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Renormalization for breakup of invariant tori

A. Apte*, A. Wurm, P.J. Morrison

Department of Physics and Institute for Fusion Studies, University of Texas, Austin, TX 78712, USA

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Abstract

We present renormalization group operators for the breakup of invariant tori with winding numbers that are quadratic irrationals. We find the simple fixed points of these operators and interpret the map pairs with critical invariant tori as critical fixed points. Coordinate transformations on the space of maps relate these fixed points, and also induce conjugacies between the corresponding operators.

Keywords: Renormalization; Invariant tori; Breakup

1. Introduction

Area-preserving maps have been used extensively as low-dimensional models for physical systems, e.g., magnetic field lines in tokamaks, experimental devices designed for confining hot plasmas to produce fusion energy. Maps that are not too far away from being integrable, the map version of the existence of action-angle variables, exhibit phase space plots with several different types of orbits: *periodic orbits*, which are discrete sets of points that are invariant under map iterations, *invariant tori*, which are one-dimensional curves that traverse phase space, and *chaotic orbits*, which randomly sample regions of phase space.

Of particular interest from a physics viewpoint are the invariant tori, because they are *transport barriers*, i.e., orbits with initial conditions on one side of an invariant torus will remain on that side under an arbitrary number of map iterations. Mathematical KAM theory guarantees the existence of a dense set of invariant tori for small perturbations away from integrability, but large perturbations destroy most or all of these tori. Here we are interested in the behavior of a certain kind of map under perturbations that push the system far away from the integrable limit.

* Corresponding author. Tel.: +1 512 471 4588; fax: +1 512 471 6715.
E-mail address: apte@physics.utexas.edu (A. Apte).

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Numerical studies [1,2] of invariant tori that are pushed to the point of break-up, so-called *critical invariant tori*, have found universal scaling behavior (i.e., the same behavior for different maps and different winding numbers) of the torus and its phase space vicinity (see further below for details). This motivated the introduction of renormalization techniques, analogous to those used in the study of phase transitions in condensed matter physics, for the study of area-preserving maps [1,3,4]. In contrast to KAM theory, the renormalization group approach addresses the problem of destruction of an invariant torus with a specific winding number under strong perturbation. The detailed implementation is based on *Greene's residue criterion* for which we refer the reader to, e.g., [6].

An interesting and still open problem is the question of universality in the break-up of invariant tori: What classes of area-preserving maps show the same scaling behavior of tori at break-up for a specific winding number? For a specific map, what classes of winding numbers of invariant tori show the same scaling behavior at break-up?

In renormalization group language these questions are rephrased in terms of fixed points of renormalization group operators (RGOs) defined on the space of area-preserving maps. There are two kinds of fixed points: *simple* fixed points and *critical* fixed points. In the case of area-preserving maps one arrives at the following interpretation: A *simple* fixed point is an integrable map, and its basin of attraction contains all the maps for which the invariant torus exists. A *critical* fixed point is a map for which the invariant torus under consideration is at criticality, i.e., at the point of break-up. All the maps in its basin of attraction exhibit the same universal behavior at break-up.

The most studied renormalization group operator (RGO) is that for the winding number $1/\gamma$, where $\gamma = (1+\sqrt{5})/2$ is the golden mean [3,4]. The purpose of the present work is to construct RGOs for other winding numbers, to describe coordinate transformations on the space of maps that relate fixed points of RGOs, and to describe conjugacy relations between RGOs. The RGO for $1/\gamma^2$ was previously constructed [5,6] using the method presented here. The RGOs we construct are similar to those of [7], which have been applied extensively to study renormalization in Hamiltonian flows. In the remainder of this section we review the RGO for $1/\gamma$, as both a means of introduction and to set our notation.

An area-preserving map M of a cylinder, $M : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, can be represented by its lift M', which is a map of $\mathbb{R}^2$ to itself that commutes with $R(x, y) = (x - 1, y)$, $M'R = RM'$. Commutation implies that n iterations of M' followed by an appropriate number of applications of R is equivalent to n iterations of M. Thus, instead of studying M, we can equivalently study the pair of commuting maps (R, M') of $\mathbb{R}^2$ to itself. The RGOs are defined on the space of such commuting map pairs. The introduction of commuting map pairs is convenient because it removes complications associated with spatial rescaling and periodicity (of the cylinder) [4,8].

Let (U, T) be a pair of commuting maps. The *orbit* of a point $\mathbf{z} = (x, y) \in \mathbb{R}^2$ is the set of points $\{U^m T^n \mathbf{z}\}$ for $m, n \in \mathbb{Z}$. An orbit has the *winding number* ω if for some sequence $\{p_i/q_i\}$ of rationals with $q_i \to \infty$ and $p_i/q_i \to \omega$, the sequence $(\pi U^{p_i} T^{q_i} \mathbf{z})/q_i \to 0$. (Here $\pi\mathbf{z} = x$ is the projection from $\mathbb{R}^2$ to $\mathbb{R}$.) An orbit is *periodic* of type (p, q) if $U^p T^q \mathbf{z} = \mathbf{z}$ and (p, q) are the smallest such integers. Such periodic orbits have winding number p/q. An *invariant torus* is a curve that extends from $x = -\infty$ to $+\infty$ that is invariant under both U and T. Generically, invariant tori have irrational winding numbers.

Invariant tori with irrational winding numbers ω are studied numerically by finding the periodic orbits with winding numbers that approximate ω. To facilitate calculation it is important to find the sequence of rationals that converges the fastest to ω. The convergents obtained by successive truncations of the continued fraction expansion of ω provide such a sequence. Also, as we shall see, such sequences of convergents are intimately connected to renormalization.

