Hamiltonian and action principle formulations of plasma physics[a)]

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(Received 23 November 2004; accepted 12 January 2005; published online 20 April 2005)

Hamiltonian and action principle formulations of the basic equations of plasma physics are reviewed. Various types of Lagrangian and Poisson bracket formulations for kinetic and fluid theories are discussed, and it is described how such formulations can be used to derive and approximate physical models. Additional uses are also described. Two applications are treated in greater detail: an algorithm based on Dirac brackets for the calculation of V-states of contour dynamics and the calculation of fluctuation spectra of Vlasov theory and shear flow dynamics by methods of statistical mechanics. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1882353]

I. INTRODUCTION

Since antiquity there have been many attempts to explain nature by means of various kinds of minimization principles.[1] For example, in classic times Hero of Alexandria (ca. 75 AD) is reputed to have attributed the path taken by reflected light to be that which minimizes the distance traveled, while in the Renaissance Fermat produced his principle of least time to offer an explanation for the path of light rays in optics. Many great thinkers incorporated such principles into their philosophies, natural and otherwise, but it is said that this line of thought reached an apex with Hamilton who developed (ca. 1830) a mathematical formalism for both light and particles. It is Hamilton's principle, the prototype action principle, that is most prevalent in modern expositions of mechanics, and below (Sec. II A) we take this as the starting point for our exposition of the Hamiltonian and action principle formulations of equations that describe plasma physics. Of particular interest here are infinite-dimensional systems or field theories such as fluid theories and the Vlasov equation.

There are many reasons thinkers have been attracted to the formulation of physical laws in terms of minimization or extremal principles, which in contemporary language amounts to the idea that laws of nature should come from setting a derivative of some quantity, the action, to zero. Early researchers espoused the teleological idea that "nature does nothing in vain," and their thought processes were usually imbued with theological opinions. Also aesthetics has always played a role in physics, and the beauty and simplicity of action principles motivated many. It is fair to say that action principles provide a framework for 20th century physics: the most successful models of physics, Maxwell's equations, Einstein's equations for general relativity, Schrödinger's equation, Yang–Mills and other theories of particle physics, etc. all have action principle and associated Hamiltonian formulations. The same is true for the most important models of plasma physics, and we will describe some of these in this paper. The subject is vast, and so we can only touch on some aspects. More extended treatments are obtained, for example, for fluid models in Ref. 2 and Vlasov theory in Ref. 3, and many references therein.

There are many practical reasons for Hamiltonian and action principle (HAP) formulations. One is that they provide a convenient setting for constructing theories, in particular, a setting for building in symmetries that one believes a physical system should possess. For example, one can systematically build in symmetries such as the Poincaré group (Lorentz invariance etc.) into an action principle, and this is much simpler than attempting to do this on the equations of motion level. Conversely, if one already has equations of motion, derived perhaps by the approximation of some known general model, then if one constructs an action principle one can use Noether's theorem to obtain constants of motion, constants that may not be evident. In very general terms this is the only way, e.g., that energy can be unambiguously defined.

Another attractive feature of HAP formulations is that there are certain properties that are the same for all such formulations. For example, if one knows that a system is Hamiltonian, then one is assured by Liouville's theorem that phase space volume is conserved. This has ramifications which are basic for statistical mechanics (cf. Sec. IV), and it also places a firm constraint on the types of eigenvalues one can obtain from any linear theory calculation, e.g., it precludes the possibility that all eigenvalues represent decay. (This is described more fully in Sec. VI of Ref. 2.) Thus, in a very large sense, demonstrating HAP form taps one into an enormous lore that has been developed over centuries. Conversely, it can occur that if one shows something new about a particular system with HAP form, then one has demonstrated something for a large class of problems. The nature of the destruction of invariant tori, both near to integrability as described by Kolmogorov–Arnold–Moser (KAM) theory (e.g., Ref. 4) and far from integrability as described by Greene[5] is a good example of this.

HAP formulations provide a setting for consistent approximations. If one abides by the idea that a HAP system ought to be approximated by another HAP system, then there

[a)]Paper CI1A 1, Bull. Am. Phys. Soc. **49**, 57 (2004). In memory of Dieter Pfirsch, colleague and friend, 1927–2004.
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are algorithmic ways of achieving this. In action principle formulations one has at one's disposal the direct method of the calculus of variations (e.g., chap. 4 of Ref. 6). With this method one inserts trial functions with undetermined parameters into the action, integrates out information, varies with respect to the parameters, and thereby obtains a reduced description. (See, e.g., Ref. 7 where this is done to describe coherent structures in a simple drift-wave model.) Another method is to time average the action[8,9] in order to remove fast time scales. In Hamiltonian systems many different kinds of perturbation theories based on the Hamiltonian form have been developed. A classic description of perturbation theory based on Hamilton–Jacobi theory is contained in Ref. 10, while more recent perturbation theory using Lie transforms is discussed in Refs. 11 and 12. A nice feature of all these methods is that one is basically dealing with approximation of a single function that defines the action principle or the Hamiltonian system, rather than all the functions that define the equations of motion.

A particularly nice feature of Hamiltonian systems is that they have an associated means for obtaining sufficient and sometimes necessary conditions for stability. An example of this is Lagrange's theorem of mechanics, which is the essence of the magnetohydrodynamics (MHD) energy principle. Other energy principles such as that of Gardner[13] and others for Vlasov and other theories, which sometimes go by the name of the energy-Casimir method, are based upon Dirichlet's theorem (and extensions thereof) of mechanics (e.g., Sec. VI of Ref. 2). (Examples abound. For one presented at this meeting see Ref. 14 where stability for Hall-MHD is treated.)

In a somewhat less explicit way, HAP form motivates calculations by providing a mindset. When one is aware of the HAP lore, then one knows of certain things to try. In this way HAP form serves as a beacon for illuminating paths one might take. An example of this is provided by the ongoing quest to understand how much of the lore of finite degree-of-freedom Hamiltonian systems carries over to infinite degree-of-freedom Hamiltonian systems. Because of soliton theory[15] we know infinite degree-of-freedom Hamiltonian systems can be integrable, and thus we are led naturally to the question of whether there is an infinite-dimensional version of the KAM theorem. The HAP form can even provide insight into dissipative systems, such as quasilinear theory, by suggesting a natural set of coordinates.[16]

Clearly the line of discussion of this Introduction could continue, but we bring it to a close by mentioning two more areas where research based on HAP form has grown. The first concerns numerical methods that preserve structure. Examples of these are symplectic (e.g., Ref. 17) and conservative (e.g., Ref. 18) integrators, differential equation iteration schemes that preserve phase space volume (and appropriate subvolumes), and constants of motion, respectively. In Sec. III we will discuss a numerical method based on infinite-dimensional Hamiltonian structure for computing V-states which are solutions of Euler's fluid equation. The second area where Hamiltonian methods are of basic importance is statistical mechanics. In Sec. IV we generalize ideas from finite degree-of-freedom statistical mechanics to obtain the fluctuation spectra of Vlasov and fluid theory. In Sec. II we review some of the HAP formulations of plasma physics and en route to this end describe some of the tools that are needed to understand it. Even this is an impossible task, so we try to hit the high points and suggest some references. There are many references given, but even so the list should not be taken to be complete. Rather, our goal has been to provide a gateway into this mode of research.

II. HAMILTONIAN AND ACTION PRINCIPLE FORMULATIONS

In this section we begin with some introductory ideas from classical mechanics, and then we describe some more complicated formalisms for infinite-dimensional plasma models.

