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On the fluctuation spectrum of plasma

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Abstract

The spectrum of electron phase space density fluctuations of a plasma is calculated by a novel method that parallels conventional calculations of the partition function in statistical physics. Expressions for the electric field fluctuations and the closely related form factor agree with existing results. The method clears up ambiguities about equipartition and provides a new expression for the spectrum of electrostatic phase space density fluctuations about stable non-Maxwellian equilibria.

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1. Introduction

Several approaches can be taken for calculating the electrostatic fluctuation spectrum of a homogeneous plasma. In the Klimontovich approach [1,2] a one-point phase space density, concentrated on the phase space positions of N point particles, is smoothed by ensemble averaging and then the resulting hierarchy is truncated. Alternatively, one can begin with an N-point Liouville equation and construct and truncate the BBGKY hierarchy (see e.g. [3,4]), or one can follow a third approach which is to consider the superposition of dressed test particles (see e.g. [5–8]). Lastly, a direct statistical mechanical approach can be taken where one constructs and coarse grains the partition function for N point particles interacting through the Coulomb potential (see e.g. Chapter VIII of [4]). In this paper we present a new method that is based on the partition function, where van Kampen modes [4,9] are taken to be the basic degrees of freedom, and, consequently, transient (nonwave) dynamics is included.

Our approach parallels the specific heat calculations of Maxwell, Einstein, and others. For example, Einstein calculated the specific heat of a solid (later refined by Debye) by supposing it to be an equilibrium lattice configuration of point masses connected by springs. He then considered the statistical mechanics of the linear

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vibrations about the equilibrium configuration. He quantized the energy associated with the vibrational degrees of freedom and calculated a resulting discrete sum that appears in the partition function, whence he obtained the specific heat. In the classical limit his result agreed with the Dulong–Petit relation, which accounts for $k_B T$ for each degree of freedom. Our approach is philosophically the same: we begin with a homogeneous stable equilibrium solution of the Vlasov–Poisson system, which is analogous to Einstein's lattice, and then calculate the classical partition function where the basic degrees of freedom are the normal modes of the plasma, which are analogous to the lattice vibrations.

Below in Section 2 we review some basics about partition functions and their use. This is followed in Section 3 by a review of features of the Vlasov–Poisson system and its Hamiltonian formulation that are needed for subsequent calculations. Section 4 contains the main part of the paper, where we evaluate the functional integral that represents the partition function and thereby obtain expressions for fluctuation spectra. In Section 4 it is also shown that the results satisfy equipartition when viewed properly and expressions for form factors are obtained. Because the problem we are considering is a classic one of plasma physics, effort has been spent comparing our approach and results with previous work. This is discussed in Section 5 along with an interpretation of our results and mention of future work.

2. Partition functions

Classical partition function calculations lead to the evaluation of the following integral:

$$Z = \int \mathrm{d}\mu e^{-\beta E},$$

where $\beta = 1/k_B T$. Evidently, two things are required to evaluate this integral: an expression for the energy, E, and a notion of invariant measure, $\mathrm{d}\mu$. Both are provided by the Hamiltonian form of classical physics, where the energy is given by the Hamiltonian and, according to Boltzmann and Gibbs [10], the appropriate measure is given in terms of canonical variables, $\mathrm{d}\mu = \prod \mathrm{d}q\mathrm{d}p$. In practice, one evaluates the integral by diagonalizing the Hamiltonian by a canonical transformation and then calculates the resulting product of Gaussian integrals. This gives the well-known equipartition theorem, which states that the average value of the energy has a contribution of $\frac{1}{2}k_B T$ for each quantity appearing as a square in the energy. Thus, for a gas that has no interaction potential energy the average energy is half that of a solid.

Our calculation is significantly more difficult than that described above because the 'vibrations' are governed by the linearized Vlasov equation which is a field theory. Thus we must evaluate the functional integral:

$$Z = \int \mathscr{D}p\mathscr{D}q e^{-\beta H}, \tag{1}$$

and this requires the Hamiltonian, H, the canonical field variables, (q, p), and a means for calculating the functional integral. This calculation is hampered by the fact that the basic variable in Vlasov theory, the phase space density f, does not constitute a set of canonical variables, and the fact that the linear normal modes of interest, the van Kampen modes, have a continuous eigenvalue spectrum and associated singular eigenfunctions. Two advances make this calculation possible – techniques to canonize and to diagonalize. In [11] it was shown how the Vlasov equation is a Hamiltonian theory in terms of noncanonical variables, and in [12] (refined and extended in [13–15]) it was shown how to make sense out of the energy associated with van Kampen modes and how to diagonalize this energy by constructing an integral transform that is a generalization of the Hilbert transform. The diagonalization procedure turns (1) into a Gaussian functional integral that is rudimentary to evaluate.

3. Review of Vlasov system and Hamiltonian structure

Now turn to a review of the basic equtions of the Vlasov system, its noncanonical Hamiltonian structure, and the map to normal coordinates.