The convergents obtained by truncating the continued fraction expansion of $1/\gamma = [0, 1, 1, 1, \ldots] =: [0, \overline{1}]$ (where we use standard continued fraction notation as in, e.g. [9]) are F_i/F_{i+1} where F_i are the Fibonacci numbers defined by

$$F_0 = 0, \qquad F_1 = 1, \quad \text{and} \quad F_i = F_{i-1} + F_{i-2}. \tag{1}$$

Numerical studies of the break-up of invariant tori (see, e.g., [2,4,6] and [16]) have discovered two important features of the phase space near a critical torus:

1. The torus itself is invariant under rescaling of phase space, i.e., if (x, y) lies on the torus, then so does $(\alpha x, \beta y)$ for the specific values of $\alpha, \beta \in \mathbb{R}$ that depend on the universality class under consideration.
2. The lower order periodic orbits that approximate the torus coincide with the higher order ones after rescaling phase space by exactly the same factors α and β.

Thus, in renormalization group language, the map at criticality is on the stable manifold of a critical fixed point of the RGO.

These observations motivate the explicit construction of the RGOs in Section 2. To give a simple example, the RGO for tori of winding number $1/\gamma$, $\mathcal{R}_{1/\gamma}$, maps a pair (U, T) of commuting maps onto a new pair (U', T') as follows:

$$U' := B_{1/\gamma} \circ T \circ B_{1/\gamma}^{-1}, \qquad T' := B_{1/\gamma} \circ U \circ T \circ B_{1/\gamma}^{-1}, \tag{2}$$

where '$\circ$' denotes composition. For convenience this is written compactly as

$$\begin{pmatrix} U' \\ T' \end{pmatrix} = \mathcal{R}_{1/\gamma} \begin{pmatrix} U \\ T \end{pmatrix} := B_{1/\gamma} \begin{pmatrix} T \\ UT \end{pmatrix} B_{1/\gamma}^{-1}. \tag{3}$$

This operator has two parts:

- *Space renormalization*, which is achieved by the coordinate change $B_{1/\gamma}$ on $\mathbb{R}^2$, which rescales the phase space coordinates (x, y) by $(x, y) \to B_{1/\gamma}(x, y) = (\alpha x, \beta y)$.
- *Time renormalization*, which is achieved by the specific combination of U and T that maps a periodic orbit with period equal to a convergent of $1/\gamma$ onto another one with period equal to a lower order convergent. To be specific, an orbit of (U, T) with winding number F_i/F_{i+1} is also an orbit of $(U', T') = (T, UT)$ with winding number F_{i-1}/F_i:

$$(U')^{F_{i-1}}(T')^{F_i} = (T)^{F_{i-1}}(UT)^{F_i} = U^{F_i}T^{F_{i-1}+F_i} = U^{F_i}T^{F_{i+1}}. \tag{4}$$

  The time renormalization also keeps the torus with winding number $1/\gamma$ invariant, i.e., if $\mathbf{z}$ lies on the $1/\gamma$-torus of (U, T), then it also lies on the $1/\gamma$-torus of (U', T').

This paper is organized as follows. In Section 2, we present the construction of RGOs for winding numbers that are quadratic irrationals. The main idea is to construct an operator such that the time renormalization part maps into each other those periodic orbits that have winding numbers that are convergents of desired irrationals. We find the coordinate transformations, on the space of maps, that relate the fixed points of these operators. These transformations, presented in Section 3, also induce conjugacy relations between the different RGOs. In Section 4.1, we find the simple (integrable) fixed points of the RGOs. In Section 4.2, the critical maps for various winding numbers are interpreted as critical fixed points of these RGO operators. Finally, in Section 5 we summarize and indicate directions for further studies.

2. RGO for quadratic irrational winding numbers

Any quadratic irrational ω has an eventually periodic continued fraction expansion [9]. Here we choose $0 < \omega < 1$. Then

$$\omega = [0, q_1, q_2, \ldots, q_l, \overline{p_1, p_2, \ldots, p_k}], \tag{5}$$

where $p_i, q_i \in \mathbb{N}$. For such winding numbers, the RGO is defined as follows:

$$\begin{pmatrix} U' \\ T' \end{pmatrix} = \mathcal{R}_\omega \begin{pmatrix} U \\ T \end{pmatrix} := B_\omega \begin{pmatrix} U^r T^t \\ U^s T^u \end{pmatrix} {B_\omega}^{-1}, \tag{6}$$

where B_ω is a coordinate change on phase space. Here $r, s, t, u \in \mathbb{Z}$, with $|ru - st| = 1$, are the elements of a matrix that relates approximants of ω. In particular, convergents obtained by truncating the continued fraction expansion of (5) are related to each other by a matrix as follows:

$$\begin{pmatrix} m_{i+k} \\ n_{i+k} \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} m_i \\ n_i \end{pmatrix} \quad \text{for } i > l + 1. \tag{7}$$

The existence of such a matrix was noted in [7]. A method for constructing it in the present context is given in Appendix A.

It can be verified easily that $\mathcal{R}_\omega$ maps a m_i/n_i-periodic orbit for $i > l + 1$ to a m_{i+k}/n_{i+k}-periodic orbit. Let $\mathbf{z}$ be a m_{i+k}/n_{i+k}-periodic orbit of (U, T), i.e.,

$$U^{m_{i+k}} T^{n_{i+k}} \mathbf{z} = \mathbf{z}. \tag{8}$$

Then

$$U'^{m_i} T'^{n_i} B_\omega \mathbf{z} = (B_\omega U^r T^t {B_\omega}^{-1})^{m_i} (B_\omega U^s T^u {B_\omega}^{-1})^{n_i} B_\omega \mathbf{z} = B_\omega U^{rm_i + sn_i} T^{tm_i + um_i} {B_\omega}^{-1} B_\omega \mathbf{z} = B_\omega \mathbf{z}.$$

Thus, $B_\omega \mathbf{z}$ is a m_i/n_i-periodic orbit of (U', T'). We can also verify that the torus with winding number ω is invariant (up to a coordinate change B_ω) under this operator, i.e., if $\mathbf{z}$ lies on the ω-torus of (U, T) then $B_\omega \mathbf{z}$ lies on the ω-torus of (U', T').