A. Hamilton's principle and Hamilton's equations

We start by describing Hamilton's principle in more detail, and the associated *procedure* for writing down the equations of motion of a mechanics problem. In this procedure one begins by determining the configuration space, i.e., determining the coordinates, angles or displacements, necessary to describe the system's configuration. We denote these generalized coordinates by $q^i(t)$, where $i=1,2,\ldots,N$ and N is the number of degrees of freedom of the system. The second step is to construct functions of these coordinates that represent the kinetic and potential energies, and thus obtain the Lagrangian, $L:=T-V$, which for good reason has been in the past referred to as the kinetic potential. Given the Lagrangian, one is then able to write down the action functional as follows:

$$S[q]=\int_{t_0}^{t_1} L(q,\dot{q},t)dt. \tag{1}$$

The action is an example of a functional, which given a path $q(t)$ returns a number upon substitution into L and integration over time. Different paths usually give different numbers. In Hamilton's principle one fixes the beginning and end points of the path, $q(t_0)$ and $q(t_1)$, and searches over such a space of paths for minimum or extremal values. Originally it was believed that physical paths were those for which the action obtains a minimum but nowadays we are content with the path being extremal. (This issue was not settled until the 20th century by Morse.) Extremal means that the functional derivative of the action vanishes, $\delta S[q]/\delta q^i=0$, and this results in Lagrange's equations of motion

$$\frac{\partial L}{\partial q^i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}=0.$$

(Since functional differentiation plays a big role in HAP formulations we review this in Sec. II B, a review that can be skipped by the cogniscenti.)

From Hamilton's principle one obtains Hamilton's equations by first constructing the canonical momenta, $p_i:=\partial L/\partial \dot{q}^i$, and then obtaining the Hamiltonian as, $H(q,p)=p_i\dot{q}^i-L$, where now the repeated indices are summed. This procedure is generally known as the Legendre transforma-

tion, and it is a means of converting N Lagrange's equations, which are usually of second order, into the $2N$ first-order equations of Hamilton

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}. \tag{2}$$

The Legendre transformation only works for Lagrangians that possess a convexity property, but when this fails Dirac's constraint theory (see, e.g., Ref. 19) can be used. We will not pursue this further here, however, we will use a product of this theory in Sec. III where we discuss the V-state calculations.

If we introduce the phase space coordinates, $z=(q,p)$, then Hamilton's equations can be compactly written as follows:

$$\dot{z}^i = J_c^{ij}\frac{\partial H}{\partial z^j}, \tag{3a}$$

where

$$(J_c^{ij}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}, \tag{3b}$$

and now the label $i=1,2,\ldots,2N$. This form can be taken as a gateway into geometric mechanics where the (symplectic two-form)=(cosymplectic form)$^{-1}$, i.e., $\omega_{ij}^c J_c^{jk} = \delta_i^k$. We will also not pursue this further, but refer the interested reader to Ref. 4.

B. Functional differentiation

In order to understand action principles and infinite-dimensional Hamiltonian systems it is necessary to understand the notion of functional derivative, which as we will show is a kind of gradient. In elementary analysis we learn that a function is a mathematical object that returns a number in its range given an element of its domain, or in mathematical notation for a function f we would write $f:\mathbb{R}^n\to\mathbb{R}$, where the domain in this case is a region in n-dimensional space. In common notation we would write this as $f(z_1,z_2,\ldots,z_n)$ and, for example, if we were interested in the temperature in a region in three space we might write this as $T(x,y,z)$.

Alternatively, a functional is a function where the domain is a set of functions and the range is again composed of numbers, i.e., it is a function of functions. For a functional evaluated on a function u we write $F[u]$, or in mathematical notation we write $F:\mathcal{B}\to\mathbb{R}$, where $\mathcal{B}$ denotes a function space, which could (but need not) be equal to Hilbert space. Examples of functionals are the action $S[q]$ of Hamilton's principle of (1) and the energy of the Vlasov–Poisson system $H[f]=m\int fv^2dxdv/2+\int E^2dx/8\pi$, where the first term is the integral over the phase space kinetic energy density and the second is the electrostatic energy, which upon making use of Poisson's equation, can be seen to be an expression quadratic in the phase space density f. A general form for functionals in one dimension is given by $F[u]=\int\mathcal{F}(x,u,u_x,u_{xx},\ldots)dx$, where $\mathcal{F}$ is an arbitrary function of a function u and its derivatives u_x, u_{xx}, etc. We also allow it to be an explicit function of x.

On a formal level evaluation of functional derivatives is no more difficult than evaluation of ordinary derivatives. We will present an algorithm for doing this along with a demonstration of what is meant by a functional derivative being a gradient in function space. We do this by comparing the first variation of functions and functionals.

If we make a small change in the point of the domain of a function, say $f(z_1,z_2,\ldots z_n)$, then there will be an induced change in value of the range. This induced change is given by the first variation

$$\delta f(z;\delta z) = \sum_{i=1}^{n}\frac{\partial f(z)}{\partial z_i}\delta z_i = :\nabla f\cdot\delta z. \tag{4}$$

The analogous first variation for a functional $F[u]$ is given by

$$\delta F[u;\delta u] = \frac{d}{d\epsilon}F[u+\epsilon\delta u]_{\epsilon=0} = \int_{x_0}^{x_1}\delta u\frac{\delta F}{\delta u(x)}dx = :\left\langle\frac{\delta F}{\delta u},\delta u\right\rangle, \tag{5}$$

where $\delta u(x)$ is a function that represents the change in the point of the domain. The operations indicated by the second term of (5) constitute a way of linearizing the functional in δu. Taking the derivative with respect to the parameter ϵ and evaluating the result at $\epsilon=0$, plucks out the linear term in δu.

Let us compare these two expressions. In (4) we see that there is a scalar product as indicated by the sum on the label i or the "$\cdot$" between the variation of the domain variable δz and the gradient ∇f. In (5) the role of the scalar product is played by the integration over the variable x, denoted by $\langle,\rangle$. Note, the variable x is a continuous version of the label i. The domain variable analogous to δz is of course δu, and the only remaining term to be compared is the gradient. The ith component of the gradient of (4), $\partial f(z)/\partial z_i$, is analogous to $\delta F[u]/\delta u(x)$. One last comment is needed. In the last expression of (5) it is most important that δu stands alone. When one varies a functional with an integrand that involves derivatives one obtains an expression involving δu_x etc. All derivatives or other operators acting on δu must be removed by taking their adjoint. Thus, to take a functional derivative one must do two things: vary and isolate. After isolating, the integral is stripped away and the functional derivative is extracted.

If you can convince yourself that the functional derivative of the Korteweg de Vries Hamiltonian, $H[u]=\int_{-\pi}^{\pi}(u^3/6-u_x^2/2)dx$, is given by $\delta H/\delta u=u^2/2+u_{xx}$ and that $\delta u(x)/\delta u(x')=\delta(x-x')$, then you understand all you need to know about functional derivatives for this paper. In the course of taking $\delta H/\delta u$ it is necessary to integrate by parts and eliminate the integrated (surface) terms. When the domain is chosen properly these vanish, but in some problems boundary terms can be important.

C. The parent action principle of plasma physics

To the extent that plasma physics is governed by classical physics, the action principle we present in this section

embodies *all* of plasma physics. This parent action principle is that which describes the dynamics of N charged, relativistic particles, with charges e_i and masses m_i, coupled to Maxwell's equations, an action principle that only includes the electromagnetic fundamental force.

The dynamical variables are the particle positions, $q_i(t)$, where $i=1,2,\ldots,N$, the electrostatic potential $\phi(x,t)$, and the vector potential $A(x,t)$. We could divide up our particles into electrons and ions, or any number of species, and easily require that the total charge sum to zero, but we will not do this. The action is given by

$$
\begin{aligned}
&S[q,\phi,A] \\
&\quad = -\sum_{i=1}^{N}\int_{t_0}^{t_1} dt m_i c^2 \sqrt{1-\frac{\dot{q}_i^2}{c^2}} \\
&\qquad -\int_{t_0}^{t_1} dt \sum_{i=1}^{N} e_i \int dx \left[\phi(x,t)-\frac{\dot{q}_i}{c}\cdot A(x,t)\right]\delta(x-q_i(t)) \\
&\qquad +\frac{1}{8\pi}\int_{t_0}^{t_1} dt \int dx [E^2(x,t)-B^2(x,t)]. \qquad (6)
\end{aligned}
$$

The first term of (6) is the kinetic part of the action; in the nonrelativistic limit this term becomes the usual expression for the kinetic energy. Note that this term involves only the particle variables. The second term accounts for the coupling between the particles and the fields ϕ and A. Note that it involves both integration of space and summation over i. In the last term E and B are to be viewed as shorthands for their expressions in terms of the potentials: $E=-\nabla\phi-(\partial A/\partial t)/c$ and $B=\nabla\times A$. Observe that the last term only involves the field variables.