3.1. Vlasov dynamics

We consider a plasma with immobile ions and electron dynamics governed by the Vlasov–Poisson system.

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}+\frac{e}{m}\frac{\partial \phi[x,t;f]}{\partial x}\frac{\partial f}{\partial v}=0,\quad \frac{\partial^2\phi}{\partial x^2}=4\pi\left[e\int \mathrm{d}v f(x,v,t)-\rho_{\mathrm{B}}\right], \tag{2}$$

where $f(x,v,t)$ is the phase space density, ϕ is the electrostatic potential, and ρ_{B} is a uniform neutralizing background charge density. Here we have integrated out all but the longitudinal variables, reducing the problem to one dimension. In one dimension the energy is given by

$$H=\frac{m}{2}\int\int \mathrm{d}x\mathrm{d}v v^2 f+\frac{1}{8\pi}\int \mathrm{d}x\left(\frac{\partial\phi}{\partial x}\right)^2, \tag{3}$$

a quantity that is conserved by the system (2).

Because we are considering the statistical mechanics of linear vibrations, we assume a periodic spatial domain and expand the phase space density as $f(x,v,t)=f_0(v)+\delta f$, with $\delta f=\frac{1}{2}\sum_k f_k(v,t)\mathrm{e}^{\mathrm{i}kx}$ and the equilibrium $f_0(v)$ assumed to be linearly stable. Moreover, we assume f_0 is a monotonically decreasing function of v^2, a sufficient but not a necessary condition for stability. This includes the special case of a Maxwellian, but allows for more general distributions. The linear dynamics of f_k is governed by

$$\frac{\partial f_k}{\partial t}+\mathrm{i}kvf_k+\mathrm{i}k\phi_k\frac{e}{m}\frac{\partial f_0}{\partial v}=0,\quad k^2\phi_k=-4\pi e\int \mathrm{d}v f_k(v,t), \tag{4}$$

where the linear electrostatic potential is expanded as $\delta\phi=\frac{1}{2}\sum_k\phi_k(t)\mathrm{e}^{\mathrm{i}kx}$. The energy of the linearized system, the Kruskal–Obermann energy [16], is given by

$$H_L=-\frac{m}{2}\int\int \mathrm{d}v\mathrm{d}x v\frac{(\delta f)^2}{f_0'}+\frac{1}{8\pi}\int \mathrm{d}x\left(\frac{\partial\delta\phi}{\partial x}\right)^2, \tag{5}$$

where $f_0'=\mathrm{d}f_0/\mathrm{d}v$. In Fourier space H_L becomes

$$H_L=\sum_{k,k'}\int\int \mathrm{d}v\mathrm{d}v' f_k(v)\mathscr{O}_{k,k'}(v|v')f_{k'}(v'), \tag{6}$$

where $\mathscr{O}=\delta_{k,-k'}[-\delta(v-v')m/(2f_0')+2\pi e^2/k^2]$.

The linear system (4) has been solved by three methods: the original Laplace Transform technique used by Landau (1946) and others; the expansion in terms of normal modes initiated by van Kampen (1955); and the introduction of a special integral transform in velocity space initiated in Ref. [12] that was refined in Refs. [13–15]. We use this last method, which is reviewed in the Hamiltonian context below where it is seen to be intimately related to van Kampen modes and amounts to a transformation to normal mode coordinates.

3.2. Hamiltonian structure

The Hamiltonian structure of the Vlasov–Poisson system in terms of noncanonical Poisson brackets was introduced in Ref. [11]. An extensive discussion of why the Poisson bracket is noncanonical and how this relates to conventional Lagrangian and Hamiltonian formulations can be found in [12,15,17,18]. Here we give the needed expressions.

The main item of the formalism is the noncanonical Poisson bracket, which has the form

$$\{F,G\}=\int \mathrm{d}x\mathrm{d}v\frac{\delta F}{\delta f}\mathscr{J}\frac{\delta G}{\delta f},$$

where F,G are functionals, $\delta F/\delta f$ is the functional or variational derivative, and $\mathscr{J}$ is the cosymplectic operator that must endow $\{F,G\}$ with antisymmetry and the Jacobi identity. For the Vlasov–Poisson system

$$\{F,G\} = \int \mathrm{d}x\mathrm{d}v f\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right], \tag{7}$$

the cosymplectic operator is given by $\mathscr{J}\cdot = [f,\cdot] = \frac{1}{m}\left(\frac{\partial f}{\partial x}\frac{\partial}{\partial v}\cdot - \frac{\partial f}{\partial v}\frac{\partial}{\partial x}\cdot\right)$, and in terms of (7) the Vlasov–Poisson system can be written as

$$\frac{\partial f}{\partial t} = \{f,H\} = [\mathscr{E},f],$$

where $\mathscr{E} = mv^2/2 + e\phi = \delta H/\delta f$.