The operator $\mathcal{R}_\omega$ is of the same form as that given for maps associated with Hamiltonian flows in Proposition 1.1 of Koch [7]. We also note that $\mathcal{R}_\omega$ defined here is slightly different from the one originally defined by MacKay [3]. MacKay introduced an operator N_m defined by

$$\begin{pmatrix} U' \\ T' \end{pmatrix} = N_m \begin{pmatrix} U \\ T \end{pmatrix} := B \begin{pmatrix} T \\ T^m U \end{pmatrix} B^{-1}. \tag{9}$$

with the property that an orbit of (x, y) has the winding number $\omega = [m, m_1, m_2, \ldots]$ under (U, T) iff $B(x, y)$ has winding number $\omega' = [m_1, m_2, \ldots]$ under (U', T'). As a consequence (U, T) has an invariant curve of winding number ω iff (U', T') has an invariant curve of winding number ω', where ω and ω' are related by $\omega = m + 1/\omega'$.

The key difference between MacKay's operator and the one used here is that in his case the winding number of the orbit under consideration changes after each application of N_m, while the operator $\mathcal{R}_\omega$ preserves the winding number. As a consequence, in each step of the renormalization a different N_m is used, while here the *same* $\mathcal{R}_\omega$ is applied every time. For numbers of the form $\omega = [0, \overline{p}]$, the two operators coincide. On the other hand, the use of N_m is not restricted to quadratic irrational winding numbers. The operator N_m has been further studied in, e.g. [10–12] and references therein.

For later use, we also write down the RGOs for winding numbers of the form

$$\Omega_l = [0, q_1, q_2, \ldots, q_l, \overline{p}]. \tag{10}$$

Let us denote

$$\omega_p = [\overline{p}] = \frac{p + \sqrt{p^2 + 4}}{2}. \tag{11}$$

Then,

$$\Omega_l = \frac{a + b\omega_p}{c + d\omega_p}, \tag{12}$$

where $a, b, c, d \in \mathbb{Z}$ are given by

$$\tfrac{a}{c} = [0, q_1, q_2, \ldots, q_{l-1}], \qquad \tfrac{b}{d} = [0, q_1, q_2, \ldots, q_l] \tag{13}$$

and $\delta := ad - bc = (-1)^l$ [9]. The relation between the successive approximants of Ω_l is given by

$$\begin{pmatrix} m_{i+1} \\ n_{i+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} m_i \\ n_i \end{pmatrix} \quad \text{for } i > l + 1, \tag{14}$$

where

$$A = (1/\delta)(bd - ac - bcp), \qquad B = (1/\delta)(a^2 - b^2 + abp), \qquad C = (1/\delta)(d^2 - c^2 - cdp),$$
$$D = (1/\delta)(ac - bd + adp), \tag{15}$$

and $AD - BC = -1$ (see Appendix A). The RGO for this winding number is given by

$$\mathcal{R}_{\Omega_l} \begin{pmatrix} U \\ T \end{pmatrix} = B_{\Omega_l} \begin{pmatrix} U^A T^C \\ U^B T^D \end{pmatrix} B_{\Omega_l}^{-1}. \tag{16}$$

Thus, for the winding number $\Omega_0 = 1/\omega_p = [0, \overline{p}]$,

$$\mathcal{R}_{1/\omega_p} \begin{pmatrix} U \\ T \end{pmatrix} = B_{1/\omega_p} \begin{pmatrix} T \\ UT^p \end{pmatrix} B_{1/\omega_p}^{-1}. \tag{17}$$

3. Conjugacy relations between the RGOs

In this section, we find the coordinate changes, on the space of commuting map pairs, that induce conjugacies between the RGOs of different winding numbers that have the same periodic 'tail' of the continued fraction expansion. For simplicity, we will work with numbers of the form given in Eqs. (10)–(12). This can be generalized easily to numbers with arbitrary periodic tails if we replace ω_p by $\omega_0 = [\overline{p_k, p_{k-1}, \ldots, p_1}]$.

We will show that for Ω_l and ω_p as defined in Eqs. (10) and (11), the following conjugacy holds between the operators $\mathcal{R}_{1/\omega_p}$ and $\mathcal{R}_{\Omega_l}$:

$$\mathcal{R}_{\Omega_l} = \mathcal{C}^{-1} \circ \mathcal{R}_{1/\omega_p} \circ \mathcal{C}, \tag{18}$$

where

$$\mathcal{C} \begin{pmatrix} U \\ T \end{pmatrix} = S \begin{pmatrix} U^a T^c \\ U^b T^d \end{pmatrix} S^{-1}, \qquad \mathcal{C}^{-1} \begin{pmatrix} U \\ T \end{pmatrix} = S^{-1} \begin{pmatrix} U^{d/\delta} T^{-c/\delta} \\ U^{-b/\delta} T^{a/\delta} \end{pmatrix} S, \tag{19}$$

and S is a particular phase space coordinate change.

To prove the above statement, note that the RHS of Eq. (18) acting on (U, T) yields

$$\begin{aligned} \mathcal{C}^{-1} \circ \mathcal{R}_{1/\omega_p} \circ \mathcal{C} \begin{pmatrix} U \\ T \end{pmatrix} &= \mathcal{C}^{-1} \circ \mathcal{R}_{1/\omega_p} \begin{pmatrix} SU^a T^c S^{-1} \\ SU^b T^d S^{-1} \end{pmatrix} \\ &= (S^{-1} B_{1/\omega_p} S) \begin{pmatrix} U^{(bd-ac-bcp)/\delta} T^{(d^2-c^2-cdp)/\delta} \\ U^{(a^2-b^2+abp)/\delta} T^{(ac-bd+adp)/\delta} \end{pmatrix} (S^{-1} B_{1/\omega_p} S)^{-1}, \end{aligned}$$

while the LHS of Eq. (18) yields:

$$\mathcal{R}_{\Omega_l}\begin{pmatrix} U \\ T \end{pmatrix} = B_{\Omega_l}\begin{pmatrix} U^A T^C \\ U^B T^D \end{pmatrix} B_{\Omega_l}^{-1}. \tag{20}$$

Thus, if $S^{-1}B_{1/\omega_p}S = B_{\Omega_l}$, then Eq. (18) holds.