When we take the derivative $\delta S/\delta q^i(t)$ it is clear that only the first two terms contribute. The first term is straightforward, but the second deserves comment. When we vary and isolate this term, the process of varying presents no difficulty, but when we isolate we have to bear in mind that the functional derivative notation tells us which variables are involved in the isolation and eventual stripping away of the integral or sum. When we write $\delta S/\delta q^i(t)$ with the argument of q_i displayed it is to signify that the isolation should take place with respect to the integration over t and the summation on i. We leave it to the reader to show that setting $\delta S/\delta q^i(t)=0$ yields the ith component of the relativistic version of Newton's second law with the Lorentz force given by the derivative of the coupling term. Similarly, the derivatives $\delta S/\delta\phi$ and $\delta S/\delta A(x,t)$ only involve the coupling terms and the last term of (6). This gives the two Maxwell equations that involve the sources, the two not determined by the introduction of the potentials. The coupling terms provide the sources while the last term provides the fields. Isolation in this case leads to stripping away the x and t integrals.

Given the claim that the action of (6) contains all of plasma physics, it is natural to wonder how dissipation enters into various plasma models. In general dissipation is not "fundamental," but arises from some sort of approximation or modeling. For example, the Lenard–Balescu collision operator is entirely electromagnetic in origin, and a route can be traced from the equations produced by (6) to it. If one is interested in including more physics, such as quantum mechanical or general relativistic phenomena, then action principles, for example that couple to Einstein's equation, are also available.

Alas, given the great level of generality contained in (6) one might conclude that we are all done. However, we have the standard problem that is typically used to motivate statistical mechanics: because N is so large, even if we could solve the equations of motion the wealth of information would be too great to be of much use. Thus we seek simplifications that usually follow from approximations. Approximations that remove irrelevant information can introduce or remove constraints, either explicitly or implicitly, and this makes the discipline interesting.

D. Approximations, reductions, and mutilations: Action extration

There exist many kinds of approximations. We reserve the word *reduction* for a procedure that is in some sense exact. Reductions may result from a variable change that exactly separates out information and allows one to solve a complicated problem in stages. A standard example is the system that describes a free rigid body in mechanics (e.g. Ref. 20). A complete specification of the motion would require the solution of the six coupled differential equations for the Euler angles and their conjugate momenta. However, if one writes the system in terms of the appropriate angular velocities one obtains the three Euler's equations for the rigid body which is a closed system. After solving these three one can then solve three more equations for the Euler angles. The same kind of reduction occurs for the infinite-dimensional equations of fluid mechanics, where the Lagrangian variable description is reduced to the Eulerian variable description. The solution of the Lagrangian variable description contains more information than the Eulerian, namely, the positions of the fluid elements.

Another kind of reduction can occur when one has a class of initial conditions that is invariant under the dynamics. An example of this is point vortex dynamics associated with the two-dimensional Euler fluid equation for vorticity evolution. If one assumes that an initial vorticity distribution is located on delta-function spikes, point vortices, at some locations, then the spikes are maintained under the dynamics. As is well-known (e.g., Ref. 21), ordinary differential equations govern the positions of the point vortices. Upon solving these differential equations one obtains an exact solution of Euler's equation, albeit one with a singular initial condition. The family of fluid closures, based on the water bag reduction of the Vlasov equation, which were found in Ref. 22 constitute another example. A third example is contour dynamics, which we treat in Sec. III.

Because of their exactness, both types of reductions described above inherit a Hamiltonian structure from their parent models. This is not the case for many systems that are obtained by approximation or other kinds of modeling. Sometimes one is explicitly interested in deriving dissipative models by coarse graining or other means, and the physical

source of the dissipation is clear. These inherently dissipative models should not be expected to have HAP form, and it is generally clear which terms cause the dissipation, for they are associated with transport coefficients, viscosities, etc. However, when these terms are eliminated the model is expected to have HAP form. Any process that results in models for which this is not the case we refer to as *mutilation*. Such non-Hamiltonian models could be useful, but one needs to worry about the introduction of unphysical dissipation that could result in the violation of energy or other conservation laws. Unlike physical dissipation there may be no phenomenological coefficients that govern the amount of dissipation and in general this kind of dissipation would not be expected to reflect any physical process.

In the remainder of this section we discuss a means, which is essentially the direct or trial function method of the calculus of variations described above, for extracting from (6), actions that govern models that contain less physics. These actions yield finite degree-of-freedom systems upon variation.

The equations that describe magnetic field lines have been known for a long time to possess HAP form (e.g., Refs. 23 and 24). This form is easily extracted from (6) by specifying the B, setting $\phi=0$, writing the results in dimensionless form, and then letting the gyroradius go to zero. This yields the Cary–Littlejohn action, $S_{CL}[r]=\int A(r)\cdot dr$, which was given in Ref. 25. Variation of the above gives the equations for the field lines of $B=\nabla\times A$.

In a similar manner the HAP form for particle orbits can be gotten by specifying both ϕ and B, inserting them into (6), and integrating by parts. This yields the standard non-self-consistent action for particles in given electric and magnetic fields. A variant of this can be used to obtain models that are partly self-consistent. This proceeds by specifying the spatial dependence of ϕ and B but with time dependent parameters. For example, one can specify ϕ to correspond to a single sine wave with a time dependent amplitude and phase. The single-wave model, originally obtained for describing beam-plasma instability (e.g., Refs. 26–31), can be derived in this manner if one separates the particles into beam and background components and expands.[32] One can also derive by this means multiwave models (e.g., Ref. 33) that include electromagnetic waves as well (cf. Refs. 34 and 35).

The procedures described above, in which the field variables were restricted, produced systems that, unlike the system described by the parent action of (6), have a finite number of degrees of freedom. Consequently, as alluded to in the Introduction, we have at our disposal the large body of knowledge of finite degree-of freedom Hamiltonian systems. This knowledge has produced a vocabulary that is commonplace in plasma physics. For example, we can ask whether such a finite system is *integrable*, and we can construct various kinds of *Poincaré sections* to address this question. If the system is nearly integrable and the frequencies of periodic motion are disparate, we can use the theory of *adiabatic invariants* as made rigorous in plasma physics.[36,37] If the system has few degrees of freedom, then in Poincaré sections we can look for *invariant tori* (good surfaces) and investigate their destruction by the *island overlap* or considerably more precisely by *Greene's residue criterion*. We have an understanding of the behavior of a *chaotic* phase space in the vicinity of tori at criticality, because of notions of *renormalization* and *universality*. And, if we are lucky enough to identify an integrable system, because of KAM-type theorems we know how the system will behave under small perturbations. Our understanding of phase space also allows us to construct maps, such as the *standard* (Chirikov–Taylor) or *standard nontwist maps*, which, e.g., describe qualitatively the nature of magnetic field lines in tokamaks with monotonic and nonmonotonic q proflles. We know, on a very basic level, that *shearless tori* are resilient. We know that none of these systems can have an attractor, strange or otherwise, and that the eigenvalues obtained in any linear analysis must conform to the constraints imposed by the Hamiltonian form: for every growing mode there must be a damped mode, for every overdamped mode there must be an underdamped mode, etc. Thus, none of these systems can relax to an equilibrium point, i.e., we never have *asymptotic stability*. Also, as mentioned in the Introduction, we have *energy stability criteria* that can be used to prove nonlinear stability. We have a theory of *normal forms*, which states that all linear systems with the same eigenvalues can be transformed by a coordinate change into the same system. For example, all stable Hamiltonian systems can be mapped to the Hamiltonian system with Hamiltonian $H=\Sigma\omega(q^2+p^2)/2$, which is merely that of a collection of independent simple harmonic oscillators. And, this is only a fraction of the lore at our disposal.

