Annals of Physics 324 (2009) 1747–1762

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

The Hamiltonian description of incompressible fluid ellipsoids

P.J. Morrison [a,*], Norman R. Lebovitz [b], Joseph A. Biello [c]

[a] *Department of Physics and Institute for Fusion Studies, The University of Texas at Austin, Austin, TX 78712, USA*
[b] *Department of Mathematics, The University of Chicago, Chicago, IL 60637, USA*
[c] *Department of Mathematics, University of California at Davis, USA*

ARTICLE INFO

Article history:
Received 26 November 2008
Accepted 9 April 2009
Available online 3 May 2009

PACS:
111.1
222.2

Keywords:
Hamiltonian reduction
Noncanonical Poisson bracket
Lie–Poisson bracket
Fluid dynamics
Riemann ellipsoids
Dirac brackets
Dirac constraints

ABSTRACT

We construct the noncanonical Poisson bracket associated with the phase space of first order moments of the velocity field and quadratic moments of the density of a fluid with a free-boundary, constrained by the condition of incompressibility. Two methods are used to obtain the bracket, both based on Dirac's procedure for incorporating constraints. First, the Poisson bracket of moments of the unconstrained Euler equations is used to construct a Dirac bracket, with Casimir invariants corresponding to volume preservation and incompressibility. Second, the Dirac procedure is applied directly to the continuum, noncanonical Poisson bracket that describes the compressible Euler equations, and the moment reduction is applied to this bracket. When the Hamiltonian can be expressed exactly in terms of these moments, a closure is achieved and the resulting finite-dimensional Hamiltonian system provides exact solutions of Euler's equations. This is shown to be the case for the classical, incompressible Riemann ellipsoids, which have velocities that vary linearly with position and have constant density within an ellipsoidal boundary. The incompressible, noncanonical Poisson bracket differs from its counterpart for the compressible case in that it is not of Lie–Poisson form.

1. Introduction

The Euler equations governing the velocity field $\mathbf{v}$, density ρ and pressure p of an inviscid fluid are

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\rho^{-1}\nabla p + \mathbf{f} \quad \text{and} \tag{1}$$

$$\partial_t \rho + \operatorname{div}(\rho\mathbf{v}) = 0 \tag{2}$$

* Corresponding author.
E-mail address: morrison@physics.utexas.edu (P.J. Morrison).

0003-4916/$ - see front matter
doi:10.1016/j.aop.2009.04.003

where **f** denotes an as yet unspecified force. These equations must be augmented by boundary and initial data, and by further conditions relating the variables $\mathbf{v}$, p and ρ: either an equation of state (in the compressible case) or the condition $\operatorname{div}\mathbf{v} = 0$ (in the incompressible case). We shall be interested in exploring their Hamiltonian structure in a particular context. Our principal reference for a general discussion of this structure and the derivations of the corresponding brackets will be [1].

Exact solutions of the Euler equations are possible only under simplifying assumptions and in simple contexts. A family of solutions in the context of astrophysics, namely, where the force term **f** includes the self-gravitational effects of the fluid mass, exists under the assumption of a fluid of uniform density confined to an ellipsoidal domain, with a velocity field linear in the coordinates. These assumptions reduce the Euler equations to a finite system of ordinary differential equations. The equations for these *Riemann ellipsoids* have been widely investigated: their study goes back to the work of Dirichlet [2] and of Riemann [3], but our principal reference for this will be [4]. We summarize their properties in Appendix A, which we will refer to often (in Appendix B. we describe four natural frames of reference for the ellipsoids, which are included here because they do not appear to have been published together elsewhere).

In none of these references was the Hamiltonian nature of the finite-dimensional system emphasized. This was first addressed by Rosensteel [5]. His starting point was the so-called *virial* method originally introduced to investigate the stability of steady solutions of the Euler equations. The *virial* is a moment of the form

$$\mathcal{M}_{ij} = \int_D \rho x_i v_j \, d^3x = \int_D x_i M_j \, d^3x, \tag{3}$$

where $i,j = 1,2,3$, and the second form introduces the specific momentum of the fluid $\mathbf{M} = \rho\mathbf{v}$. This moment is considered together with another moment, equivalent to the moment-of-inertia tensor,

$$\Sigma_{ij} = \int_D \rho x_i x_j \, d^3x \, . \tag{4}$$

Rosensteel presents an algebra for these moments, i.e., bracket relations among them that are closed, and that provide a noncanonical Hamiltonian description of the Riemann ellipsoids with a certain choice of the Hamiltonian function $H(\Sigma, \mathcal{M})$; we present these relations below, in Eqs. (12)–(14).

We call attention to two features of Rosensteel's description of the incompressible case:

1. The bracket relations are presented without reference to the fluid-dynamics Eqs. (1) and (2) above, and
2. The formulation requires a Hamiltonian function other than the total energy as well as the imposition of extraneous constraints.

The feature (1) is addressed in Section 2 below, where we derive Rosensteel's bracket relations in a straightforward way via a moment reduction of the general fluid-dynamical bracket (7). Feature (2) is discussed in detail in Sections 5 and 6.

We view the fluid as incompressible. This is natural because the density of the Riemann ellipsoids is spatially uniform.[1] However, Rosensteel's bracket does not constrain the fluid to be incompressible, and we therefore modify it via Dirac's procedure for incorporating constraints. Dirac's method is described in Section 3. We observe in Section 4 that one can alternatively first apply Dirac's procedure and subsequently effect a moment reduction, with the same result. The resulting Dirac bracket is no longer of Lie–Poisson type: the bracket relations depend nonlinearly on the moments.

In Section 5 we relate the noncanonical Hamiltonian equations obtained from Rosensteel's bracket to the equations describing the Riemann ellipsoids and show that, if the Hamiltonian is taken to be the total energy, the pressure term from fluid dynamics is missing. It can be restored by adding an extra term to the Hamiltonian. In Section 6, we show that the Hamiltonian equations obtained from the Dirac bracket using the total energy as Hamiltonian give the full equations for the Riemann ellipsoids,

[1] There are also applications allowing for compressibility wherein $\rho = \rho(t)$, i.e., the density is spatially uniform but varies with time. We do not address these cases here.

and, moreover, avoid the necessity of imposing any further constraints. Finally, in Section 7 we summarize and discuss these results.