The bracket formulation for the linear dynamics is obtained by expanding the above in the smallness of $\delta f/f_0$. This yields the noncanonical Poisson bracket

$$\{F,G\}_L = \int \mathrm{d}x\mathrm{d}v f_0\left[\frac{\delta F}{\delta\delta f},\frac{\delta G}{\delta\delta f}\right], \tag{8}$$

which with the Kruskal and Oberman energy of (5) gives the representation of the linearized Vlasov–Poisson system as

$$\frac{\partial\delta f}{\partial t} = \{\delta f, H_L\}_L,$$

which can be demonstrated by straightforward manipulation.

3.3. Canonization and diagonalization

The bracket of (8) does not have the usual canonical form in terms of generalized coordinates and momenta, but this is easy to obtain in terms of the Fourier transformed variable. Using the chain rule, (8) becomes

$$\{F,G\}_L = \sum_{k=1}^{\infty}\frac{\mathrm{i}k}{m}\int \mathrm{d}v f_0'\left(\frac{\delta F}{\delta f_k}\frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k}\frac{\delta F}{\delta f_{-k}}\right). \tag{9}$$

(See [17] for a tutorial on calculations such as this.) To canonize we define

$$q_k(v,t) = f_k \quad \text{and } p_k(v,t) = \frac{mf_{-k}}{\mathrm{i}kf_0'}, \tag{10}$$

where $k > 0$. In terms of these variables (9) takes the usual canonical form:

$$\{F,G\}_L = \sum_{k=1}^{\infty}\int \mathrm{d}v\left(\frac{\delta F}{\delta q_k}\frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k}\frac{\delta F}{\delta p_k}\right).$$

From (6) it is clear that the Hamiltonian H_L can easily be written in terms of the variables (q_k,p_k). However, because the electrostatic contribution to H_L (the second term of $\mathscr{O}$) possesses two integrals over v, H_L is not diagonal, and this complicates the evaluation of (1). The task of diagonalization is significantly more difficult than that of cannonization. The details were worked out in [12–15] and are most easily explained in the present context in terms of a mixed variable generating functional, which we do here.

Diagonalization is achieved by transforming from the variables $(q_k(v,t),p_k(v,t))$ to a new set of variables $(Q_k(u,t),P_k(u,t))$. We find it convenient to introduce an intermediate set of variables $(Q_k'(u,t),P_k'(u,t))$ where the new coordinate Q_k' is obtained from the old by a transformation $\mathscr{G}_k$ given by

$$q_k(v,t) = \mathscr{G}_k[Q_k'] = \epsilon_R(k,v)Q_k'(v,t) + \epsilon_I(k,v)\overline{Q_k'}. \tag{11}$$

Here $\epsilon_I = -\pi\omega_p^2 f_0'/k^2$, the Hilbert transform of a function g is $\bar{g} = \mathscr{P}/\pi\int \mathrm{d}u g(u)/(u-v)$, with $\mathscr{P}$ indicating Cauchy principal value, and $\epsilon_R = 1 + \overline{\epsilon_I}$. Note, $|\epsilon|^2 = \epsilon_I^2 + \epsilon_R^2 \neq 0$ because the equilibrium is stable. It can be shown that this transformation is invertible (on appropriate Banach spaces; see [15]). Eq. (11) is the coordinate part of the canonical transformation generated by the type-two mixed variable generating functional

$$\mathscr{F}[q_k, P'_k] = \sum_{k=1}^{\infty} \int \mathrm{d}u P'_k \mathscr{G}_k^{-1}[q_k], \tag{12}$$

which follows from $Q'_k = \delta\mathscr{F}/\delta P'_k$. The momentum part of the canonical transformation is obtained from $p_k = \delta\mathscr{F}/\delta q_k$, which gives $p_k(v,t) = (\mathscr{G}_k^{-1})^\dagger[P'_k]$, where $\mathscr{G}_k^{-1}$ is given by

$$Q'_k(u,t) = \mathscr{G}_k^{-1}[q_k] = \frac{\epsilon_R(k,u)}{|\epsilon|^2(k,u)} q_k(u,t) - \frac{\epsilon_I(k,u)}{|\epsilon|^2(k,u)} H[q_k]. \tag{13}$$

Now, the diagonalizing coordinates are given by the simple transformation

$$Q_k = \frac{1}{\sqrt{2}}(Q'_k - \mathrm{i}P'_k) \quad \text{and} \quad P_k = \frac{1}{\sqrt{2}}(P'_k - \mathrm{i}Q'_k), \tag{14}$$

in terms of which the Hamiltonian for the linearized system becomes

$$H_L = \frac{1}{2}\sum_{k=1}^{\infty} \int \mathrm{d}u\, ku(Q_k^2 + P_k^2). \tag{15}$$

Observe that H_L is the Hamiltonian for a sum over a continuum of simple harmonic oscillators indexed by the discrete label k and the continuum label u, with the frequency of a given oscillator being ku.