It is natural to wonder to what extent these notions carry over to infinite degree-of-freedom systems. All these ideas survive, and some new things can happen too, such as the existence of a continuous spectrum as discussed in Sec. IV. So, let us turn to such infinite systems.

E. Continuum systems—Particles to fields

In this section we show how one can smooth out the particles of (6) and obtain an action principle where the particles are treated as a continuum. In this way we will obtain the action principle of Low[38] of (1958) for the Maxwell–Vlasov equations, which treats the particles by means of a Lagrangian or material variable labeled by its initial condition, a continuum phase space variable. For simplicity we consider only a single species. There are several other kinds of action principles and ways of describing a continuum of particles, in various kinds of fluid and kinetic theories; at the end of this section we will briefly describe some of them, many of which are treated in greater detail in Ref. 3.

At $t=0$ we suppose a particle is located at every point $z_0:=(x_0,v_0)$ of phase space. Thus, in the action of (6) we replace the discrete label by this continuum label, $q_i \rightarrow q(z_0,t)$, and we replace the sums as follows $\Sigma_{i=1}^{N} \rightarrow \int f_0(z_0)dz_0$, where f_0 can be viewed as a probablility or number density attached to each point of phase space. This procedure results in the following action:

$$S[q,\phi,A] = \int dt \int dz_0 f_0(z_0) \frac{m}{2} \dot{q}^2(z_0,t) - e \int dt \int dz_0 f_0(z_0) \int dx \left[\phi(x,t) - \frac{\dot{q}}{c} \cdot A(x,t) \right] \delta(x - q(z_0,t)) + \frac{1}{8\pi} \int dt \int dx (E^2(x,t) - B^2(x,t)), \quad (7)$$

where now all variables are fields, the particle phase space field $q(z_0,t)$, and the electromagnetic fields $\phi(x,t)$ and $A(x,t)$. This is essentially the action principle of Low. The derivative $\delta S[q,\phi,A]/\delta q(z_0,t)$ produces the equation for Lagrangian particle orbits, which can be shown to be equivalent to the Vlasov equation if we consider the Eulerian variable $f(z,t) := f_0(z_0)$, where z_0 is the initial condition of the particle that is located at z at time t, i.e., z_0 is determined by inverting $z = q(z_0,t)$, which is always possible because of uniqueness. Faraday's law and $\nabla \cdot B = 0$ follow from the introduction of the potentials, ϕ and A, and the remaining two Maxwell equations follow as for (6), with the sources seen to be the usual expressions of Vlasov theory by using $f(z,t) := f_0(z_0)$ and the fact that the map $z_0 \to z$ has unit Jacobian. Note that (7) reverts to (6) if we suppose that initially particles are located at N isolated points, i.e., upon using $f_0(z_0) = \Sigma_i^N \delta(z_0 - z_0^i)$ and $q^i(t) := q(z_0^i,t)$

In order to bring a sense of order to the many different types of continuum action principles, we divide them into two dichotomies: Lagrangian vs Eulerian and fluid vs kinetic. Since we are on the topic we will first discuss kinetic theories and then make some brief remarks about fluid action principles.

1. Action principles for kinetic theories

Kinetic theories are distinguished by the nature of the particle orbits described. For example, Vlasov theory has characteristics that satisfy the usual electromagnetic force law, while gyrokinetic or oscillation-center theories have characteristics based on guiding-center (e.g., Ref. 39) or oscillation-center equations. In any case, the basic variable is a phase space density, a naturally Eulerian variable, and the treatment of this variable distinguishes the different action principles.

Lagrangian variable actions. The Low action (see also Refs. 40–42) is the prototype Lagrangian variable action, and, as described above, one treats a continuum of particle orbits and uses the relation $f(z,t) := f_0(z_0)$ to map back to the Eulerian variable. The Low action is essentially Hamilton's principle in infinite dimensions. An alternative to this is to consider the phase space action[3,43–46] in which $\dot{q}$ is replaced by a canonical momentum, π, and these quantities are varied independently. The derivatives $\delta S/\delta q = 0$ and $\delta S/\delta \pi = 0$ produce directly equations that are of Hamiltonian form.

Hamiltonian–Jacobi actions. These actions, which were introduced by Pfirsch in Ref. 47 and improved and extended in Refs. 48 and 49, use a mixed variable generating function and an associated density function to describe particles. It was described in Ref. 50 how this formulation is a mixed Lagrangian–Eulerian variable theory. In Ref. 3 the Leaf action, a Hamiltonian–Jacobi action that manifestly preserves Casimir invariants (cf. Sec. II F), was introduced. It was shown there how the density variable can be removed.

Kinetic Clebsch actions. Clebsch obtained an action principle for the ideal fluid by introducing potentials to describe the velocity field. It was shown in Refs. 3, 51, and 52 that this idea could be adapted to obtain a purely Eulerian action principle for the Vlasov equation. By writing the phase space density as $f = [\alpha,\beta]$, where α and β, the potentials, are functions defined on phase space and $[,]$ denotes the ordinary Poisson bracket, one can write a phase space action for kinetic theories.

Sometimes one can eliminate the electromagnetic field variables from the action altogether. For example, if all the forces are electrostatic, then one can solve Poisson's equation for the electric field in terms of the phase space density and insert this into the action, and thereby obtain a pure particle action. This was done for all the actions above in Ref. 3. It can also be done in the Darwin approximation[53] where relativistic effects are included.

The above action principles have been used for many things, but of special importance is the relationship between symmetries and constants of motion. Noether's theorem has been used to obtain various constants of motion, including the energy-momentum tensor, in Refs. 44, 45, 48, 54, and 66. This is the only consistently reliable way to physically identify energy and other invariants.

2. Action principles for fluid theories

The situation for ideal fluid theories such as MHD, the two fluid equations, gyrofluid models, etc. is similar to that above. We briefly describe some action principles for these systems in Lagrangian and Eulerian variables.

Lagrangian variable actions. The basic Lagrangian fluid variable is $q(a,t)$, which unlike the Lagrangian variable for kinetic theory is labeled by a configuration space variable, a, which can be taken to be the position of a fluid element at $t = 0$. Descriptions of these action principles can be found in Refs. 56 and 57 and 2 for classical fluids, and a particularly nice discussion is given in Ref. 58 for MHD.

Eulerian variable actions. While action principles for fluids in terms of Lagrangian variables are a natural extension of Hamilton's principle of classical mechanics, the situation for Eulerian variables, which are not canonical variables, requires some artifice. We describe three ways to accomplish this. First, one can introduce so-called Lin constraints, in which one enforces the continuity and entropy equations (and possibly others) by the method of Lagrange multipliers (see, e.g., Ref. 56). A second approach is to introduce the Clebsch potential representation of the velocity field (see, e.g., Refs. 2, 52, 56, and 59). The third method is to directly constrain the Eulerian variable variations to a form induced by underlying Lagrangian variable variations. There is a good description of this method in Ref. 57.

Various fluid action principles have been used for various physical applications, including the construction of various kinds of fluid theories, gyro, and otherwise.[60] One can also construct generalized fluid action principles based on symmetry.[61]

F. Noncanonical Hamiltonian structure

Because Lagrangian variables are a continuum particle description, both action principles and Hamiltonian structure in terms of these variables match those of classical particle mechanics. However, the transformation from Lagrangian variables to Eulerian variables is not a canonical transformation, and consequently the form of Hamilton's equations in Eulerain variables is not of the canonical form of (2) or (3a) and (3b), or its infinite-dimensional generalization.[62] In this section we will describe the universal Hamiltonian form that plasma (and other continuous media) equations take when expressed in terms of Eulerian variables. Since this material has been extensively covered elsewhere (e.g., Ref. 2) we give a brief review, as is needed for the applications of Secs. III and IV. We begin with finite systems and then turn to infinite systems.