We have presented the operators for winding numbers $0 < \omega < 1$. The operator for $\omega > 1$ can be shown to be conjugate to that for $0 < 1/\omega < 1$ by

$$\mathcal{R}_\omega = \mathcal{I}^{-1} \circ \mathcal{R}_{1/\omega} \circ \mathcal{I}, \tag{21}$$

where

$$\mathcal{I}\begin{pmatrix} U \\ T \end{pmatrix} = S_I \begin{pmatrix} T \\ U \end{pmatrix} S_I^{-1}, \qquad \mathcal{I}^{-1}\begin{pmatrix} U \\ T \end{pmatrix} = S_I^{-1}\begin{pmatrix} T \\ U \end{pmatrix} S_I. \tag{22}$$

If $\omega < 0$, then we can write $\omega = -p + \tilde{\omega}$, where $0 < \tilde{\omega} < 1$ and $p \in \mathbb{N}$. The operator for ω is conjugate to that for $\tilde{\omega}$ by

$$\mathcal{R}_\omega = \mathcal{N}^{-1} \circ \mathcal{R}_{\tilde{\omega}} \circ \mathcal{N}, \tag{23}$$

where

$$\mathcal{N}\begin{pmatrix} U \\ T \end{pmatrix} = S_N \begin{pmatrix} U \\ TU^{-p} \end{pmatrix} S_N^{-1}, \qquad \mathcal{N}^{-1}\begin{pmatrix} U \\ T \end{pmatrix} = S_N^{-1}\begin{pmatrix} U \\ TU^{p} \end{pmatrix} S_N. \tag{24}$$

In Eqs. (22) and (24), S_I and S_N are phase space coordinate changes.

The important consequence of Eq. (18) is that the fixed points $(U_{\Omega_l}, T_{\Omega_l})$ and $(U_{1/\omega_p}, T_{1/\omega_p})$ of $\mathcal{R}_{\Omega_l}$ and $\mathcal{R}_{1/\omega_p}$, respectively, are related by the coordinate change $\mathcal{C}$ on the space of maps, i.e.,

$$\mathcal{C}\begin{pmatrix} U_{\Omega_l} \\ T_{\Omega_l} \end{pmatrix} = \begin{pmatrix} U_{1/\omega_p} \\ T_{1/\omega_p} \end{pmatrix}. \tag{25}$$

Recall that there are two kinds of fixed points of these RGOs: *simple* and *critical*. A *simple* fixed point is an integrable map, and its basin of attraction contains all the maps for which the invariant torus exists. A *critical* fixed point is a map for which the invariant torus under consideration is at criticality. All the maps in its basin of attraction exhibit the same universal behavior at critical breakup. The relation (25) also implies that the maps corresponding to the fixed points $(U_{\Omega_l}T_{\Omega_l})$ and $(U_{1/\omega_p}T_{1/\omega_p})$ belong to the same universality class, and exhibit the same universal critical behavior. In general, since the maps with critical invariant tori of winding number with the same continued fraction expansion tail are related to each other by coordinate transforms on the space of maps, they belong to the same universality class.

4. Fixed points and cycles of RGOs for specific maps

In this section, we present some specific cycles of the RGOs presented above. The form of the simple cycles we find is the same as the simple fixed point and the simple two-cycle of $\mathcal{R}_{1/\gamma}$ found in [3] and [13], respectively. This specific form is motivated by the integrable ($k = 0$) limit of the following widely studied maps:

$$y' = y - \tfrac{k}{2\pi}\sin(2\pi x), \qquad x' = x + y', \tag{26}$$

and

$$y' = y - k\sin(2\pi x), \qquad x' = x + a(1 - y'^2), \tag{27}$$

where $(x, y) \in \mathbb{T} \times \mathbb{R}$ and $a, k \in \mathbb{R}$ are parameters. We refer the reader to [3,5,14] for a detailed discussion of these maps. Here we only recall that a map M is said to satisfy the *twist condition* if

$$\frac{\partial x'}{\partial y} \neq 0, \quad \text{where}\,(x', y') = M(x, y). \tag{28}$$

The former is the *standard map*, the most common example of a twist map while the latter is the *standard non-twist map* [15].

In Section 4.1, we find the simple fixed points [with linear twist as in Eq. (26)] and two-cycles [with quadratic twist as in Eq. (27)] of $\mathcal{R}_{\Omega_l}$. In Section 4.2, we find the critical nontwist 12-cycles for the RGOs for a few noble winding numbers and find the critical conjugacy relations between them.

4.1. Simple fixed points and two-cycles

In this section, we present the integrable fixed points and two-cycles of $\mathcal{R}_{\Omega_l}$ (Eq. (16)). Motivated by the maps of Eqs. (26) and (27), we find the twist fixed point $(U_{\Omega_l}, T_{\Omega_l})$ and nontwist two-cycle $(U_{\Omega_l\pm}, T_{\Omega_l\pm})$ of the following form:

$$\begin{aligned} &U_{\Omega_l}(x, y) = (x + ey + f, y) && \text{and } T_{\Omega_l}(x, y) = (x + gy + h, y) \\ &U_{\Omega_l\pm}(x, y) = (x + e_\pm y^2 + f_\pm, y) && \text{and } T_{\Omega_l\pm}(x, y) = (x + g_\pm y^2 + h_\pm, y) \end{aligned} \tag{29}$$