2. The Lie–Poisson bracket and its moment reduction

The Euler Eqs. (1) and (2) can be re-expressed in terms of the momentum density, $\mathbf{M} := \rho\mathbf{v}$, as

$$\partial_t\mathbf{M} + \mathbf{v}\cdot\nabla\mathbf{M} + (\mathrm{div}\,\mathbf{v})\mathbf{M} = -\nabla p + \rho\mathbf{f} \quad \text{and} \tag{5}$$

$$\partial_t\rho + \mathrm{div}\,\mathbf{M} = 0. \tag{6}$$

These, like Eqs. (1) and (2), will be referred to as unconstrained, since neither the constraint of incompressibility nor that of an equation of state has yet been imposed.

The Hamiltonian description of these equations is reviewed in [1]. The noncanonical Poisson bracket, as given in [6], is[2]

$$\{F,G\}_M = \int_{IR^3} M_i\left(\frac{\delta G}{\delta M_j}\frac{\partial}{\partial x_j}\frac{\delta F}{\delta M_i} - \frac{\delta F}{\delta M_j}\frac{\partial}{\partial x_j}\frac{\delta G}{\delta M_i}\right)d^3x + \int_{IR^3}\rho\left(\frac{\delta G}{\delta\mathbf{M}}\cdot\nabla\frac{\delta F}{\delta\rho} - \frac{\delta F}{\delta\mathbf{M}}\cdot\nabla\frac{\delta G}{\delta\rho}\right)d^3x. \tag{7}$$

This is a Lie–Poisson bracket (i.e., is linear in the variables $\mathbf{M}$ and ρ). It is implicit in the derivation of this bracket that the integrals are convergent, i.e., the density and momentum variables, and the functions of them that appear in the integrals, fall off sufficiently fast at large distances. The subscript M indicates that this version employs the momentum (as opposed to the velocity) as a dynamical variable. Since several brackets appear below, we will use subscripts to distinguish among them. This bracket, like the versions of the Euler equations given above, is unconstrained. It allows for compressibility, which is, however, expressed explicitly only in the Hamiltonian:

$$H = \int_{IR^3}\left(\frac{|\mathbf{M}|^2}{2\rho} + \rho U(\rho) + \rho\chi\right)d^3x, \tag{8}$$

where $\mathbf{f} = \nabla\chi$ and U represents the internal energy. This bracket and this Hamiltonian generate the compressible Euler equations with the pressure given by

$$p = \rho^2\frac{\partial U}{\partial\rho}\,. \tag{9}$$

We can apply the bracket (7) above to the functionals $\mathcal{M}_{ij}, \Sigma_{ij}, i,j = 1,3$. Since (see Eqs. (3) and (4) above)

$$\frac{\delta\mathcal{M}_{ij}}{\delta M_k} = x_i\delta_{jk} \quad \text{and} \quad \frac{\delta\mathcal{M}_{ij}}{\delta\rho} = 0; \tag{10}$$

$$\frac{\delta\Sigma_{ij}}{\delta M_k} = 0 \quad \text{and} \quad \frac{\delta\Sigma_{ij}}{\delta\rho} = x_i x_j, \tag{11}$$

we easily find the following bracket relations:

$$\{\Sigma_{ij},\Sigma_{kl}\}_R = 0 \tag{12}$$

$$\{\mathcal{M}_{ij},\mathcal{M}_{kl}\}_R = \delta_{il}\mathcal{M}_{kj} - \delta_{jk}\mathcal{M}_{il} \tag{13}$$

$$\{\Sigma_{ij},\mathcal{M}_{kl}\}_R = \delta_{il}\Sigma_{jk} + \delta_{jl}\Sigma_{ik}. \tag{14}$$

These are precisely the relations obtained by Rosensteel by other means ([5], Eq. (134)); hence the index R. Since they are obtained directly from the unconstrained bracket (7), they must be likewise unconstrained.

3. Dirac bracket for the moment formulation

We address here the constraint of incompressibility, which is not incorporated in the bracket relations (12)–(14) above. We do this with the aid of the Dirac bracket formalism. We begin this section by

[2] Unless otherwise indicated, repeated Latin indices are summed from 1 to 3.

defining this and conclude by giving the relations (25)–(27) for the Dirac bracket obtained from Rosensteel's bracket.

3.1. The Dirac bracket

Given a bracket $\{\cdot,\cdot\}$, canonical or noncanonical, and an even number $2k$ of phase-space functions $\{C^\mu\}_1^{2k}$, one can define a new bracket for which these functions are Casimir invariants. This (so-called) Dirac bracket is constructed as follows:

$$\{F,G\}_D = \{F,G\} - \sum_{\mu,\nu=1}^{2k} \{F,C^\mu\}\,\omega^{-1}_{\mu\nu}\,\{C^\nu,G\} \tag{15}$$

where

$$\omega^{\mu\nu} = \{C^\mu, C^\nu\}; \tag{16}$$

it is further assumed that ω, an antisymmetric matrix function of the dynamical variables, is invertible. The following observations follow directly from this definition:

1. Each of the functions C^μ is a Casimir invariant for the Dirac bracket
2. Any Casimir invariant of the original bracket $\{\cdot,\cdot\}$ is likewise a Casimir invariant for $\{\cdot,\cdot\}_D$.
3. $\{\cdot,\cdot\}_D$ is antisymmetric and satisfies the Leibnitz rule (as in Eq. (20) below).

It is less obvious but also true that it satisfies the Jacobi identity. This is proved in Appendix C below.