The usual dictum of classical mechanics is that one only considers transformations of Hamilton's equations that are canonical transformations, these being transformations that preserve the form of Hamilton's equations. However, a Hamiltonian system in noncanonical coordinates is still a Hamiltonian system, albeit with a possibly obscured form. Because the cosymplectic form of (3a) and (3b) transforms as a second rank contravariant tensor and a Hamiltonian transforms as a scalar, Hamilton's equations and the Poisson bracket take the forms

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j} = [z^i, H], \quad [f,g] = \frac{\partial f}{\partial z^i}J^{ij}(z)\frac{\partial g}{\partial z^j}.$$

The essence of being Hamiltonian lies in two properties of the Poisson bracket that are transformation invariant

(1) antisymmetry: $[f,g] = -[g,f]$,
(2) Jacobi identity: $\mathbf{[}f,[g,h]\mathbf{]} + \mathbf{[}g,[h,f]\mathbf{]} + \mathbf{[}h,[f,g]\mathbf{]} = 0$,

which are to be satisfied for all functions of phase space f, g, and h.

Given the above two properties and the requirement that det $J \neq 0$, a 19th century theorem due to Darboux says that there exists a transformation that takes $J \to J_c$. Thus we can get back to canonical coordinates and the usual form of Hamilton's equations. The understanding of this and the use of noncanonical coordinates for perturbation theory in particle orbit theory were introduced into plasma physics in Ref. 63.

For the case where det $J=0$, another 19th century theorem was proven by Lie. This theorem states that one can transform to a set of coordinates, part of which are canonical and part of which are in a sense redundant. The canonical coordinates describe a space of dimension equal to the rank of J and the remaining coordinates are described by a set of functions that have become known as Casimir invariants, because they are invariant under the dynamics. Casimir invariants are in this sense built into the original phase space, and this structure explains the "mysterious" invariants that are sprinkled throughout plasma theory.

The case where det $J=0$ is of interest because it transpires universally when one has equations for media in terms of Eulerian variables. Moreover, it is usually the case that the Poisson bracket takes a form where J is linear in the dynamical variables. In finite dimensions this form is given by $J^{ij} = c_k^{ij} z^k$, where the constants c_k^{ij} are structure constants of a Lie algebra, but this will not concern us here. Brackets of this form have become known as Lie–Poisson brackets (e.g., Ref. 64), and below we briefly describe this form in infinite dimensions.

In infinite dimensions we represent a general field by $\psi(\mu,t)$ labeled by μ, where, e.g., for fluid theories $\mu = x$ and Vlasov-type kinetic theories $\mu = (x,v)$. The Poisson brackets of interest have the form

$$\{F,G\} = \int \frac{\delta F}{\delta \psi}\mathcal{J}(\psi)\frac{\delta G}{\delta \psi}d\mu,$$

where we now have a cosymplectic operator $\mathcal{J}$ and if the field ψ has multicomponents additional sums are implied. We now see why it is necessary to understand the functional derivatives of Sec. II B. The Lie–Poisson form of continuous media is given by

$$\{F,G\} = \left\langle \psi, \left[\frac{\delta F}{\delta \psi}, \frac{\delta G}{\delta \psi}\right]\right\rangle,$$

where $[,]$ is a Lie product and the cosymplectic operator is $\mathcal{J}\cdot = [\psi, \cdot]^{\dagger}$. We give an example below in Sec. II G that will give some meaning to this formula.

This is the general noncanonical Hamiltonian form possessed by ideal fluid equations, the Vlasov equation, the Liouville equation, the BBGKY hierarchy, gyrokinetic theories, MHD, tokamak reduced fluid models, such as reduced MHD, the Hasagawa–Mima equation, ITG models, etc. Indeed it is a universal form.

G. Vlasov and two-dimensional Euler Hamiltonian structure

The noncanonical Poisson bracket for the Vlasov–Poisson system[65] is given by

$$\{F,G\} = \int f\left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]dxdv, \tag{8}$$

where the cosymplectic operator has the form

$$\mathcal{J}\cdot = \frac{1}{m}\left(\frac{\partial f}{\partial x}\frac{\partial \cdot}{\partial v} - \frac{\partial \cdot}{\partial x}\frac{\partial f}{\partial v}\right).$$

Note that $\{,\}$ is a bracket defined on functionals, while $[,]$ is a bracket defined on functions. With the bracket of (8) the one-dimensional Vlasov equation is represented by

$$\frac{\partial f}{\partial t} = \{f,H\} = [f,\mathcal{E}],$$

with the Hamiltonian being the total energy

$$H[f]=\frac{m}{2}\int fv^2dxdv+\frac{1}{8\pi}\int E^2dx.$$

It is easy to show that the particle energy $\mathcal{E}$ results as follows: $\mathcal{E}=mv^2/2+e\phi=\delta H/\delta f$.

The bracket for Euler's fluid equation in two dimensions[52,65] is identical to the above with the independent variables (x,v) replaced by (x,y), f replaced by the scalar vorticity ω, and the Hamiltonian being $H[\omega]=-\int\omega\psi\,dx\,dy$. We will use this structure in Secs. III and IV.

H. Approximation and organization

Noncanoncial Poisson bracket such as those above have been used in a variety of Hamiltonian approximation schemes. An example using Fourier expansion was used to describe beam-plasma instability in Refs. 67 and 68. Alternatively, Hamiltonian reductions can be achieved by using moments of the dynamical variables. This was recognized, perhaps first, in Ref. 69. Moment reductions have been used to describe beam systems,[70] vortex dynamics,[71,72] and recently models of short-pulse laser-plasma interactions.[73]

Poisson brackets also provide an organizational scheme. For many years, reduced fluid models have been derived in order to describe as much physics as possible in tractable models of plasma devices and systems. The reduction process, based on aspect ratio and other quantities, produces two-dimensional (2D) or quasi-three-dimensional models with Poisson bracket nonlinearities. In Refs. 74 and 75 it was shown that RMHD (both 2D and 3D, high and low β) has a Lie–Poisson bracket. Subsequently, many other systems were shown to have this form, which constitutes an algebraic extension of brackets involving the group of canonical transformations. Based on Ref. 76 it was shown that this structure could be used to derive a model with FLR physics, e.g., the four-field model of Ref. 77. The Hamiltonian theory provides a way to obtain constants of motion and to use them for constructing equilibria, coherent structures, and stability arguments. Various other models were considered in Refs. 78–80. A general theory for such brackets was worked out in Ref. 81.

III. CALCULATION OF V-STATES OF CONTOUR DYNAMICS

The water bag model and contour dynamics are examples of reductions of the Vlasov–Poisson and two-dimensional Euler fluidlike equations, respectively. Both are based on initial conditions where the dynamical variable is constant in a region bounded by a contour. Because both the Vlasov and Euler equations have the common Hamiltonian form of Sec. II G a similar formulation applies to both. In this section we will consider the 2D Euler case. We present a Poisson bracket for contours that bound a region that need not be star shaped, and show how this bracket can be used to obtain a numerical (relaxation or simulated annealing) algorithm for calculating equilibrium states.[82] In particular V-states,[83] rigidly rotating vortex states that are claimed to be exact solutions of Euler's fluid equations, are obtained by searching for critical points of the Hamiltonian in a rotating reference frame, $\delta(H+\Omega L)=0$, where here L is the angular momentum. V-states have been found numerically with m-fold symmetry. Examples for the case of twofold symmetry (the Kirchoff ellipse) and threefold symmetry are shown in Fig. 1.