The calculation leading to (15) rests on the following two identities:

$$\mathscr{G}^{-1}[vf] = u\mathscr{G}^{-1}[f] - \frac{\epsilon_I}{\pi|\epsilon|^2}\int \mathrm{d}v f \quad \text{and} \quad \mathscr{G}^{-1}[\epsilon_I] = \frac{\epsilon_I}{|\epsilon|^2}, \tag{16}$$

which are based on properties of the Hilbert transform. These were first proven in [12] and later with greater rigor in [15]. We take them as given and direct the reader to these references for their proofs.

In terms of the canonical variables (q_k, p_k) the first term of H_L is given by

$$A = -\frac{m}{2}\sum_k \int \mathrm{d}v v \frac{|f_k|^2}{f_0'} = -\sum_{k=1}^{\infty} \mathrm{i}k \int \mathrm{d}v v q_k p_k. \tag{17}$$

Upon inserting $p_k(v,t) = (\mathscr{G}_k^{-1})^\dagger[P'_k]$ into (17), flipping the '†', and making use of the identity of (16), we obtain

$$A = -\sum_{k=1}^{\infty} \mathrm{i}k \int \mathrm{d}u P'_k \left(u\mathscr{G}_k^{-1}[q_k] - \frac{\epsilon_I}{|\epsilon|^2}\frac{1}{\pi}\int \mathrm{d}v f_k \right). \tag{18}$$

The second term of H_L is

$$B = \sum_{k=1}^{\infty} \frac{4\pi e^2}{k^2} \int \mathrm{d}v f_k \int \mathrm{d}u f_{-k} = \sum_{k=1}^{\infty} \mathrm{i}k \int \mathrm{d}v f_k \frac{1}{\pi}\int \mathrm{d}u p_k \epsilon_I, \tag{19}$$

into which we insert $p_k(v,t) = (\mathscr{G}_k^{-1})^\dagger[P'_k]$ in the last term, flip the '†', and make use of the second identity of (16), to obtain

$$B = -\sum_{k=1}^{\infty} \mathrm{i}k \int \mathrm{d}v f_k \frac{1}{\pi}\int \mathrm{d}u P'_k \mathscr{G}_k^{-1}[\epsilon_I] = -\sum_{k=1}^{\infty} \mathrm{i}k \int \mathrm{d}v f_k \frac{1}{\pi}\int \mathrm{d}u P'_k \frac{\epsilon_I}{|\epsilon|^2}. \tag{20}$$

Adding (18) and (20) gives

$$H_L = -\sum_{k=1}^{\infty} \mathrm{i}k \int \mathrm{d}u u P'_k \mathscr{G}_k^{-1}[q_k] = -\sum_{k=1}^{\infty} \mathrm{i}k \int \mathrm{d}u u P'_k Q'_k. \tag{21}$$

Finally, inserting the transformation of (14) into (21) produces (15).

4. Calculating the functional integral spectra

Having achieved diagonal form, we can evaluate Z and use it to obtain averages. In this section we first obtain electric field fluctuation spectra or equivalently charge density fluctuation spectra, and the concomitant form factors. Following this we obtain phase space density fluctuation spectra.

4.1. Electric field spectra

We evaluate (1) by writing it in terms of the canonical coordinates (Q_k, P_k): these canonical variables are inserted into H_L, the resulting expression is used for H, and the measure $\mathscr{D}p\mathscr{D}q$ is taken to be $\prod_k \mathscr{D}q_k\mathscr{D}p_k$. The continuum labels of Q_k and P_k are discretized on $u_j = j\Delta u$, for $j = -N, \ldots, N$, where $\Delta u = v_*/N$, and then the limits $v_*, N \to \infty$ are taken (see e.g. [19] and references therein). The ensemble average of a quantity $\mathscr{O}$ is given by $\langle \mathscr{O} \rangle = \int \mathscr{D}Q\mathscr{D}P\mathscr{O}\mathrm{e}^{-\beta H}/Z$. In terms of the above discretization,

$$\langle \mathscr{O} \rangle = \lim_{\substack{N\to\infty \\ v_*\to\infty}} \frac{1}{Z} \prod_{k=1} \prod_{j=-N}^{N} \int \mathrm{d}P_k(u_j)\mathrm{d}Q_k(u_j)\mathscr{O}\mathrm{e}^{-\beta H_L}. \tag{22}$$

After a straightforward calculation we obtain

$$\langle E_k(u)E^*_{k'}(u') \rangle = \frac{16k_\mathrm{B}T}{V} \frac{\epsilon_I}{u|\epsilon|^2} \delta_{k,k'}\delta(u-u'). \tag{23}$$

where V is the volume of the system and the units are chosen to compare with the usual expression for the dielectric energy (cf. [12]). The details of this calculation are given below in Section 4.1.1, which is followed by a discussion of the forumla of (23) in Sections 4.1.2 and 4.1.3.