Suppose now that P is a constant of the motion in the dynamics provided by a particular Hamiltonian function H under the original bracket, but not a Casimir: $\{P,H\}=0$ but $\{P,G\}\neq 0$ for some phase-space function G. Then it is not guaranteed that P will be a constant of the motion in the dynamics provided by H under the modified bracket $\{\cdot,\cdot\}_D$; it is possible in principle that $\{P,H\}_D\neq 0$. An example of this is given in Appendix D. However, *some* constants of the motion P remain constants of the motion under the modified bracket. The following proposition is easily verified:

Proposition 1. *If $P=H$, the Hamiltonian, or if $\{P,C^\mu\}=0$ for each $\mu=1,2,\ldots,2k$, then P is a constant of the motion in the dynamics provided by H also under the Dirac bracket $\{\cdot,\cdot\}_D$.*

In the application of the present paper we find that the constants of the motion are in fact unchanged. We have $k=1$ and the functions C^1 and C^2 are given by Eq. (17) below. The only constants of the motion that are not Casimirs of the original bracket are the Hamiltonian H and the three components of the angular momentum

$$L_i = \epsilon_{ijk}\mathcal{M}_{jk}, \quad i=1,2,3.$$

These commute with C^1 and C^2 by virtue of the formulas (21) and (22) below. The persistence of the constants of the motion under the change of bracket follows therefore from the Proposition (1).

3.2. A pair of constraints

We choose for the original bracket that of Rosensteel, whose relations are given in Eqs. (12)–(14) above.

As discussed in Section 7 below, this bracket has a Casimir whose fluid-dynamical interpretation is the magnitude of the circulation vector. In order to construct the incompressible bracket we shall augment this algebra by adding two additional Casimirs C^1 and C^2 expressing the constancy of the volume and constancy of the divergence of the velocity field. We may express these in the forms

$$C^1 = \ln(\mathrm{Det}(\Sigma)) \text{ and } C^2 = \mathrm{Tr}\left(\Sigma^{-1}\mathcal{M}\right). \tag{17}$$

The explanation for these choices originates in the context of a fluid confined to an ellipsoidal domain and having velocity components that are linear in the cartesian coordinates. Consider C^1 first. The moment tensor Σ is symmetric and, when transformed to a principal-axis frame for the ellipsoid, takes the form

$$Q = (m/5)\mathrm{diag}\left(a_1^2, a_2^2, a_3^2\right) \tag{18}$$

where $a_1,\ a_2$, and a_3 are the principal-axis lengths and m is the total mass.[3] Therefore

$$\det(\Sigma) = (m/5)^3 (a_1 a_2 a_3)^2 \tag{19}$$

and C^1 as defined above is a constant of the motion as long as the volume $(4/3)\pi a_1 a_2 a_3$ is. Since for a figure of uniform density the constancy of the volume implies that of the density, the constancy of C^1 can be viewed equally as the constancy of the density ρ. Regarding C^2, we note that for a fluid having a linear velocity field $V = L(t)X$ for some matrix L, the divergence of the velocity is the trace of L, which should therefore vanish under the assumption of incompressibility. Substituting the expression for V into the moment Eq. (3), we find that $L^t = \Sigma^{-1}\mathcal{M}$. Therefore

$$\mathrm{Tr}(L) = \mathrm{Tr}(L^t) = \mathrm{Tr}(\Sigma^{-1}\mathcal{M})$$

and the velocity field is solenoidal if $C^2 = 0$.

3.3. Some useful formulas

The calculation of the Dirac bracket relations and of other related quantities needed below requires some preliminary formulas, which we record here. Two useful, general identities for matrices $A = (A_{ij})$ are

$$\frac{\partial A^{-1}_{ij}}{\partial A_{kl}} = -A^{-1}_{ik} A^{-1}_{lj} \quad \text{and} \quad \frac{\partial \det A}{\partial A_{ij}} = C_{ij}\,,$$

where C_{ij} is the cofactor of A_{ij}.

In order to apply the bracket relations to arbitrary functions of functionals, we use the derivative property of brackets: if $v_1, v_2, \ldots v_k$ are functionals and $g(v) = g(v_1, v_2, \ldots, v_k)$ is a real-valued function of them, then for any other functional u

$$\{u, g(v)\} = \sum_{i=1}^{k} \frac{\partial g}{\partial v_i}\{u, v_i\}. \tag{20}$$

We can now record the following relations for Rosensteel's bracket:

$$\left\{\Sigma_{ij}, C^1\right\}_R = 0 \quad \text{and} \quad \left\{\Sigma_{ij}, C^2\right\}_R = 2\delta_{ij}, \tag{21}$$

$$\left\{\mathcal{M}_{ij}, C^1\right\}_R = -2\delta_{ij} \quad \text{and} \quad \left\{\mathcal{M}_{ij}, C^2\right\}_R = \Sigma^{-1}_{in}\mathcal{M}_{nj} + \Sigma^{-1}_{jn}\mathcal{M}_{ni} \tag{22}$$

3.4. Dirac bracket for incompressible ellipsoids

Since there are only two constraints, the matrix ω has only one independent entry,

$$\omega^{1\,2} = -\omega^{2\,1} = \left\{C^1, C^2\right\}_R = \Sigma^{-1}_{kl}\{\Sigma_{kl}, \mathcal{M}_{ij}\}\Sigma^{-1}_{ij} = \Sigma^{-1}_{ij}\left(\delta_{jk}\Sigma_{il} + \delta_{jl}\Sigma_{ik}\right)\Sigma^{-1}_{kl} = 2\mathrm{Tr}\left(\Sigma^{-1}\right) \tag{23}$$

which implies

$$\omega^{-1} = \frac{-1}{2\mathrm{Tr}\left(\Sigma^{-1}\right)}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{24}$$

Thus, the relations for the Rosensteel–Dirac bracket become

[3] The total mass is conserved; it is a Casimir invariant of the bracket (7) (see [1]).

$$\{\Sigma_{ij}, \Sigma_{kl}\}_{RD} = 0, \tag{25}$$

$$\{\mathcal{M}_{ij}, \mathcal{M}_{kl}\}_{RD} = \{\mathcal{M}_{ij}, \mathcal{M}_{kl}\}_R + \frac{1}{\mathrm{Tr}(\Sigma^{-1})}\left(\delta_{ij}\left(\Sigma^{-1}_{kn}\mathcal{M}_{nl} + \Sigma^{-1}_{ln}\mathcal{M}_{nk}\right) - \delta_{kl}\left(\Sigma^{-1}_{in}\mathcal{M}_{nj} + \Sigma^{-1}_{jn}\mathcal{M}_{ni}\right)\right) \text{ and} \tag{26}$$

$$\{\mathcal{M}_{ij}, \Sigma_{kl}\}_{RD} = \{\mathcal{M}_{ij}, \Sigma_{kl}\}_R + \frac{2\delta_{ij}\delta_{kl}}{\mathrm{Tr}\left(\Sigma^{-1}\right)}. \tag{27}$$

These bracket relations are nonlinear, i.e., the Dirac bracket is no longer of Lie–Poisson type. We return later (Section 6) to a verification that they provide a Hamiltonian description of the equations governing the motions of the incompressible Riemann ellipsoids.