by solving the following equations

$$B_{\Omega_l} U^A_{\Omega_l} T^C_{\Omega_l} B^{-1}_{\Omega_l}(x, y) = U_{\Omega_l}(x, y), \qquad B_{\Omega_l} U^B_{\Omega_l} T^D_{\Omega_l} B^{-1}_{\Omega_l}(x, y) = T_{\Omega_l}(x, y), \tag{30}$$

and

$$B_{\Omega_l} U^A_{\Omega_l\pm} T^C_{\Omega_l\pm} B^{-1}_{\Omega_l}(x, y) = U_{\Omega_l\mp}(x, y), \qquad B_{\Omega_l} U^B_{\Omega_l\pm} T^D_{\Omega_l\pm} B^{-1}_{\Omega_l}(x, y) = T_{\Omega_l\mp}(x, y), \tag{31}$$

where $B_{\Omega_l}(x, y) = (\alpha_{\Omega_l} x, \beta_{\Omega_l} y)$ is the space renormalization. We also tried to find the nontwist fixed point and twist two-cycle. They are not presented here because the winding number for the nontwist fixed point is found to be constant (as a function of y), which we refer to as an integrable *non-twist* map and the twist two-cycle is simply a multiple of the twist fixed point.

Let us introduce some notation for representing the solutions. We will use Eqs. (10)–(12), and (15). The quadratic equation satisfied by $s := -1/\Omega_l$ can be shown to be the following:

$$B^2 s^2 + (2D - p)s + (D^2 - Dp - 1) = 0. \tag{32}$$

Let us denote by $\tilde{s}$ the other solution of this equation. The following relation can be established:

$$s = \frac{-C\omega_p}{1 + A\omega_p} \quad \text{and} \quad \tilde{s} = \frac{C}{\omega_p - A}. \tag{33}$$

Solving Eqs. (30), we get four fixed points: two of these are trivial because the winding number is a constant; one of them has $\tilde{s}$ as the winding number of the $y = 0$ orbit. Thus, the only one which has the winding number Ω_l at $y = 0$ is

$$U_{\Omega_l}(x, y) = (x + ey + f, y), \qquad T_{\Omega_l}(x, y) = \left(x + \tfrac{e}{\tilde{s}} y - \Omega_l f, y\right), \qquad B_{\Omega_l}(x, y) = (-\omega_p x, -\omega_p^2 y). \tag{34}$$

This is really a two-parameter family of fixed points. Using the definition $U^p_{\Omega_l} T^q_{\Omega_l}(x, y) = (x, y)$ of a periodic orbit of winding number p/q, we obtain the winding number as a function of y:

$$w(y) = -\frac{\frac{e}{\tilde{s}} y - \Omega_l f}{ey + f} = \Omega_l \left(1 - \frac{e}{\tilde{s}\Omega_l f} y\right) \left(1 + \frac{e}{f} y\right)^{-1} = \Omega_l \left[1 - \left(1 + \frac{1}{\Omega_l \tilde{s}}\right) \frac{e}{f} y + \mathcal{O}\left(y^2\right)\right].$$

We see that the invariant torus at $y = 0$ has the winding number Ω_l, and the winding number changes linearly with y. Thus, this is the twist fixed point of $\mathcal{R}_{\Omega_l}$.

Solving Eqs. (31) results in 16 two-cycles: half of these are trivial because they have constant winding numbers; four of them have $\tilde{s}$ as the winding number of the $y = 0$ orbit; for two of them, $U_{\Omega_l +}$ (respectively $T_{\Omega_l +}$) differs from $U_{\Omega_l -}$ (respectively $T_{\Omega_l -}$) only in the constant terms $f_\pm$ (respectively $h_\pm$). Thus, the only non-trivial two-cycles with the winding number Ω_l at $y = 0$ are

$$U_{\Omega_l \pm}(x, y) = (x \pm ey^2 + f, y), \qquad T_{\Omega_l \pm}(x, y) = (x \pm \tfrac{e}{\tilde{s}} y^2 - \Omega_l f, y), \qquad B_{\Omega_l}(x, y) = (-\omega_p x, \pm\omega_p y). \tag{35}$$

This again is a two-parameter family of two-cycles. As above, we obtain the winding number

$$w_\pm(y) = -\frac{\pm\frac{e}{\tilde{s}} y^2 - \Omega_l f}{\pm ey^2 + f} = \Omega_l \left[1 \mp \left(1 + \frac{1}{\Omega_l \tilde{s}}\right) \frac{e}{f} y^2 + \mathcal{O}\left(y^4\right)\right]. \tag{36}$$

We see that $y = 0$ is the torus with winding number Ω_l and $w_\pm(y)$ has an extremum at $y = 0$. Thus, this is the non-twist two-cycle of $\mathcal{R}_{\Omega_l}$.

Replacing Ω_l by $(1/\omega_p)$, i.e., setting $a = 1$, $b = 0$, $c = 0$, and $d = 1$ gives the integrable twist fixed point and non-twist two-cycle of $\mathcal{R}_{1/\omega_p}$. These are given by

$$U_{1/\omega_p}(x, y) = (x + ey + f, y), \qquad T_{1/\omega_p}(x, y) = (x + e\omega_p y - \tfrac{f}{\omega_p}, y), \qquad B_{1/\omega_p}(x, y) = (-\omega_p x, -\omega_p^2 y), \tag{37}$$

and

$$\begin{aligned} &U_{1/\omega_p \pm}(x, y) = (x \pm ey^2 + f, y), \qquad T_{1/\omega_p \pm}(x, y) = (x \pm e\omega_p y - \tfrac{f}{\omega_p}, y), \\ &B_{1/\omega_p}(x, y) = (-\omega_p x, \pm\omega_p y). \end{aligned} \tag{38}$$

