi\left(\frac{s}{K - \Gamma}\right).$$

Eliminating ϕ and V yields an equation for s

$$\nabla_\perp^2 s - Q(s, x) = 0, \tag{65}$$

where

$$Q(s,x) = \frac{1-u}{K-\Gamma+(1-\Gamma)u} + \left(\frac{\nu(K-\Gamma)}{[K-\Gamma+(1+\Gamma)u]u} + \frac{(K-\Gamma)^2\Psi'^2}{u^2}\right)[\Psi(s)-\psi(x)]. \tag{66}$$

where $Q(s,x)$ is a wave potential. In systems with magnetic shear the wave potential is reversed and induces shear damping. The shear damping results in the decay of the modon. For modons of radius $a \ll \omega_*/k'_\parallel c_\mathrm{s} = \rho_\mathrm{s} L_\mathrm{s}/L_\mathrm{n}$, the damping is asymptotically weak and has been calculated perturbatively by Meiss and Horton [9] for cold ions and by Hong *et al* [14] for ITG modes. Here we neglect the effect of the magnetic shear and consider instead the effect of the shear in the parallel velocity (parametrized by ν).

For a constant magnetic field $\psi(x) = \psi' x$ and ψ' is constant. It follows that Q is linear with respect to s where

$$Q(s,x) = \beta^2 s$$

and

$$\beta^2 = \frac{1-u+\nu\psi'/u}{K-\Gamma+(1-\Gamma)u} + \frac{\psi'^2}{u^2}. \tag{67}$$

In this case the general solution of (65) is a sum of modified Bessel functions. We select the lowest order mode,

$$s = AK_1(\beta r)\cos\theta, \tag{68}$$

where $r^2 = x^2 + y^2$, $\cos\theta = x/r$ and $\beta = \sqrt{-Q}$. Note that β must be real in order that the modons be localized. We thus recover the familiar result that modons can only propagate at phase velocities such that the linear waves are spatially evanescent.

We next note that some of the surfaces of constant s in the solution of equation (68) do not extend to infinity. In the corresponding region the profile functions need not satisfy (62)–(64) and may in fact be chosen freely, subject to continuity requirements. Taking these profile functions to be linear, but with a different slope compared to that in the exterior region,

$$Q_\mathrm{int} = -\left(1+\frac{\gamma^2}{\beta^2}\right),$$

and, demanding continuity of s and ∇s, we obtain the following generalization of the classic modon solution

$$s = \begin{cases} AK_1(\beta r) & r > a, \\ CJ_1(kr)\cos(\theta) + ac\left(1+\dfrac{\beta^2}{\gamma^2}\right)x & r < a, \end{cases} \tag{69}$$

where the coefficients are determined by

$$A = \frac{ac}{K_1(\beta)}, \qquad C = -\left(\frac{\beta}{\gamma}\right)^2 \frac{ac}{J_1(\gamma)}.$$

The constant γ is determined by

$$\frac{K_2(\beta)}{\beta K_1(\beta)} = -\frac{J_2(\gamma)}{\gamma J_1(\gamma)}. \tag{70}$$

These results differ from those found in cold ion models in the dependence of β on the plasma parameters and the propagation speed given in equation (67).

4. Parallel adiabatic model

We now consider the Hamiltonian formulation of the KHH model using the parallel adiabatic response described by equation (6). The pressure equation for this model takes the form of a convection equation identical to equation (21), but with s now given by

$$s = p - \Gamma\tilde{\phi} + (K - \Gamma)x \tag{71}$$

eliminating p from the quasi-neutrality equation yields

$$\frac{\partial \Omega}{\partial t} = [\Omega, \phi] - [\nabla_\perp \phi; \nabla_\perp (s + \Gamma\tilde{\phi})] + [\psi, v], \tag{72}$$

where

$$\Omega = \nabla_\perp^2((\phi + \Gamma\tilde{\phi} + s) - \tilde{\phi}) - x.$$

The new term $\Gamma[\nabla_\perp\phi; \nabla_\perp\tilde{\phi}]$ does not suggest the Lie–Poisson form of section 2.2. It is easy to see that there is no combination of operations acting on the dynamical equations that will provide the necessary term without introducing many more inappropriate terms than it eliminates. This suggests that the combination of the KHH model with the parallel adiabatic response model is non-Hamiltonian. In order to obtain a Hamiltonian model, we henceforth set $\Gamma = 0$, so that $s = p + Kx$ is the ion pressure fluctuation.