4. Dirac bracket for the fluid formulation

We obtain a different route to the Dirac bracket for the incompressible, Riemann ellipsoids if we first constrain the fluid bracket (7) and only subsequently perform the moment reduction. For this purpose we carry out the procedure embodied in Eq. (15) but for the original bracket we employ the fluid-dynamical bracket (7). We impose the same constraints C^1 and C^2 as defined in Eq. (17) and therefore need expressions for the brackets $\{F, C^1\}_M, \{F, C^2\}_M$ and $\{C^1, C^2\}_M$. For this we need the variational derivatives of C^1 and C^2 with respect to the variables $\mathbf{M}$ and ρ. Straightforward calculations lead to the following:

$$\frac{\delta C^1}{\delta M_k} = 0 \quad \text{and} \quad \frac{\delta C^1}{\delta \rho} = \Sigma^{-1}_{ij} x_i x_j; \tag{28}$$

$$\frac{\delta C^2}{\delta M_k} = \Sigma^{-1}_{kj} x_j \quad \text{and} \quad \frac{\delta C^2}{\delta \rho} = -\mathcal{M}_{kl}\Sigma^{-1}_{ik}\Sigma^{-1}_{jl} x_i x_j. \tag{29}$$

The expressions needed for modifying the bracket are easily obtained with the aid of Eqs. (28) and (29):

$$\left\{F, C^1\right\}_M = -2\Sigma^{-1}_{lm}\int_{R^3} \rho x_m \frac{\delta F}{\delta M_l}\, d^3x, \tag{30}$$

$$\left\{F, C^2\right\}_M = \int_{R^3} M_i\left[\Sigma^{-1}_{jl} x_l \frac{\partial}{\partial x_j}\frac{\delta F}{\delta M_i} - \Sigma^{-1}_{ij}\frac{\delta F}{\delta M_j}\right] d^3x + \int_{R^3} \rho\left[\Sigma^{-1}_{kj} x_j \frac{\partial}{\partial x_k}\frac{\delta F}{\delta \rho} + x_i \frac{\delta F}{\delta M_j}(A_{ji} + A_{ij})\right] d^3x, \tag{31}$$

where $A = \Sigma^{-1}\mathcal{M}\Sigma^{-1}$; and, from either of the preceding equations,

$$\left\{C^1, C^2\right\}_M = 2\mathrm{Tr}\left(\Sigma^{-1}\right). \tag{32}$$

The Dirac-constrained fluid bracket is therefore

$$\{F, G\}_{MD} = \{F, G\}_M + \frac{1}{2\mathrm{Tr}\left(\Sigma^{-1}\right)}\left(\left\{F, C^1\right\}_M\left\{C^2, G\right\}_M - \left\{F, C^2\right\}_M\left\{C^1, G\right\}_M\right), \tag{33}$$

where the index MD denotes the momentum-Dirac bracket, and the index M denotes the unconstrained fluid bracket (7). We next carry out the moment reduction with the aid of equations (10) and (11). We find

$$\left\{\Sigma_{ij}, C^1\right\}_M = 0 \quad \text{and} \quad \left\{\Sigma_{ij}, C^2\right\}_M = 2\delta_{ij}; \tag{34}$$

$$\left\{\mathcal{M}_{ij}, C^1\right\}_M = -2\delta_{ij} \quad \text{and} \quad \left\{\mathcal{M}_{ij}, C^2\right\}_M = \Sigma^{-1}_{ik}\mathcal{M}_{kj} + \Sigma^{-1}_{jk}\mathcal{M}_{ki}. \tag{35}$$

These are exactly the same expressions as found in the preceding section where the braces referred to the finite system of Rosensteel's relations (12)–(14). Since the moment reduction of the first term on

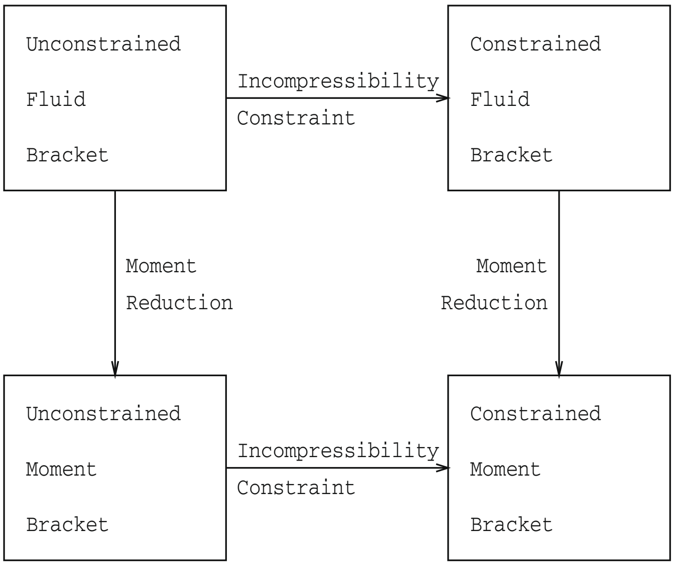


Fig. 1. This commuting diagram indicates that the final result is obtained on following either path.

the right-hand side of Eq. (33) leads as we have seen to Rosensteel's bracket, we arrive at the same constrained, moment bracket via either route, as indicated in Fig. 1.