Here, we explicitly verify that the fixed point (two-cycle) of $\mathcal{R}_{1/\omega_p}$ is related to the fixed point (two-cycle) of $\mathcal{R}_{\Omega_l}$ by the coordinate change $\mathcal{C}$ of Eq. (19). Thus, we show that

$$SU^a_{\Omega_l} T^c_{\Omega_l} S^{-1}(x, y) = U_{1/\omega_p}(x, y) \quad \text{and} \quad SU^b_{\Omega_l} T^d_{\Omega_l} S^{-1}(x, y) = T_{1/\omega_p}(x, y). \tag{39}$$

Evaluating the LHS of each of these equations, we get

$$\begin{aligned} SU^a_{\Omega_l} T^c_{\Omega_l} S^{-1}(x, y) &= SU^a_{\Omega_l} T^c_{\Omega_l}(x/\mu, y/\nu) = S\left(\tfrac{x}{\mu} + c\left(\tfrac{e}{\tilde{s}}\tfrac{y}{\nu} - \Omega_l f\right) + a\left(e\tfrac{y}{\nu} + f\right), \tfrac{y}{\nu}\right) \\ &= \left(x + \left(\tfrac{c}{\tilde{s}} + a\right)\tfrac{e\mu}{\nu} y + f\left(a - c\Omega_l\right)\mu\right) =: \left(x + e'y + f'\right), \end{aligned}$$

and

$$SU^b_{\Omega_l} T^d_{\Omega_l} S^{-1}(x, y) = \left(x + \left(\tfrac{d}{\tilde{s}} + b\right)\tfrac{e\mu}{\nu} y + f\left(b - d\Omega_l\right)\mu\right) =: \left(x + g'y + h'\right).$$

For Eqs. (39) to hold, we need that $h'/f' = -1/\omega_p$ and $g'/e' = \omega_p$. Using the relations

$$\tilde{s} = \frac{d - c\omega_p}{-b + a\omega_p}, \quad \text{and} \quad \Omega_l = \frac{a + b\omega_p}{c + d\omega_p}, \tag{40}$$

we obtain

$$\frac{h'}{f'} = \frac{b - d\Omega_l}{a - c\Omega_l} = \frac{b - d\frac{a+b\omega_p}{c+d\omega_p}}{a - c\frac{a+b\omega_p}{c+d\omega_p}} = -\frac{1}{\omega_p}, \tag{41}$$

and

$$\frac{g'}{e'} = \frac{d + b\tilde{s}}{c + a\tilde{s}} = \frac{d + b\frac{d-c\omega_p}{-b+a\omega_p}}{c + a\frac{d-c\omega_p}{-b+a\omega_p}} = \omega_p. \tag{42}$$

An almost identical calculation shows that the two-cycles of $\mathcal{R}_{\Omega_l}$ and $\mathcal{R}_{1/\omega_p}$ are also related by $\mathcal{C}$ of Eq. (19). Note that the phase space scalings B_{Ω_l} and B_{1/ω_p} for the fixed points of $\mathcal{R}_{\Omega_l}$ and $\mathcal{R}_{1/\omega_p}$, respectively, are identical while the scaling given by S to relate them to each other is undetermined. We will see later that for the critical fixed points of different winding numbers, the phase space scalings (the Bs) are the same whereas the scalings that relate them to each other (the Ss) have different specific values.

4.2. Critical fixed points and their relation to each other

We have studied the breakup of shearless invariant tori in the standard non-twist map for the following winding numbers:

$$\omega^{(1)} = [0, 1, 1, 1, \ldots] = 1/\gamma,$$
$$\omega^{(2)} = [0, 2, 1, 1, 1, \ldots] = 1/\gamma^2,$$
$$\omega^{(3)} = [0, 2, 2, 1, 1, 1, \ldots] = \gamma^2/(1 + 2\gamma^2).$$

Here we will not present the details of the numerical results but refer the reader to [6,13,16]. It is observed that the convergence pattern (the six-cycle) of Greene's residues [2], spatial scalings, and the eigenvalues of the RGOs at these fixed points are all the same within numerical accuracy for these three cases. This is expected because they are all nobles and hence the fixed points (or higher order cycles; in this case, 12-cycles) must be related to each other by coordinate transformations on the space of maps. Here we present evidence that such is indeed the case.

We discuss in detail the relation between the $\omega^{(1)}$-invariant torus and the $\omega^{(2)}$-invariant torus. The relation between the $\omega^{(1)}$-invariant torus and the $\omega^{(3)}$-invariant torus is the same (except that the numerical values for the scalings are different). We also note that the operators for $\omega^{(2)}$ and $\omega^{(3)}$ are

$$\mathcal{R}_{\omega^{(2)}}\begin{pmatrix} U \\ T \end{pmatrix} = B_{\omega^{(2)}}\begin{pmatrix} U^{-1}T^{-1} \\ UT^2 \end{pmatrix} B^{-1}_{\omega^{(2)}}, \qquad \mathcal{R}_{\omega^{(3)}}\begin{pmatrix} U \\ T \end{pmatrix} = B_{\omega^{(3)}}\begin{pmatrix} U^4T^{-1} \\ U^{11}T^{-3} \end{pmatrix} B^{-1}_{\omega^{(2)}}. \tag{43}$$

$\mathcal{R}_{\omega^{(2)}}$ was first presented in [5] and [6].