4.1. Hamiltonian formulation for $\Gamma = 0$

For the purpose of modelling transport barriers, it is of interest to generalize the KHH model by allowing the background density to have an arbitrary profile [40],

$$n(\boldsymbol{x}, t) = \bar{n}(x) + \tilde{\phi}(\boldsymbol{x}, t).$$

The equations of motion are then

$$\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}t} - \nabla_\perp \cdot \frac{\mathrm{d}}{\mathrm{d}t}\nabla_\perp(\phi + s) + [\phi, \bar{n}] + \nabla_\parallel v = 0, \tag{73}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0, \tag{74}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \nabla_\parallel(\phi + s) = 0, \tag{75}$$

where s differs from p by including the spatial variation of the background pressure.

We next demonstrate the conservation of energy by multiplying the above equations by $\phi + s$, $\tilde{\phi} + \bar{n}$ and v, respectively, integrating over space, and summing. There follows

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0,$$

where

$$H = \tfrac{1}{2}\langle\tilde{\phi}^2 + (\nabla\phi + \nabla s)^2 + 2\tilde{\phi}s + 2s\bar{n} + v^2\rangle. \tag{76}$$

We may eliminate the undesirable $[\nabla\phi; \nabla s]$ term from equation (73) by adding half of the Laplacian of equation (74) to equation (73) as before. There follows

$$\frac{\partial N}{\partial t} = [N, \phi] + \frac{1}{2}[s, \nabla_\perp^2\phi] + [v, \psi], \tag{77}$$

where

$$N = \bar{n} + \tilde{\phi} - \nabla_\perp^2\phi - \tfrac{1}{2}\nabla_\perp^2 s. \tag{78}$$

The functional derivatives of H with respect to the new variables are

$$H_N = \phi + s, \tag{79}$$

$$H_s = N + \tfrac{1}{2}\nabla_\perp^2 \phi, \tag{80}$$

$$H_V = V + (1+\Gamma)\psi. \tag{81}$$

Expressing the arguments in equations (74), (75) and (77) in terms of the above functional derivatives of H, we find that these equations are Hamiltonian and have the same Lie–Poisson bracket as that given in (36). It follows that the Casimirs will have the same dependence on the fields N, s and V as those found in section 3.3, although the field N has a different interpretation, and in particular a different dependence on ϕ, in each of the two models.

The Casimirs for the parallel adiabatic response model can be used to simplify the equilibrium equations as demonstrated in the previous section for the model with the full adiabatic response. The dependence of N on nonlocal information entering through $\tilde{\phi} = \phi - \bar{\phi}$, however, causes the conventional methods for solving the equilibrium equations to fail.

4.2. Nonexistence of equilibria for $\Gamma \neq 0$

In view of the considerable simplification of the equilibrium problem that the Hamiltonian formulation purchases, it is natural to inquire as to the existence and nature of CS in non-Hamiltonian systems. We may investigate this issue by using our result that for the model with a parallel adiabatic response, it is necessary to have $\Gamma = 0$ in order that the dynamics be Hamiltonian. Assuming the existence of a nontrivial CS solution for $\Gamma = 0$, we attempt to solve the equilibrium equations perturbatively for Γ small and positive.

For simplicity we take $v = 0$ and consider the solutions of the equations obtained by setting the time derivatives to zero in the equations of motion, equations (21) and (26):

$$[\phi, s] = 0, \tag{82}$$

$$[\phi, N] - \tfrac{1}{2}[s, \nabla^2\phi] - \Gamma[\nabla\bar{\phi}, \nabla\phi] = 0. \tag{83}$$

The first of these, equation (82), is easily integrated:

$$s = f(\phi),$$

where f is an arbitrary function. For equation (83), we look for a solution of the form

$$\phi = \phi_0 + \Gamma\phi_1, \qquad s = s_0 + \Gamma s_1, \tag{84}$$

where (ϕ_0, s_0) is the solution for $\Gamma = 0$ and (ϕ_1, s_1) indicate small corrections of order Γ.