In the next two sections, we investigate the structure of Hamilton's equations first using Rosensteel's bracket and then using the Dirac bracket based on it.

5. Dynamical equations under Rosensteel's bracket

In this section, we work out the dynamical equations obtained under Rosensteel's bracket, using as the Hamiltonian function the total energy of a Riemann ellipsoid. We shall find (see the last sentence of this section) that a key term is missing.

The symmetric matrix Σ can be transformed to the diagonal form Q, as in Eq. (18) above. We have implicitly assumed in this description that Σ is positive-definite: this represents a choice of initial data and, once made, will persist for at least a finite time interval. We assign the potential energy appropriate to an ellipsoid with semiaxes a_1, a_2, a_3:

$$\mathcal{W} = -(1/2)\int \rho V(x) d^3x = -(3/10)m^2 G\mathcal{I}, \tag{36}$$

where the potential function V is given by Eq. (63) and $\mathcal{I}$ by Eq. (64) of Appendix A. We have further used Eq. (22) of chapter 3 of [4] to complete the integration. This potential energy is therefore a function only of the squares of the semiaxes, i.e., of the eigenvalues of the matrix Σ. If we write

$$\Sigma = T^t Q T, \tag{37}$$

we may think of the six independent entries of Σ as consisting of the three eigenvalues together with the three angles needed to specify the rotation matrix T. We may equally regard the potential energy as a function of Σ:

$$\mathcal{V}(\Sigma) = \mathcal{W}(Q). \tag{38}$$

We use the total energy for the Hamiltonian function:

$$H = (1/2)\mathrm{Tr}\left(\mathcal{M}^t \Sigma^{-1} \mathcal{M}\right) + \mathcal{V}(\Sigma). \tag{39}$$

That this function depends only on the moments $\mathcal{M}$ and Σ shows that a reduction has been achieved.[4] Derivatives of the Hamiltonian are given by the formulas

[4] This should be compared with Rosensteel's Eq. (4), where an extra term, proportional to the fluid pressure, appears. It is this extra term that leads to the correct dynamical equations under Rosensteel's bracket.

$$\frac{\partial H}{\partial \Sigma} = -\frac{1}{2}\Sigma^{-1}\mathcal{M}\mathcal{M}^t\Sigma^{-1} + \frac{\partial \mathcal{V}}{\partial \Sigma} \tag{40}$$

and

$$\frac{\partial H}{\partial \mathcal{M}} = \Sigma^{-1}\mathcal{M}, \tag{41}$$

where indices have been suppressed: to get the ij derivative on the left, one takes the ij entry of the matrix on the right.

We now find, using Rosensteel's bracket relations (12)–(14), the equations of motion

$$\dot{\mathcal{M}}_{ij} = \left(\mathcal{M}^t\Sigma^{-1}\mathcal{M}\right)_{ij} + \frac{\partial \mathcal{V}}{\partial \Sigma_{kl}}\{\mathcal{M}, \Sigma_{kl}\}_R \quad \text{and} \tag{42}$$

$$\dot{\Sigma} = \mathcal{M} + \mathcal{M}^t. \tag{43}$$

These represent a dynamical system of dimension 18 that has a 15-dimensional invariant manifold expressed by the symmetry of Σ, and we henceforth restrict consideration to this system of dimension 15. With the aid of the relation (14) we may rewrite the first of these equations as

$$\dot{\mathcal{M}} = \mathcal{M}^t\Sigma^{-1}\mathcal{M} - 2\Sigma\frac{\partial \mathcal{V}}{\partial \Sigma}. \tag{44}$$

From the relations $\Sigma_{ij} = T_{ri}Q_{rs}T_{sj}$ (see Eq. (37) above) and the chain rule, we find that

$$\frac{\partial \mathcal{W}}{\partial Q_{ij}} = \frac{\partial \mathcal{V}}{\partial \Sigma_{kl}}\frac{\partial \Sigma_{kl}}{\partial Q_{ij}} = \frac{\partial \mathcal{V}}{\partial \Sigma_{kl}}T_{ik}T_{jl} = \left(T\frac{\partial \mathcal{V}}{\partial \Sigma}T^t\right)_{ij},$$

where, in the next-to-last term, we have exploited the fact that T and Q may be regarded as independent. Eq. (44) therefore takes the form

$$\dot{\mathcal{M}} = \mathcal{M}^t\Sigma^{-1}\mathcal{M} - 2\Sigma T^t\frac{\partial \mathcal{W}}{\partial Q}T.$$

Next writing $\mathcal{M} = T^t\mathcal{N}T$ and $\Sigma = T^tQT$ we obtain the equations in the rotating frame:

$$\dot{\mathcal{N}} + [\mathcal{N}, \Omega] = \mathcal{N}^tQ^{-1}\mathcal{N} - 2Q\frac{\partial \mathcal{W}}{\partial Q} \quad \text{and} \quad \dot{Q} + [Q, \Omega] = \mathcal{N} + \mathcal{N}^t. \tag{45}$$

Here $\Omega = \dot{T}T^t$ is the antisymmetric angular-velocity matrix and the square bracket is the commutator: $[A, B] = AB - BA$. At first glance Eq. (45) do not look like a well-determined dynamical system. The variables $\mathcal{N}, Q$ appear on the left-hand side but on the right are $\mathcal{N}, Q$ and Ω, so this system is well-determined only if Ω is a function of $\mathcal{N}$ and Q. However, the second equation consists of three differential equations for Q_{11}, Q_{22}, Q_{33} and three equations expressing $\Omega = \Omega(\mathcal{N}, Q)$. They therefore indeed represent a dynamical system of dimension 12 for these variables. The remaining three variables of the original 15 define the rotation matrix T and may be recovered if desired from the equation $\dot{T} = \Omega T$ after the time dependence of Ω has been found. Thus we can think of the transformation from $(\Sigma, \mathcal{M})$ to $(Q, T, \mathcal{N})$ as a change of coordinates.