It is observed that the positions $(x_n^{(a)}, y_n^{(a)})$ of the nth convergent of $\omega^{(a)}$ for $a = 1$–3 scale in the following manner:

$$x_n^{(a)} = A_n^{(a)}\alpha^{-2n}, \quad \text{and} \quad y_n^{(a)} = B_n^{(a)}\beta^{-2n}, \tag{44}$$

where $A_n^{(a)}$ and $B_n^{(a)}$ are period-six functions of n. Note that n is a dummy index, i.e., it can be shifted by an integer without changing the above statement. Also, though the result is true for all the symmetry lines, we will be using the

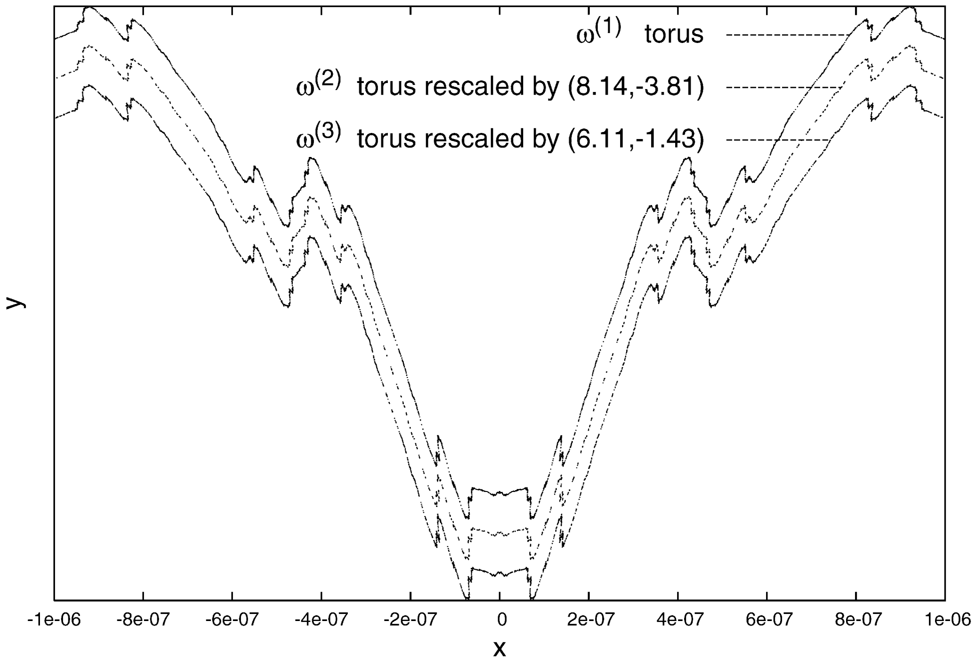

Fig. 1. Mapping of critical invariant tori of different winding numbers onto each other. Note that all three tori lie on top of each other but have been shifted in y for clarity. (Hence the scale on y-axis has not been shown.)

s_3 symmetry line in the following discussion (see [13,16] for details about defining symmetry lines and symmetry line coordinates). It follows from Eq. (44) that

$$A_n := \frac{x_n^{(1)}}{x_n^{(2)}} = \frac{A_n^{(1)}}{A_n^{(2)}} \tag{45}$$

is a period-six function of n. (Similarly for y and B_n.) But we observe numerically that A_n is in fact a constant, A, independent of n if, in Eq. (45), we choose the x coordinates of orbits which have the same value of residue. This is shown in Table 1. We see that $A \approx 8.14$ and $B \approx -3.81$.

We also observe that the critical $\omega^{(2)}$-torus maps, locally around the s_3 symmetry line, onto the critical $\omega^{(1)}$-torus under exactly the rescalings (A, B) of the phase space. Numerical investigation shows that the periodic orbits, not just their positions along the symmetry lines, also map locally onto each other. This is depicted in Figs. 1 and 2.

Table 1
Scaling of periodic orbits approximating tori with winding numbers $\omega^{(1)}$ and $\omega^{(2)}$

q_1	x_1	y_1	q_2	x_2	y_2	$A = \frac{x_1}{x_2}$	$B = \frac{y_1}{y_2}$
75025	2.012943e−5	3.932996e−6	17711	1.638693e−4	−1.497908e−5	8.140783	−3.808567
196418	4.783568e−6	2.466482e−6	46368	3.894257e−5	−9.393067e−6	8.140903	−3.808286
514229	1.932103e−6	1.874730e−6	121393	1.572945e−5	−7.139952e−6	8.141104	−3.808524
1346269	7.241581e−7	2.572146e−7	317811	5.895735e−6	−9.718763e−7	8.141502	−3.778464
3524578	2.600689e−7	9.238172e−8	832040	2.117285e−6	−3.450858e−7	8.141246	−3.735433
9227465	1.187355e−7	3.614236e−8	2178309	9.667462e−7	−1.319071e−7	8.142016	−3.649653
24157817	6.258871e−8	9.617356e−9	5702887	5.100913e−7	−3.147145e−8	8.149892	−3.272360

(x_a, y_a) are coordinates of the orbit with period q_a at criticality for breakup of $\omega^{(a)}$ torus ($a = 1$ and 2).

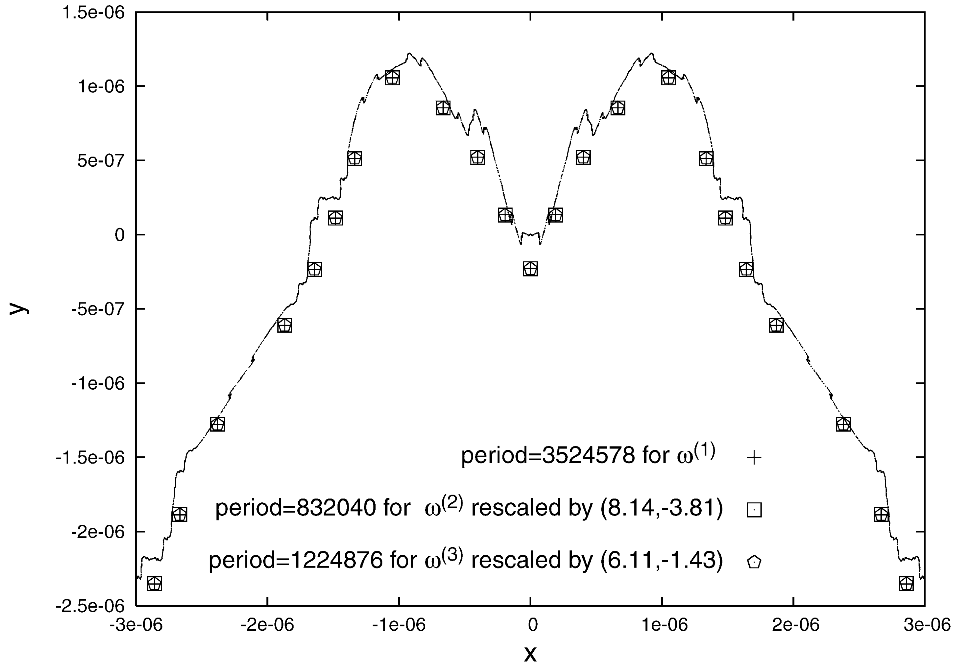

Fig. 2. Mapping of periodic orbits approximating the three invariant tori. The invariant torus of $\omega^{(1)}$ is shown for reference.