4.1.1. The calculation

With the discretization described above, the calculation of (23) proceeds as follows: the Hamiltonian becomes

$$H_L = \frac{1}{2} \prod_{k=1} \int \mathrm{d}u\, ku(P_k^2 + Q_k^2) = \frac{1}{2} \lim_{\substack{v_*\to\infty \\ N\to\infty}} \Delta u \prod_{k=1} \sum_{j=-N}^{N} ku_j(P_{kj}^2 + Q_{kj}^2), \tag{24}$$

where $Q_{kj} = Q_k(u_j)$ and $P_{kj} = P_k(u_j)$. Since the energy density vanishes at $u = 0$, the discretized dynamics corresponding to $j = 0$ is trivial. Thus we exclude $j = 0$ from the functional integral. Using the above, the partition function becomes

$$\begin{aligned} Z &= \lim_{\substack{v_*\to\infty \\ N\to\infty}} \prod_{k=1} {\prod_{j=-N}^{N}}' \int \mathrm{d}P_{kj}\mathrm{e}^{-\beta\Delta uku_jP_{kj}^2/2} \int \mathrm{d}Q_{kj}\mathrm{e}^{-\beta\Delta uku_jQ_{kj}^2/2} \\ &= \lim_{\substack{lv_*\to\infty \\ N\to\infty}} \prod_{k=1} {\prod_{j=-N}^{N}}' \sqrt{\frac{2\pi}{ku_j\Delta u\beta}} \sqrt{\frac{2\pi}{ku_j\Delta u\beta}} = \lim_{\substack{v_*\to\infty \\ N\to\infty}} \prod_{k=1} \left(\frac{2\pi}{k\Delta u\beta}\right)^{2N} {\prod_{j=-N}^{N}}' \frac{1}{u_j}, \end{aligned} \tag{25}$$

where primes denote $j = 0$ is excluded from the product. Now $u_j = j\Delta u$ and

$${\prod_{j=1}^{N}}' \frac{1}{u_j} = \frac{1}{\Delta u^N N!} = \left(\frac{N}{v_*}\right)^N \frac{1}{N!} \tag{26}$$

leading to

$$Z = \lim_{\substack{v_*\to\infty \\ N\to\infty}} \left(\frac{2\pi}{k\beta}\right)^{2N} \left(\frac{N}{v_*}\right)^{4N} \frac{1}{N!^2}. \tag{27}$$

For this discretization, depending on the order in which the limits are taken, we obtain either $Z = 0$ or $Z = \infty$. (We regularize these expressions by first taking the limit $v_* \to \infty$, followed by $N \to \infty$.) We may still perform

the usual formal manipulations to obtain ensemble averages by postponing the limits until the final step of the calculation.

En route to (23) we calculate the following ensemble average

$$\langle Q_k(u)Q_{k'}(u')\rangle = \lim_{\substack{v_*\to\infty\\ N\to\infty}} \frac{1}{Z}\prod_{\substack{\ell\neq k\\ \ell\neq k'}}\prod_{\substack{j\neq m\\ j\neq m'}}{}' \int \mathrm{d}P_{\ell j}\mathrm{d}Q_{\ell j}\mathrm{e}^{-\beta\Delta u\ell u_j\left(P_{\ell j}^2+Q_{\ell j}^2\right)/2} \times \int \mathrm{d}P_{km}\mathrm{d}Q_{km}Q_{km}\mathrm{e}^{-\beta\Delta uku_m\left(P_{km}^2+Q_{km}^2\right)/2} \times \int \mathrm{d}P_{k'm'}\mathrm{d}Q_{k'm'}Q_{k'm'}\mathrm{e}^{-\beta\Delta uk'u_{m'}(P_{k'm'}^2+Q_{k'm'}^2)/2} \tag{28}$$

where m and m' are chosen such that, in the continuum limit, $u_m = u$ and $u_{m'} = u'$. Unless $m = m'$ and $k = k'$, the above vanishes. Thus we have

$$\langle Q_k(u)Q_{k'}(u')\rangle = \delta_{kk'}\delta_{mm'}\lim_{\substack{v_*\to\infty\\ N\to\infty}}\frac{1}{Z}\int \mathrm{d}P_{km}\mathrm{d}Q_{km}Q_{km}^2\mathrm{e}^{-\beta\Delta uku_m\left(P_{km}^2+Q_{km}^2\right)/2} \times \prod_{\ell\neq k}\prod_{j\neq m}{}' \int \mathrm{d}P_{\ell j}\mathrm{d}Q_{\ell j}\mathrm{e}^{-\beta\Delta u(P_{\ell j}^2+Q_{\ell j}^2)/2}. \tag{29}$$

Upon canceling integrals in the numerator with those of Z in the denominator, we obtain the ratio,

$$\frac{\int \mathrm{d}Q_{km}Q_{km}^2\mathrm{e}^{-\beta\Delta uku_mQ_{km}^2/2}}{\int \mathrm{d}Q_{km}\mathrm{e}^{-\beta\Delta uku_mQ_{km}^2/2}} = \frac{1}{ku_m\Delta u\beta}, \tag{30}$$

whence we obtain $\langle Q_k(u)Q_{k'}(u')\rangle = \lim \delta_{kk'}\delta_{mm'}/ku_m\Delta u\beta$. Now $\lim_{\substack{v_*\to\infty\\ N\to\infty}}(\delta_{mm'}/\Delta u) = \delta(u-u')$ gives the compact expression

$$\langle Q_k(u)Q_{k'}(u')\rangle = \frac{k_\mathrm{B}T}{ku}\delta_{k,k'}\delta(u-u'). \tag{31}$$

The expected UV catastrophe results upon summing and integrating over the arguments.