The equations of motion for the Riemann ellipsoids in their standard form as given in Eq. (62) below likewise represents a 12-dimensional system. We next bring the moment system (45) into this standard form. Recall that the matrix $\mathcal{N}$ represents the set of first moments of the momentum $\rho\mathbf{U}$ where $U = Kx$ is linear in x:

$$\mathcal{N}_{ij} = \int \rho x_i U_j \, d^3x = \int \rho x_i K_{jl} x_l \, d^3x = Q_{il}K_{jl}.$$

Accordingly, we replace the matrix $\mathcal{N}$ with K through the transformation

$$\mathcal{N} = QK^t. \tag{46}$$

The moment Eq. (45) take the forms

$$\dot{K} + K^2 - \Omega K + K\Omega = -2\frac{\partial \mathcal{W}}{\partial Q} \quad \text{and} \tag{47}$$

$$\dot{Q} = QK^t + KQ^t + \Omega Q - Q\Omega. \tag{48}$$

We focus our attention first on the second of these equations, Eq. (48). Eq. (48), which is unchanged under transposition, may be regarded as six equations for the nine entries of K. We introduce the matrix

$$A = \mathrm{diag}(a_1, a_2, a_3), \tag{49}$$

the matrix of semiaxes, so that $Q = (m/5)A^2$. The diagonal entries of K are easily found to be (for example) $K_{11} = \dot{a}_1/a_1$, by virtue of Eq. (18). Among the off-diagonal entries there must be three that are as yet undetermined. If we define a matrix Λ through the formula

$$K = \dot{A}A^{-1} + A\Lambda A^{-1} - \Omega, \tag{50}$$

we find that Eq. (48) is satisfied if and only if the matrix Λ is antisymmetric. This prescribes the nine entries of K through the three entries of $\dot{A}A^{-1}$, the three independent entries of Ω, and the three independent entries of Λ. This should be compared with [4], chapter 4, Eq. (42), where the same result is arrived at in a different way.

With the choice (50) for K, we can now express the left-hand side of Eq. (47) in terms of the variables $A, \dot{A}, \Omega, \Lambda$. We find

$$\dot{K} + K^2 + [K, \Omega] = \left[\frac{d^2A}{dt^2} + \frac{d}{dt}(A\Lambda - \Omega A) + \dot{A}\lambda - \Omega\dot{A} + A\Lambda^2 + \Omega^2 A - 2\Omega A\Lambda\right]A^{-1}, \tag{51}$$

i.e., the left-hand side of Eq. (47) agrees exactly with that of Eq. (62) of Appendix A. The right-hand side of Eq. (47) is diagonal with, for example, the 11 entry

$$-2\frac{\partial \mathcal{W}}{\partial Q_{11}} = -(10/m)\frac{\partial \mathcal{W}}{\partial a_1^2} = +3mG\frac{\partial \mathcal{I}}{\partial a_1^2} = (3/2)mG\mathcal{A}_1, \tag{52}$$

where we have used the definition (64) of $\mathcal{I}$.

This gives agreement with Eq. (62) with the important exception that the pressure term is missing.

6. Dynamical equations under the Dirac bracket

We address here two aspects of results of the preceding section that are not wholly satisfactory. One is the apparent need for a Hamiltonian that is not the total energy as usually defined, and the other is that the system obtained is not self-contained but needs to be augmented by the further constraints alluded to above regarding the density and the divergence. The latter may seem an innocent requirement since such augmentation is needed also in the fluid-dynamical derivation as presented in [3] or [4]; see also the discussion in Appendix A below. However, the Hamiltonian version, as embodied in the bracket (7) above, incorporates not only the law of conservation of momentum but also that of conservation of mass.[5] We should therefore expect the dynamics to be fully described by a Hamiltonian description without the need for any augmentation.

Consider the Dirac bracket $\{\cdot,\cdot\}_{RD}$ presented in Eqs. (25)–(27), and again employ as the Hamiltonian the total energy (Eq. (39)). The additional terms added by the Dirac procedure to the right-hand sides of the bracket relations provide corresponding additional terms on the right-hand sides of the dynamical equations. The dynamical equations corresponding to Eqs. (42) and (43) therefore become (after a series of tedious but straightforward calculations)

[5] It would also include conservation of energy (or entropy) if we used the full bracket as given in [1].

$$\dot{\mathcal{M}} = \mathcal{M}^t \Sigma^{-1}\mathcal{M} + \frac{1}{\mathrm{Tr}(\Sigma^{-1})}\mathrm{Tr}\left(K^2 + 2\frac{\partial \mathcal{W}}{\partial Q}\right)I - \frac{C^2}{\mathrm{Tr}\left(\Sigma^{-1}\right)}\left(\Sigma^{-1}\mathcal{M} + \mathcal{M}^t\Sigma^{-1}\right) \quad \text{and} \tag{53}$$

$$\dot{\Sigma} = \mathcal{M} + \mathcal{M}^t - 2\frac{C^2}{\mathrm{Tr}\left(\Sigma^{-1}\right)}I, \tag{54}$$

where I denotes the unit matrix and C^2 is one of the two Casimir invariants of the Dirac bracket defined in Eq. (17) and is therefore a constant of the motion for the preceding dynamical system. Since it is proportional to the divergence of the velocity field, it is supposed to vanish, and we choose the initial data so that this is so; this simplifies the preceding equations.