This is interpreted as follows: the map $M^{(1)}$ with critical $\omega^{(1)}$-torus is related to the map $M^{(2)}$ with critical $\omega^{(2)}$-torus by the coordinate transformation $\mathcal{C}$ (Eq. (19)) on the space of maps: $M^{(2)} = \mathcal{C}M^{(1)}$. The phase space coordinate change S involved in $\mathcal{C}$ is diagonal: $S(x, y) = (Ax, By) = (8.14x, -3.81y)$. We note that the coordinate transform $\mathcal{C}$ provides not only a phase space rescaling but also *a mapping of periodic orbits of different winding numbers onto each other*. This is necessary because we are relating periodic orbits with winding numbers equal to convergents of $\omega^{(1)}$ to those with winding numbers equal to convergents of $\omega^{(2)}$.

5. Conclusions and future work

We have presented renormalization group operators for studying the breakup of invariant tori with any quadratic irrational winding number. The simple (integrable) cycles were calculated. We also presented coordinate transformations on the space of maps inducing conjugacies between different RGOs. The evidence for extending this picture to the critical fixed points is presented for the case of the standard non-twist map.

These results prompt a re-examination of the breakup of tori in twist maps. For example, we note that the residue behavior for quadratic irrationals for the standard map (see [17] and references therein) can be interpreted as follows: if a quadratic irrational has periodicity k in its continued fraction expansion [as in Eq. (5)], the residues of convergents of the continued fraction expansion converge to a k-cycle at criticality. This is interpreted as a critical fixed point of $\mathcal{R}_\omega$ and not a k-cycle of $\mathcal{R}_\omega$ because the RGOs for such numbers relate the ith convergent to the $(i+k)$th convergent [see the discussion following Eq. (6)]. Studying twist maps is also useful because the numerical results for them can be obtained much faster, giving us an opportunity to study a multitude of winding numbers. Some steps in this direction have been taken.

Another direction for investigating these RGOs is to study sequences of rationals other than the convergents of continued fraction expansions such that consecutive elements are still related by the RGO. Preliminary results show that the residue convergence patterns for such sequences are similar to those of continued fraction convergents, but the limiting residue values are different [18]. These results will be reported elsewhere. Finally, quoting John Greene, whose pioneering numerical studies and deep insights peppered throughout his work were a constant motivation for this investigation, "much work remains to be done" [8].

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Appendix A

Here, we derive the relation (15) and present a method to calculate the integer matrix given by r, s, t, u in (7).

1. To get (15), let g_i/h_i be the convergents of $1/\omega_p$. Then,

$$\begin{pmatrix} g_{i+1} \\ h_{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix} \begin{pmatrix} g_i \\ h_i \end{pmatrix}. \tag{A.1}$$

The convergent m_{i+1}/n_{i+1} of Ω_l, given by

$$\frac{m_{i+1}}{n_{i+1}} = [0, q_1, \dots, q_l, p, \dots, p], \tag{A.2}$$

where p appears $(i - l + 1)$ times, is related to g_{i-l+1}/h_{i-l+1} by

$$\frac{m_{i+1}}{n_{i+1}} = \frac{a g_{i-l+1} + b h_{i-l+1}}{c g_{i-l+1} + d h_{i-l+1}}, \tag{A.3}$$

where a–d are the same as in Eq. (13) [9]. From Eqs. (A.1)–(A.3) it follows that

$$\begin{pmatrix} m_{i+1} \\ n_{i+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} m_i \\ n_i \end{pmatrix} \tag{A.4}$$

which leads to (15) by multiplying the matrices above. Note that Eq. (A.3) is valid only for $i > l - 1$.

2. To get r–u in Eq. (7), let us denote by G_i/H_i the convergents of $[0, \overline{p_1, p_2, \dots, p_k}]$. Then,

$$\begin{pmatrix} G_{i+k} \\ H_{i+k} \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} G_i \\ H_i \end{pmatrix}, \tag{A.5}$$

where $a', b', c', d' \in \mathbb{Z}$ are given by

$$\frac{a'}{c'} = [0, p_1, p_2 \dots, p_{k-1}], \qquad \frac{b'}{d'} = [0, p_1, p_2, \dots, p_k], \tag{A.6}$$

and $a'd' - b'c' = (-1)^k$. The convergents m_{i+k}/n_{i+k} of ω are related to G_{i-l+k}/H_{i-l+k} by

$$\frac{m_{i+k}}{n_{i+k}} = \frac{aG_{i-l+k} + bH_{i-l+k}}{cG_{i-l+k} + dH_{i-l+k}}, \tag{A.7}$$

where a–d are the same as those in Eq. (13) [9]. From Eqs. (A.5)–(A.7) it follows that

$$\begin{pmatrix} m_{i+k} \\ n_{i+k} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} m_i \\ n_i \end{pmatrix}. \tag{A.8}$$

Multiplying the matrices and comparing with (7) results in explicit, though not very illuminating, expressions for r–u. We note that

$$ru - ts = \det \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = (-1)^k. \tag{A.9}$$

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