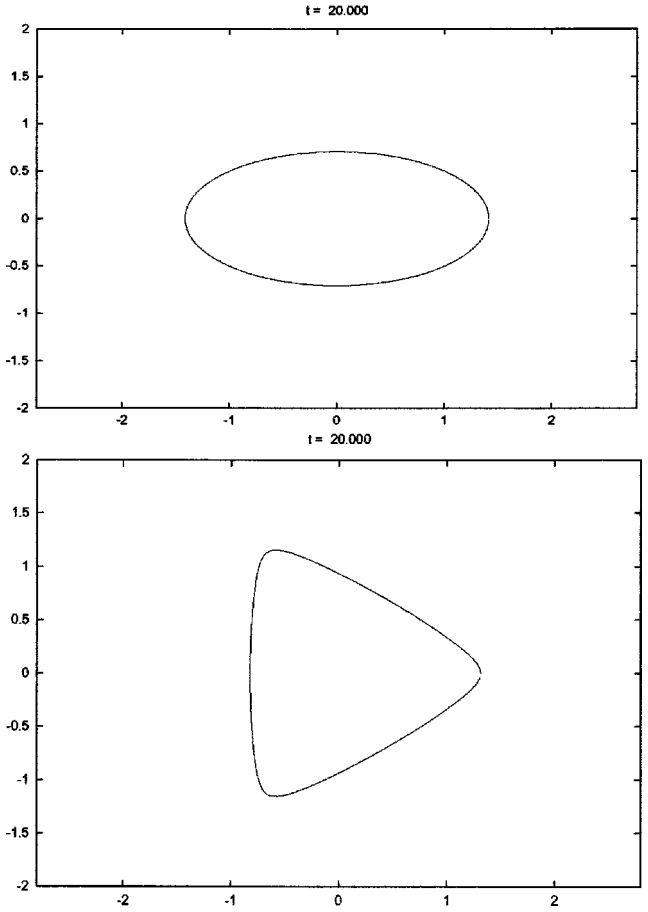

FIG. 1. Depiction of V-states with twofold (upper) and threefold (lower) symmetry.

A. Hamiltonian structure of contour dynamics

The reduction to contour dynamics proceeds by replacing the vorticity variable $\omega(x,y,t)$ by a plane curve that bounds a vortex patch, $\boldsymbol{X}(\sigma)=(X(\sigma),Y(\sigma))$. Here the curve parameter σ is not chosen to be arc length because arc length is not conserved by the dynamics of interest.

Because plane curves are geometrical objects, their Hamiltonian theory should be based on parametrization invariant functionals, i.e, functionals of the form $F[X,Y]=\oint d\sigma\,\mathcal{F}(X,Y,X_\sigma,Y_\sigma,Y_{\sigma\sigma},X_{\sigma\sigma},\ldots)$, where $X_\sigma:=\partial/\partial\sigma$, etc. and $\mathcal{F}$ has an Euler homogeneity property. A consequence of parametrization invariance is the Bianchi-like identity,

$$\frac{\delta F}{\delta X(\sigma)}X_\sigma+\frac{\delta F}{\delta Y(\sigma)}Y_\sigma\equiv 0, \tag{9}$$

relating functional derivatives, a result which follows from E. Noether's second theorem.[84]

The noncanonical Poisson bracket for the contours is given by

$$\{F,G\}=\oint d\sigma\left[\frac{Y_\sigma\frac{\delta F}{\delta X}-X_\sigma\frac{\delta F}{\delta Y}}{X_\sigma^2+Y_\sigma^2}\right]\frac{\partial}{\partial\sigma}\left[\frac{Y_\sigma\frac{\delta G}{\delta X}-X_\sigma\frac{\delta G}{\delta Y}}{X_\sigma^2+Y_\sigma^2}\right] \tag{10}$$

and the equations for the contour are generated by inserting the following compact form for the Hamiltonian into this bracket,

$$H=\oint d\sigma\oint d\sigma'\ \phi\hat{n}\cdot\hat{n}',$$

where $\hat{n}$ and $\hat{n}'$ are unit vectors tangent to the contours, $\phi(\rho)$ satisfies $\nabla'^2\phi(\rho)=G(\rho)$, where the function $\rho=|\mathbf{x}-\mathbf{x}'|$ and G is the Green's function of the two-dimensional Laplacian.

It is easy to show that this bracket has the area functional $2\Gamma:=\oint(XY_\sigma-YX_\sigma)d\sigma$ as a Casimir invariant, i.e., $\{\Gamma,F\}\equiv 0$ for all functionals F.

B. Dirac brackets and simulated annealing

Given any Poisson bracket and Hamiltonian one can construct a dynamics that relaxes to critical equilibrium points of the system. This can be done with the *simulated annealing* (SA) bracket, which in finite dimensions has the form

$$[f,g]_{SA}=[f,z^\ell][z^\ell,g]=\frac{\partial f}{\partial z^i}J^{i\ell}J^{\ell j}\frac{\partial g}{\partial z^j}. \tag{11}$$

That this bracket generates relaxation dynamics follows from $dH/dt=[H,H]_{SA}\geqslant 0$. This means we have asymptotic stability by Lyapunov's theorem, the idea behind the H-theorem relaxation of the Boltzmann equation to thermal equilibrium.

Attempts to calculate V-states using (10) in an infinite-dimensional version of (11) with the rotating frame Hamiltonian $H+\Omega L$ resulted in failure. Probably this is because there are many equilibria and different parts of the initial conditions can lie in the basins of attraction of different equilibria. To rectify this we construct a *Dirac bracket* which can be used to constrain the system.

En route to his theory of quantization, which required generalizing the Legendre transformation to cases that were degenerate, Dirac discovered a way to construct degenerate Poisson brackets that satisfy the two conditions of Sec. II F, while building in invariance of pairs of functions. The pairs then satisfy the same condition as Casimir invariants, i.e., $[C_{1,2},f]\equiv 0$ for all functions f. Dirac's theory was built on canonical Poisson brackets, and in the case of a single pair, $C_{1,2}$, his bracket has the following form:

$$[f,g]_{GD}=\frac{1}{[C_1,C_2]}([C_1,C_2][f,g]-[f,C_1][g,C_2]+[g,C_1][f,C_2]). \tag{12}$$

We have shown[85] that one can construct good brackets with Dirac's construction out of Poisson brackets that need not be canonical. To obtain practicable Poisson brackets for simulated annealing, we consider the infinite-dimensional generalization of Dirac's construction with the bracket [,] of (12) replaced by that of (10). We choose one constraint to be the angular momentum $L=\int(x^2+y^2)dxdy$, and the other is chosen to enforce the symmetry of the desired V-state. For example, for Kirchoff's ellipse we choose the second constraint to be the xy-moment, $K=\int_D xydxdy$. Figure 2 shows a sequence of states relaxing into Kirchoff's ellipse. We have successfully applied this algorithm to a variety of V-states and other known rotating equilibrium contour dynamics solutions. These results will be reported elsewhere.

IV. FLUCTUATION SPECTRA

We now turn to our last application,[86] the calculation of the spectrum of electron phase space density fluctuations of a plasma and vorticity fluctuations about shear flow. This is done by a novel method that parallels conventional calculations using the partition function in statistical physics and the Hamiltonian formulation. Expressions for the electric field fluctuations in a Maxwellian plasma agree with known results. New results are obtained for non-Maxwellian equilibria and vorticity fluctuations. The latter results have been compared to fluid experimental results, which we briefly discuss below.

Fluctuation spectra in plasmas can be calculated by many means that are treated in standard plasma physics textbooks. For example, one can follow Klimontovich and smooth δ-functions located on particles, construct the BBGKY hierarchy from Liouville's equation, follow Thompson[87] and Rostoker[88] and treat dressed test particles, or one can approximate the N-body partition function. In fluids, statistical mechanics of fluctuations have been treated by Onsager,[89] Lee,[90] and others using point vorticies, Fourier modes, or amplitudes at lattice sites as degrees of freedom (see, e.g., Refs. 21 and 91). We differ from all these approaches in that we do statistical mechanics using eigenmodes associated with the continuous spectrum (van Kampen modes) as degrees of freedom.

Our calculation is modeled after early statistical mechanical treatments of the lattice vibrations of a simple solid. We suppose the existence of stable dynamical equilibrium states for our plasma and fluid systems, which are analogous to the stationary lattice of the solid. Our equilibria are chosen so that they support a complete continuous spectrum of stable eigenmodes. Because plasmas can exist for long times in equilibrium states that are out of thermal equilibrium, these distribution functions need not be Maxwellian. Similarly, shear flow equilibria are chosen to have Rayleigh stable profiles. In experiments, such profiles can be selected by the competition between forcing and damping. The oscillations that occur on top of these equilibria are treated analogously to the solid lattice vibrations that are assumed to be weakly interacting and the partition function is evaluated. From the partition function the fluctuation spectra are obtained.