Proceeding as in Section 5, we obtain from these, with the same definition of K as in Eq. (50) above, the equation

$$\dot{K} + K^2 + [K, N] = -2\frac{\partial \mathcal{W}}{\partial Q} + \left[\frac{1}{\mathrm{Tr}\left(Q^{-1}\right)}\left(K^2 + 2\partial\mathcal{W}/\partial Q\right)\right]Q^{-1}. \tag{55}$$

This not only has the structure of Eq. (62) but also explicitly provides the expression for the pressure that is otherwise obtained by the standard fluid-dynamical procedure needed to maintain the vanishing of the divergence of the velocity field. To see this, observe that the term $2p_c/\rho$ of Eq. (62) is expressed in terms of the dynamical variables by taking the trace of each side of Eq. (62):

$$\frac{2p_c}{\rho}\mathrm{Tr}(A^{-2}) = \mathrm{Tr}\left(\dot{K} + K^2 + [K, \Omega] + 2\partial\mathcal{W}/\partial Q\right) = \mathrm{Tr}\left(K^2 + 2\partial\mathcal{W}/\partial Q\right).$$

Here we have used the identity (51), we have used the formula (52), we have observed that $Tr([K, \Omega]) = 0$, and we have set

$$\mathrm{Tr}(\dot{K}) = \frac{d}{dt}\left(\sum \dot{a}_i/a_i\right) = \frac{\partial}{\partial t}\mathrm{div}U = 0,$$

in accordance with the fluid-dynamical procedure for defining the pressure. This gives for the pressure term on the right-hand side of Eq. (62) the expression

$$\frac{2p_c}{\rho}A^{-2} = \frac{1}{\mathrm{Tr}(A^{-2})}\mathrm{Tr}\left(K^2 + 2\partial\mathcal{W}/\partial Q\right)A^{-2} = \frac{1}{\mathrm{Tr}(Q^{-1})}\mathrm{Tr}\left(K^2 + 2\partial\mathcal{W}/\partial Q\right)Q^{-1}. \tag{56}$$

The latter is exactly the extra term provided by the Dirac bracket formulation and completes the verification that the dynamics given by the Hamiltonian (39) under the Dirac bracket is exactly that of the Riemann ellipsoids.

7. Discussion

Beginning with the Hamiltonian structure of the ideal fluid, we have shown that the incompressible Riemann ellipsoids are governed by Hamiltonian equations in which the Hamiltonian function is the total energy and the constraints of incompressibility are incorporated into a nonlinear bracket via the Dirac formalism. No extraneous constraints are required in our formulation. Our results are obtained by introducing a Dirac bracket for the finite-dimensional system of moment equations governing the motions of the Riemann ellipsoid, and are related in spirit to work of Nguyen and Turski [7], who formally introduce a Dirac bracket for the purpose of achieving a Hamiltonian formulation of the full, infinite-dimensional system of incompressible Euler equations.

Below we make some additional remarks about constraints. In particular, we show that a formulation of Lewis et al. [8] for a free boundary liquid, which enforces the incompressibility constraint by requiring divergence free functional derivatives, gives the correct equations for a self-gravitating liquid mass.

7.1. A bracket for a free-boundary problem

Lewis et al. [8] have proposed the following bracket for a liquid with uniform density and a free boundary:

$$\{F, G\} = \int_D \frac{\delta F}{\delta \mathbf{v}} \cdot \left(\frac{\delta G}{\delta \mathbf{v}} \times \omega \right) d^3x + \int_{\partial D} \left(\frac{\delta F}{\delta \sigma} \frac{\delta G}{\delta \phi} - \frac{\delta F}{\delta \phi} \frac{\delta G}{\delta \sigma} \right) d^2x, \tag{57}$$

where $\omega = \operatorname{curl} \mathbf{v}$ and the variations have the following meanings. The functionals F and G depend on the velocity field $\mathbf{v}$ in the domain D and also on a variable σ determining the instantaneous shape of the boundary and defined as follows. The distance $\Delta\sigma$ is the amount that some point $\mathbf{x}$ on ∂D moves normal to itself in the time interval Δt. Therefore $\sigma_t = \hat{\mathbf{n}} \cdot \mathbf{v}$. This is the local evolution equation for the motion of the surface normal to itself. The variable σ is therefore a function of surface coordinates on ∂D and of time. The variational derivative $\delta F/\delta \mathbf{v}$ is clear, but what is less obvious is the requirement that it, like the velocity $\mathbf{v}$, be solenoidal:

$$\operatorname{div} \frac{\delta F}{\delta \mathbf{v}} = 0. \tag{58}$$

The remaining functional derivative is given by the formula $\delta F/\delta\phi = \hat{\mathbf{n}} \cdot \delta F/\delta \mathbf{v}$. It is evaluated only on the boundary and is not an independent variation but depends on $\delta F/\delta \mathbf{v}$.

In their paper, Lewis et al. show how this bracket yields the equations of motion for a liquid drop held together by surface tension. We now verify that it does the same if surface tension in the Hamiltonian is replaced by self-gravitation. The Hamiltonian is then $H[\mathbf{v}, \sigma] = T[\mathbf{v}, \sigma] + W[\sigma]$, where

$$T[\mathbf{v}, \sigma] = \int_D (1/2)|\mathbf{v}|^2 \, d^3x \quad \text{and} \quad W[\sigma] = -(1/2) \int_D V(x) \, d^3x, \; V(x) = \int_D \frac{d^3y}{|\mathbf{x} - \mathbf{y}|}.$$

The dependence on σ arises because the domain D depends on the shape of the boundary. Straightforward calculations show that

$$\frac{\delta H}{\delta \mathbf{v}} = \mathbf{v}, \quad \frac{\delta H}{\delta \phi} = \hat{\mathbf{n}} \cdot \mathbf{v}, \quad \frac{\delta H}{\delta \sigma} = (1/2)|\mathbf{v}|^2 - V(x). \tag{59}$$

These variations have been made without explicitly imposing the solenoidal constraint (58), but note that $\delta H/\delta \mathbf{v}$ satisfies this constraint anyway by virtue of the solenoidal character of $\mathbf{v}$. Therefore

$$\begin{aligned}
\{F, H\} &= \int_D \frac{\delta F}{\delta \mathbf{v}} \cdot (\mathbf{v} \times \omega) \, d^3x + \int_{\partial D} \hat{\mathbf{n}} \cdot \mathbf{v} \frac{\delta F}{\delta \sigma} \, d^2x - \int_{\partial D} \hat{\mathbf{n}} \cdot \frac{\delta F}{\delta \mathbf{v}} \Big((1/2)|\mathbf{v}|^2 + V(x) \Big) \, d^2x \\
&= \int_D \frac{\delta F}{\delta \mathbf{v}} \cdot (-\mathbf{v} \cdot \nabla \mathbf{v} + \nabla V(x)) \, d^3x + \int_{\partial D} \frac{\delta F}{\delta \sigma} \hat{\mathbf{n}} \cdot \mathbf{v} \, d^2x,
\end{aligned}$$

where we have used the fact that the divergence of $\delta F/\delta \mathbf{v}$ vanishes and a standard vector identity.