Considering first the lowest order terms in equation (83), we find that the reference solution satisfies

$$N_0 = h(\phi_0) + \tfrac{1}{2} f'(\phi_0)\nabla^2\phi_0, \tag{85}$$

where h is an arbitrary function. The first-order equation is

$$[\phi_0, N_1] + [\phi_1, N_0] - \tfrac{1}{2}[s_0, \nabla^2\phi_1] - \tfrac{1}{2}[s_1, \nabla^2\phi_0] = [\nabla\bar{\phi}_0; \nabla\phi_0].$$

We note that the operator $[\phi, \cdot] = \boldsymbol{v}_E \cdot \nabla$ has the interpretation of the derivative along the streamlines. Using the lowest order solution, we may regroup all the terms that are expressible as a derivation along the streamlines,

$$\boldsymbol{v}_E \cdot \nabla \left(N_1 - \frac{[f'(\phi_0) + 2h'(\phi_0) + f''(\phi_0)\nabla^2\phi_0]\phi_1}{2} \right) = \frac{\partial^2\bar{\phi}_0}{\partial x^2}\frac{\partial^2\phi_0}{\partial x\,\partial y}.$$

In order that this equation have a solution it is necessary that the integral of the right-hand side along any closed streamline vanish. It is easily seen that this is generally not the case for two-dimensional CS. In particular, near an extremum of the potential (corresponding to the centre of a convection cell) we find

$$\oint \frac{\mathrm{d}\ell}{|\nabla\phi|}\partial_x^2\bar{\phi}_0\partial_{xy}\phi_0 = \left[\frac{2\pi\,\partial_x^2\bar{\phi}_0\partial_{xy}\phi_0}{\sqrt{\partial_x^2\bar{\phi}_0\partial_y^2\bar{\phi}_0 - (\partial_{xy}\phi_0)^2}}\right]_{\boldsymbol{x}=\boldsymbol{x}_{\max}}.$$

This vanishes only if the major axes of the streamlines are aligned with the coordinate axes. We conclude that two-dimensional CS generally do not exist for $\Gamma \neq 0$. We observe that it is the same term, $\Gamma[\nabla\bar{\phi};\nabla\phi]$, that is responsible both for the non-Hamiltonian nature of the parallel adiabatic model with $\Gamma \neq 0$ and for the non-integrability of the equilibrium equations. This suggests that the Hamiltonian property is a necessary condition for the integrability of the equilibrium equations. If so, this would imply that non-Hamiltonian models will lead to qualitatively different predictions when used to describe properties that result from CS, such as intermittency.

5. Discussion

We have constructed a Poisson bracket that provides a Hamiltonian formulation for two models of ITG dynamics, the first, with finite adiabatic compression index Γ and a fully adiabatic electron response, and the second, with $\Gamma = 0$ and a electron response obeying Boltzmann's law only along the field lines. The Hamiltonian formulation shows that both these models possess three families of detailed conservation laws in addition to globally conserved quantities such as the energy and centre of mass. They do not, however, conserve enstrophy.

Recalling that the behaviour of soliton solutions of the KdV equation are a consequence of its complete integrability, we conjecture that the Casimirs play a similar role for CS interactions in ITG turbulence. Specifically, the infinite set of constraints imposed by Casimirs are responsible for the approximate preservation of modon identity during interactions. We note, however, that Casimir conservation is fragile, since Casimirs are subject to mixing similar to that which affects the distribution function during Landau damping (note that the conservation of the distribution function in Vlasov dynamics is itself linked to the existence of a Casimir). Mixing can occur, in particular, as the result of Kelvin–Helmholtz instability during the interaction between two otherwise stable dipole vortices.

We have provided arguments that for non-vanishing specific heat, $\Gamma \neq 0$, the model with an adiabatic electron response along the field lines is non-Hamiltonian. This has two consequences with clear physical import. First, the set of conserved quantities is limited to the energy and the internal energy $p_{\mathrm{tot}}/n_{\mathrm{tot}}^{\Gamma}$ (here we use the subscript 'tot' to denote the sum of the background and perturbed quantities). The laws of detailed conservation of potential vorticity and parallel momentum no longer apply. Second, there are no coherent convection cells solutions. One thus expects markedly different behaviour from the non-Hamiltonian model. In view of the importance to turbulent dynamics of CS in general and of zonal flows in particular, we conclude that the Hamiltonian nature of the underlying equations will play a determining role in turbulent transport.

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