A. Partition functions

Einstein and Debye calculated the specific heat of a solid by treating it as a collection of $3N$ quantized simple har-

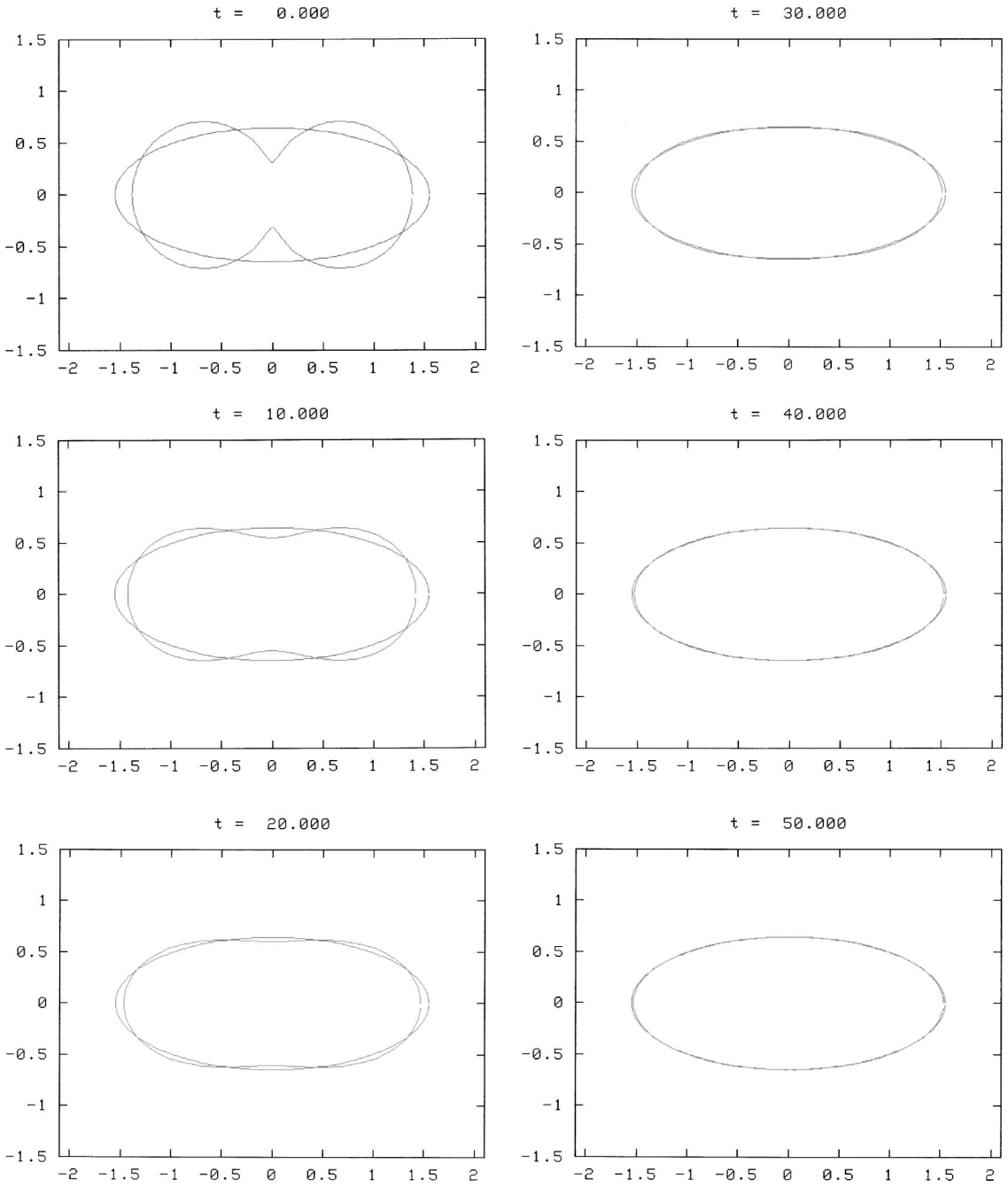

FIG. 2. (Color online) Relaxation onto Kirchoff's ellipse.

monic oscillators. They summed the partition function explicitly and then used it to obtain an expression that reproduced the Dulong–Petit relation in the classical limit, $C_v = 3Nk_B$. Thus they obtained the well-known equipartition result for a solid that the average energy contains a full $k_B T_b$ per degree of freedom, where T_b is the temperature of the heat bath.

For classical systems the expression for the partition function is given by the following:

$$\mathcal{Z} = \int e^{-\beta E} \prod_i^N dq_i\, dp_i. \tag{13}$$

From (13) it is seen that there are two requirements needed to define this object: a conserved energy $E(p,q)$ and a notion of measure, which is provided by phase space volume. Both requirement are assured when the dynamics of the systems is Hamiltonian: the Hamiltonian defines the energy of a state, and a well-defined notion of measure arises from Liouville's theorem on the preservation of phase space volume under the dynamics.

Partition functions are easily evaluated for stable Hamiltonian systems that are quadratic forms in the phase space variables, $H = \Sigma(pMp/2 + qGp + qVq/2)$, by using canonical

transformations to diagonalize and then perform the resulting Gaussian integrals. The canonical transformation results in any of the following normal forms:

$$H = \sum_i^N \omega_i (Q_i^2 + P_i^2)/2 = \sum_i^N i\omega_i \hat{Q}_i \hat{P}_i = \sum_i^N \omega_i |A_i|^2. \tag{14}$$

The equipartition theorem then states that for each quantity (up to the frequencies ω_i) that appears as a square in the Hamiltonian, there occurs a $k_B T_b/2$ contribution to the averaged energy of the system interacting with a heat bath.

We effect an analog of the calculation above for the Vlasov and Euler systems. In doing so we must evaluate the partition function for a field theory and this amounts to a functional integral of the form

$$\mathcal{Z} = \int e^{-\beta H[q,p]} \mathcal{D}q \mathcal{D}p, \tag{15}$$

where $q(z)$ and $p(z)$ are fields labeled by z. Functional integrals were introduced by Wiener[92] and used in Feynman's path integral formulation of quantum mechanics. They are not always mathematically well-defined objects, but when the Hamiltonian is a quadratic function one can consistently do calculations by discretizing and reducing the calculation to a sequence of ordinary integrals. Fortunately, this is the case we wish to do.

B. Two Hamiltonian systems

The Vlasov system of interest is defined in Sec. II G. To be definite we give the 2D Euler equation below:

$$\frac{\partial \omega}{\partial t} + [\psi, \omega] = 0, \tag{16}$$

where here $[\psi, \omega] := \psi_x \omega_y - \psi_y \omega_x$ and the streamfunction and vorticity are related by $\psi = \Delta^{-1}\omega$ or in the case of quasigeostrophy another integral relation.

We consider linear fluctuations about the following class of stable equilibria for the two systems. For Vlasov we suppose the phase space density is given by a Maxwellian, $f_0 \sim \exp(-mv^2/2k_B T_e)$ or any stable homogeneous form, $f_0(v; T_1, T_2, \ldots)$, where we note that the temperature is merely a parameter that describes the equilibrium state. In our second form we allow for the possibility of more than a single parameter. By Gardner's theorem we can be assured of stability if this function is a monotonic function of v^2. Because plasmas can reside for a long time in equilibrium states away from Maxwellian, we can distinguish between the equilibrium temperature T_e and the bath temperature of the fluctuations T_b. In the case of shear flow, we suppose the equilibrium is a flow along a finite channel with a cross stream variation: $U(y; a_1, a_2, \ldots)$. By Rayleigh's criterion we are assured of stability if $U' \neq 0$. Setting $f = f_0(v) + \delta f(x, v, t)$ and linearizing we obtain linearized Vlasov theory for plasma oscillations, the system investigated by Landau, van Kampen, and others. We follow Refs. 93 and 94 and solve it as one would solve a Hamiltonian system, albeit an infinite-dimensional one. The situation is similar for the shear flow problem.