Paralleling the above calculation leads to $\langle P_k(u)P_{k'}(u')\rangle = k_\mathrm{B}T/(ku)\delta_{k,k'}\delta(u-u')$ and $\langle Q_k(u)P_{k'}(u')\rangle = 0$, which together with (14), imply

$$-\mathrm{i}k\langle uQ_k'(u)P_{k'}'(u')\rangle = k_\mathrm{B}T\delta_{k,k'}\delta(u-u').$$

Now we know q_k is equal to f_k, but what are the physical meanings of the Q_k' and P_k'? These are the coordinates in which the linear dynamics is decoupled. In particular Q_k' can be shown to be proportional to the electric field, $E_k(u,t)$ $(k>0)$, associated with a van Kampen mode. In [12] it was shown the energy density for stable electrostatic oscillations is given by

$$\frac{V}{16}\frac{|\epsilon|^2}{\epsilon_I}u|E_k(u)|^2 = -\mathrm{i}kuQ_k'(u)P_k'(u)$$

(see Eq. (27) of [15]). Thus we obtain (23). Note we have written this expression in units to compare with the usual expression for the dielectric energy

$$\mathscr{E}_D = \frac{V}{16\pi}\frac{\partial(\omega\epsilon_R)}{\partial\omega}|E_k|^2,$$

a common but generally *incorrect* expression for the energy density of stable electrostatic oscillations described by the Vlasov equation (see [12]).

4.1.2. Equipartition

An expression similar to that of (23) appeared in the previous calculations [1,3,5–8] referred to above in Section 1. In fact, it can be argued that (23) is actually what is meant by the statistical part of the fluctuation–dissipation theorem, but we will not pursue this here. For the case of Maxwellian equilibria where the

Maxwellian temperature is equal to the heat bath temperature, our expression is identical to that of previous authors (first given by Thompson[5,6]). We discuss this agreement further in Section 5.

In [1–8] it is noted that the right hand side of (23) approaches $\frac{1}{2}k_{\mathrm{B}}T$ in the limit $k\lambda_D \ll 1$, which suggests a failure of the equipartition theorem when this limit is not taken. This arises because of the factor involving ϵ. However, one should not expect E_k to be in equipartition because it is not a canonical variable. In statistical mechanics, equipartition is a property defined in terms of the canonical variables in which the Hamiltonian is diagonal. For each quadratic term in the Hamiltonian one obtains a contribution of $k_{\mathrm{B}}T/2$ to the expectation value of the energy. Thus, the statement of equipartition in the present plasma physics context is (31), and (23) is entirely consistent with equipartition for all values of k. Our approach that uses the Hamiltonian formulation of Vlasov theory makes this clear.

4.1.3. Form factor

Using (23), we can compute both the dynamic and static form factors. The dynamic form factor, $S(k,\omega)$, is defined (e.g. [3]) in terms of the density fluctuations, $\rho_k(\omega)$, by $\langle \rho_k(\omega)\rho_k^*(\omega')\rangle = 4\pi^2\delta(\omega-\omega')S(k,\omega)$. Because $\langle \rho_k(\omega)\rho_k^*(\omega')\rangle$ is proportional to (23); this fact and some algebra yields the well-known result

$$S(k,\omega) = -\frac{N}{\pi\omega}\frac{k^2}{k_D^2}\mathrm{Im}\left(\frac{1}{\epsilon(k,\omega)}\right), \tag{32}$$

where N is the total number of particles. Thus our result is consistent in this respect with previous results.

The static form factor, $S(k)$, is defined by the sum-rule $\int \mathrm{d}\omega S(k,\omega) = NS(k)$, hence

$$S(k) = \frac{k^2}{k_D^2}\overline{\mathrm{Im}\left(\frac{1}{\epsilon(k,\omega)}\right)}(0). \tag{33}$$

Without loss of generality we choose a frame where f_0 has a maximum at $v=0$, thus $\epsilon_I(0)=0$. We define $\epsilon_R(0) = 1 + k_\theta^2/k^2$, where k_θ is a measure of the width of f_0. (For a Maxwellian equilibrium, $k_\theta = k_D$.) Using, $\overline{\epsilon_I}(0) = \epsilon_R(0) - 1$, we obtain

$$S(k) = \frac{k^2}{k^2 + k_\theta^2} \tag{34}$$

which is the standard expression that describes both self-correlation and Debye shielding (see e.g. [3]).