The linear theories of the two systems are Hamiltonian with the Poisson bracket

$$\{F, G\}_L = \int f_0 \left[\frac{\delta F}{\delta \delta f}, \frac{\delta G}{\delta \delta f} \right] dx\, dv, \tag{17}$$

and the linear Vlasov theory has

$$H_L = -\frac{m}{2} \int_\Pi \int_{\mathbb{R}} \frac{v(\delta f)^2}{f_0'} dv dx + \frac{1}{8\pi} \int_\Pi (\delta \phi_x)^2 dx, \tag{18}$$

the Kruskal and Oberman[95] energy, as Hamiltonian. It is easy to verify that the linear Vlasov theory can be written as

$$\frac{\partial \delta f}{\partial t} = \{\delta f, H_L\}_L.$$

The situation is similar for the shear flow problem, but we refer the reader to Ref. 96 for details.

C. Canonization and diagonalization

In order to calculate the functional integral of (15) we transform the bracket of (17) into canonical form and then find a canonical transformation to variables in which H_L is diagonal. Expanding $\delta f(x, v, t) = \Sigma_{k=-\infty}^{\infty} f_k(v, t) e^{ikx}$ and writing (17) and (18) in terms of the variable f_k yields

$$\{F, G\}_L = \sum_{k=1}^{\infty} \frac{ik}{m} \int_{\mathbb{R}} f_0' \left(\frac{\delta F}{\delta f_k} \frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k} \frac{\delta F}{\delta f_{-k}} \right) dv \tag{19}$$

and

$$H_L = -\frac{m}{2} \sum_k \int_{\mathbb{R}} \frac{v}{f_0'} |f_k|^2 dv + \frac{1}{8\pi} \sum_k k^2 |\phi_k|^2$$

$$=: \sum_{k,k'} \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(v) \mathcal{A}_{k,k'}(v|v') f_{k'}(v') dv dv'. \tag{20}$$

Details are given in Refs. 93 and 94 for Vlasov and in Ref. 96 for shear flow.

Equation (19) is not quite of canonical form. However, by defining $q_k = m f_k/(ikf_0')$ and $p_k = f_{-k}$ the Poisson bracket is canonized, i.e., it becomes

$$\{F, G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\frac{\delta F}{\delta q_k} \frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k} \frac{\delta F}{\delta p_k} \right) dv.$$

Because of the electrostatic energy term of H_L, the energy is not a diagonal quadratic form. This is achieved by using the type-2 mixed variable generating functional

$$\mathcal{F}[q, P] = \sum_{k=1}^{\infty} \int_{\mathbb{R}} q_k(v) G[P_k](v) dv \tag{21}$$

to effect the canonical coordinate change $(q, p) \leftrightarrow (Q, P)$ according to

$$p_k(v) = \frac{\delta \mathcal{F}[q, P]}{\delta q_k(v)} = G[P_k](v),$$

$$Q_k(u) = \frac{\delta \mathcal{F}[q, P]}{\delta P_k(u)} = G^\dagger[q_k](u). \tag{22}$$

The essential ingredient of (21) is the integral transform G defined by

$$f(v) = G[g](v) := \epsilon_R(v)g(v) + \epsilon_I(v)\mathcal{H}[g](v), \tag{23}$$

where $\epsilon_I(v) = -\pi\omega_p^2 f_0'(v)/k^2$ and $\epsilon_R(v) = 1 + \mathcal{H}[\epsilon_I](v)$, with $\mathcal{H}[g]$ being the Hilbert transform

$$\mathcal{H}[g](v) := \frac{P}{\pi}\int_{\mathbb{R}} \frac{g(u)}{u - v}du,$$

with P denoting the Cauchy principal value. The mathematics of this transform is discussed in Ref. 94. The transformation generated by $\mathcal{F}$ is designed to diagonalize the Hamiltonian, i.e., it becomes

$$\begin{aligned} H_L &= \sum_{k=1}^{\infty}\int_{\mathbb{R}} i\omega_k(u)Q_k(u)P_k(u)du \\ &= \sum_{k=1}^{\infty}\int_{\mathbb{R}} \omega_k(u)|A_k(u)|^2 du, \end{aligned} \tag{24}$$

where $\omega_k(u) = ku$. Equation (24) represents a generalization to infinite dimensions of two well-known normal forms of Hamiltonian dynamics [cf. (14)]. Now we are in a position to obtain fluctuation spectra.

D. Fluctuation spectra

The ensemble average of a quantity $\mathcal{O}$ is given in terms of $\mathcal{Z}$ according to $\langle\mathcal{O}\rangle = \int \mathcal{D}q\mathcal{D}p\ \mathcal{O}\ e^{-\beta H}/\mathcal{Z}$. We leave the details of the evaluation to a later publication, but the diagonalization allows the following:

$$\mathcal{Z} = \int e^{-\beta H_L[q,p]}\mathcal{D}q\mathcal{D}p = \int e^{-\beta\Sigma_k\int\omega_k|A_k|^2du}\prod\nolimits_k \mathcal{D}A_k\mathcal{D}A_k^*,$$

whence $\langle A_k A_k^*\rangle$ is obtained. It turns out that $A_k \propto E_k$, where E_k is the electric field associated with a van Kampen mode. Writing the result in terms of E_k, a noncanonical variable, gives

$$\langle E_k(u)E_k(u')^*\rangle = \delta_{k,k'}\frac{16}{V\beta}\frac{\epsilon_I(u)}{u|\epsilon|^2}\delta(u - u'). \tag{25}$$

We make two observations about (25). First, because of the factor involving ϵ, this result is not in an obvious equipartition form. However, if one transforms from E_k, to the correct canonical variables, the variables in which the Hamiltonian is diagonal, then equipartion is obtained for all k values. Second, in the case where f_0 is Maxwellian and the bath and equilibrium temperatures are equal, (25) agrees with a result first given by Thompson.[87] However, for general equilibria of the form $f_0(v; T_1, T_2, \ldots)$, the result differs from Thompson's. Derivations (e.g., Ref. 88) are not performed as asymptotic expansions in a dimensionless number, so it is at present unclear why the results differ.

An advantage of the present formalism is that it is possible to transform from the variable $A_k(u)$ back to $f_k(v)$ and obtain the following result for phase space fluctuations:

$$\begin{aligned} \langle f_k(v)f_{k'}^*(v')\rangle = \delta_{k,k'}\frac{k^2}{\pi^2 e^2 V\beta}\Bigg\{ &\frac{\epsilon_I(v)}{v}\delta(v - v') \\ &- \frac{1}{\pi}\frac{\epsilon_R(0)}{|\epsilon(0)|^2}\frac{\epsilon_I(v')\epsilon_I(v)}{vv'}\Bigg\}. \end{aligned} \tag{26}$$

The analogous result for the shear flow problem is given by

$$\langle\omega_k(y)\omega_k^*(y')\rangle = \delta_{k,k'}\{\mu(y)\delta(y - y') + v_k(y,y')\}. \tag{27}$$

where μ and v_k are determined by the equilibrium shear flow profile. The details of this calculation are too lengthy to present here.

We have compared the result (27) with particle tracking (PIV) and hot film probe measurements of the pumped rotating tank experiment of Swinney's laboratory (e.g., Ref. 97), with surprising agreement. The details will be presented elsewhere, but note that the function μ is independent of k. Thus, near the line $y = y'$ the δ-function term dominates and we expect the vorticity spectrum to be independent of k, which implies the velocity spectrum should vary as k^{-2}. Indeed, this is surprisingly close to what is seen.

ACKNOWLEDGMENTS

The author has learned a great many things from far too many people to acknowledge here, so he gives a blanket thanks to his mentors, colleagues, and students who have helped him throughout the years.

This work was supported by the U.S. Department of Energy Contract No. DE-FGO3-96ER-54346.

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