4.2. Phase space fluctuation spectra

Since $\mathscr{G}$ is a linear functional, E_k is linearly related to f_k and this fact can be exploited to obtain phase space fluctuation spectra. The basic point is that the ensemble average can be taken inside the functional. Thus (23) can be used to obtain $\langle f_k(v) f_{k'}^*(v')\rangle$ by mapping back from (Q_k, P_k) to (q_k, p_k) and writing the result in terms of f_k. Accounting for the scalings in the definitions of the various variables this amounts to

$$\langle f_k(v) f_{k'}^*(v')\rangle = \frac{kk'}{16\pi^2 e^2}\langle \mathscr{G}_k[E_k](v)\mathscr{G}_{k'}[E_{k'}^*](v')\rangle, \tag{35}$$

whence we obtain

$$\langle f_k(v) f_{k'}^*(v')\rangle = \delta_{k,k'}\frac{k^2}{\pi^2 e^2 V\beta}\left\{\frac{\epsilon_I(k,v)}{v}\delta(v-v') - \frac{1}{\pi}\frac{\epsilon_R(k,0)}{|\epsilon(k,0)|^2}\frac{\epsilon_I(k,v')\epsilon_I(k,v)}{vv'}\right\}. \tag{36}$$

This is a rigorous calculation, the details of which are presented below. We note that this general result is not in [1–8], although the special case where the equilibrium is Maxwellian appears in [1].

4.2.1. The calculation

Our calculation of $\langle f_k(v) f_{k'}^*(v')\rangle$ uses the fact that the two $\mathscr{G}$'s of (35) can be slipped out of the average, $\langle\ \rangle$, together with several Hilbert transform identities (see [12–15]), which we record below:

$$\overline{\frac{1}{u-u'}} = \pi\delta(u-u'), \quad \overline{\delta(u-u')} = -\frac{1}{\pi}\frac{1}{u-u'} \tag{37}$$

$$\overline{\frac{\phi(u)}{u-u'}} = \frac{\bar{\phi}(u)-\bar{\phi}(u')}{u-u'} + \pi\phi(u)\delta(u-u'), \quad \overline{\phi\psi} = \bar{\phi}\psi + \phi\bar{\psi} + \overline{\bar{\phi}\bar{\psi}}. \tag{38}$$

Defining $g(u)$ by $\langle E_k(u)E^*_{kc}(u')\rangle = 16\delta_{kk'}\delta(u-u')g(u)/(V\beta)$, then $\psi(u',v) = \mathcal{G}[g(u)\delta(u-u')](v)$ and $F(v,v') = \mathcal{G}[\psi(u',v)](v')$ is the essential part of (36). Note, for the calculation we suppress the dependence on k and write $\alpha = \epsilon_I$ and $\beta = \epsilon_R$, so $g(u) = \zeta(u)/u$ where $\zeta = \alpha/(\alpha^2+\beta^2)$. Now

$$\psi(u',v) = \frac{\alpha(v)}{\pi}\frac{g(u')}{u'-v} + \beta(v)g(v)\delta(v-u') \tag{39}$$

giving

$$F(v,v') = \alpha(v') - \frac{\beta(v)}{\pi}\frac{1}{v'-v} + \frac{\alpha(v)}{\pi}\left\{\frac{\bar{g}(v')-\bar{g}(v)}{v'-v} + \pi g(v)\delta(v-v')\right\} + \beta(v')\left\{\frac{\alpha(v)}{\pi}\frac{g(v')}{v'-v} + \beta(v)g(v)\delta(v-v')\right\}. \tag{40}$$

Rearranging, we have

$$F(v,v') = \delta(v-v')g(v)[\alpha^2(v)+\beta^2(v)] + \frac{1}{\pi}\frac{\alpha(v)}{v'-v}[\beta(v')g(v') + \alpha(v')\bar{g}(v')] - \frac{1}{\pi}\frac{\alpha(v')}{v'-v}[\beta(v)g(v) + \alpha(v)\bar{g}(v)]. \tag{41}$$

Now (41) can be rewritten using

$$\bar{g}(v) = \frac{\bar{\zeta}(0)-\bar{\zeta}(v)}{v} = \frac{\chi(0)-\chi(v)}{v}. \tag{42}$$

Using $\alpha\chi + \beta\zeta = 0$, we obtain

$$F(v,v') = \delta(v-v')\frac{\alpha(v)}{v} - \frac{1}{\pi}\chi(0)\frac{\alpha(v')\alpha(v)}{vv'} \tag{43}$$

whence finally we obtain (36).

5. Comparisons and discussion

Our approach is perhaps most akin to that of classical N-particle statistical mechanics of an electron gas neutralized by a positive charge background (see e.g. [4]), where the classical partition function is constructed for N interacting electrons. Calculation of the partition function in the N-particle approach is difficult because of the Coulomb interaction. Consequently, the partition function is expanded and coarse grained, and eventually written as a product of one-particle partition functions. En route, a diagonalization of a discrete Hamiltonian is effected. Thus the N-particle approach, like ours, involves diagonalization within a Hamiltonian context. The N-particle dynamics is most basic in plasma physics, but the partition function obtained after approximation is not that for any known dynamics. In contrast, the linearized dynamics of our approach is limiting, but our partition function calculations are exact. The N-particle approach produces the static form factor of (34) with $k_\theta = k_D$, the Maxwellian special case of our result (33).

The Liouville, Klimontovich, and test particle approaches contain discrete particle dynamics in some form, which is then smoothed and truncated, in all cases leading to more or less the same answers. Equations like (23) appear in these calculations, and fluctuation information needed for calculation of the Lenard–Balescu collision operator is calculated. However, none except Klimontovich explicitly obtain an expression for $\langle f_k(v)f^*_{k'}(v')\rangle$. If we insert a Maxwellian distribution function into our result (36), it reduces to his expression (cf. Eq. (10.38) of [1]).

One may wonder how Vlasov theory, being ostensibly collisionless, can produce results about correlations in agreement with the above calculations. Essentially this is possible because the Vlasov equation is identical to

the Klimontovich equation, and thus contains the correct dynamics on small velocity scales. The distinction between the two equations amounts to a choice of initial conditions. Physically it is the interaction with the heat bath that communicates the interaction in the plasma, and the heat bath does not distinguish between Vlasov and Klimontovich theory. (Recall, in Einstein's calculation the Hamiltonian is the sum over independent linear oscillators.) That Vlasov theory can correctly produce correlations is not a new idea, for it was used in [5] and has also been used to obtain Lenard–Balescu type collision operators (see e.g. [20,21]). However, for this to work it is essential that the linear operator equation be solved exactly. Calculations involving expansions that lead to Landau damping eliminate the essential transient effects. Landau damping sets in on the time scale $\tau_L := max\{v_e/k, v_{ic}/k\}$ (e.g. [4]), where v_e and v_{ic} are velocities characteristic of the equilbrium and initial condition, respectively. So, theories with approximations leading to Landau damping do not contain transient effects, which are continually generated by jostling by the bath, that occur on a time scale less that τ_L. Thus, these theories are incomplete and this will show up in the statistics.

Our generalization to non-Maxwellian equilibria is important because hot plasmas can exist in states different from thermodynamic equilibrium for substantial lengths of time. The temperature associated with these equilibrium states need not be the same as the heat bath temperature $T = 1/(k_B\beta)$ that characterizes the fluctuations. The thermal nature of the heat bath arises from the large number of degrees of freedom that couple to the plasma, which can be distinct from the temperature associated with a prepared equilibrium state. It is important to recognize that the equilibrium temperature in our calculation is merely a parameter that describes the equilibrium state f_0, about which the linear dynamics occurs. Non-Maxwellian equilibria could be described by general equilibria of the form $f_0(v; T_1, T_2 \ldots)$ with many parameters, and these parameters are distinct from the bath temperature T.

In the case where f_0 is not Maxwellian our expression (23) does not agree with that of [5,8], two places where a nonequilibrium expressions has been given. It is difficult to judge the result of [5,8] because in [5] the result is only stated (probably based on an understanding of the Nyquist noise theorem), while there are difficulties with the nonequilibrium derivation of [8] that uses an expansion in terms of the charge. On course a meaningful asymptotic expansion requires a dimensionless parameter, and in the equilibrium case this amounts to the plasma parameter. The plasma parameter requires the introduction of a temperature and in the equilibrium case this is supplied by the Maxwellian temperature. However, in the nonequilibrium calculations of [8], no such temperature is available and one wonders what exactly is the dimensionless parameter. Thus, at present it is unclear why the results differ. One possibility is the linearity of our approach.

The approach presented in this paper is of general applicability. It is straightforward to include multiple species, and the essential ingredients, the Hamiltonian structure and the diagonalizing integral transform, exist for electromagnetic fluctuations [14]. In fluid mechanics, functional integral calculations of partition functions for homogeneous turbulence have existed since the work of Onsager [22] and Lee [23] (see also [24–29]). The methods described here can be used to describe the fluctuations about inhomogeneous fluid states such as those that occur in shear flow and Rossby (or Drift) wave dynamics, because the diagonalizing transform has been worked out [30,31]. For the case of shear flow the calculations have been performed and it has been shown that the resulting formulas explain dominant features of experiments [35]. Approximate fluid and plasma systems, such as the single wave model (e.g. [34]) and vorticity defect dynamics [33] are also amenable. Basically, for any stable equilibrium of a general class of Hamiltonian systems with a continuous spectrum [32] an analogous calculation can be performed.

An important limitation of our approach is the use of linear dynamics; consequently large fluctuations are not properly described. One can view the calculation presented here as a first step in a perturbation calculation and continue this avenue by proceeding to higher order. Alternatively, nonlinearity can be included by other means, as it was for the shear flow problem in [29,35]. The relationship between the present approach and the approach of that work will be described elsewhere.

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