On the other hand,

$$F_t = \int_D \frac{\delta F}{\delta \mathbf{v}} \cdot \mathbf{v}_t \, d^3x + \int_{\partial D} \frac{\delta F}{\delta \sigma} \sigma_t \, d^2x.$$

Hamilton's equations hold if and only if $F_t = \{F, H\}$ for all functionals F. Comparing the expressions for the two quantities we see that we must have $\sigma_t = \hat{\mathbf{n}} \cdot \mathbf{v}$, expressing the free-boundary condition. The equality of the two integrals multiplied by $\delta F/\delta \mathbf{v}$ does not guarantee the equality of their coefficients because $\delta F/\delta \mathbf{v}$ is not entirely arbitrary but in the Lewis et al. formulation must be constrained by the solenoidal condition: if $p(x)$ is any function on D vanishing on ∂D, $\int_D \delta F/\delta \mathbf{v} \cdot \nabla p \, d^3x = 0$. Thus the equality of F_t with $\{F, H\}$ implies the correct equation of motion, $\mathbf{v}_t = -\mathbf{v} \cdot \nabla \mathbf{v} - \nabla p - \nabla V(x)$, where p is a scalar vanishing on ∂D.

In principle one should next check whether the moments $(\Sigma, \mathcal{M})$ effect a reduction with the Lewis et al. procedure. Because we know that the Hamiltonian depends only on these moments, this amounts to checking that they are closed under the brackets. With the definitions of (3) and (4) we find for the variational derivatives, on ignoring the solenoidal constraint,

$$\frac{\delta \mathcal{M}_{ij}}{\delta v_k} = x_i \delta_{jk}, \quad \frac{\delta \mathcal{M}_{ij}}{\delta \sigma} = x_i v_j, \quad \frac{\delta \Sigma_{ij}}{\delta v_k} = 0, \quad \text{and} \quad \frac{\delta \Sigma_{ij}}{\delta \sigma} = x_i x_j.$$

It is seen that $\delta\Sigma/\delta\mathbf{v}$ satisfies this constraint, but $\delta\mathcal{M}/\delta\mathbf{v}$ does not. This can be rectified by restricting also the variations $\delta\mathbf{v}$ to be solenoidal, thereby modifying the expression for $\delta\mathcal{M}/\delta\mathbf{v}$ by the addition of a certain gradient. Carrying this out, checking algebraic closure, and verifying the equations of motion of the Riemann ellipsoid would require calculations of a length and difficulty similar to those already carried out in this paper and we have not done this.

Fasso and Lewis [9] have given an alternative Hamiltonian formulation, not for fluid dynamics, but explicitly for the equations governing the Riemann ellipsoids.

7.2. The nature and number of incompressibility constraints

The Dirac procedure requires an even number of constraint functions and we have used two. It might be surmised that the goal of introducing incompressibility would require only one constraint, $\operatorname{div} v = 0$, and that the imposition of a second is an artifice needed in order to use the Dirac procedure. This is not so.

It is easiest to see this in the special context of the Riemann ellipsoids. In Eq. (62) there are two extra parameters, p_c and ρ, that need to be defined in order to make the system determinate. One of these is achieved by simply declaring ρ to be a fixed constant. The second is achieved by taking the trace of either side of the equation and setting

$$\frac{\partial}{\partial t} \operatorname{div} v = \frac{d}{dt}\left(\sum \dot{a}_i/a_i\right) = 0,$$

thereby defining p_c as a function of the velocity field. This definition of p_c ensures that the preceding equation will hold for all t and therefore that $\sum \dot{a}_i/a_i = 0$ for all t if this is chosen to be true at the initial instant. Our choice of two invariants for the Dirac bracket corresponds precisely to these choices.

In a more general fluid-dynamical framework in which velocity and density vary with position, the imposition of the constraint $\operatorname{div} v = 0$ is not a single constraint, but an infinite family of constraints indexed by the position vector x. Once imposed, it implies by virtue of mass-conservation Eq. (2) that $D\rho/Dt = 0$, where $D/Dt = \partial/\partial t + v \cdot \nabla$ represents the convective derivative. This means the initial values of the density are convected by the velocity field and necessitates the imposition of a second family of conditions, namely those determining the density at the initial instant of time.

7.3. Invariants

Notice that the mass m is the zeroth moment of the density distribution and an algebra reduction can be constructed for it. It is a Casimir invariant and, as one would expect, so is the first moment (the center-of-mass position). By restricting attention to the quadratic moments of the density we sit on the symplectic leaf of constant mass and center-of-mass position. In the algebra we have constructed, aside from the Casimirs that we have introduced, there is one more.

Rosensteel [5] shows that the magnitude of the Kelvin circulation vector

$$\Gamma^2 \equiv \operatorname{Tr}\left[\Sigma^{-1}\mathcal{M}\Sigma\mathcal{M}^t - \mathcal{M}\mathcal{M}\right] \tag{60}$$

is a Casimir for the algebra ($gcm(3)$) and it remains so for the present algebra.[6] That it is a Casimir for Rosensteel's unconstrained algebra shows that its validity does not depend on incompressibility. The angular momentum, $\epsilon_{ijk}\mathcal{M}_{jk}$ is not a Casimir for this algebra, but is conserved by the choice of Hamiltonian.

[6] This refers to the system in the rotating frame. When the equations of motion are written in the inertial frame, it is possible to identify a three-component vector of circulation, each of whose components is separately conserved.

Acknowledgments

P.J.M. was supported by the US Department of Energy Contract No. DE-FG03-96ER-54346 and J.A.B. was supported by an NSF VIGRE Postdoctoral fellowship while at Rensselaer Polytechnic Institute.

Appendix A. Summary of the equations governing riemann ellipsoids

We provide a summary of the basic equation governing the motion of a self-gravitating, liquid ellipsoid of spatially uniform density ρ and semiaxes a_1, a_2, a_3 with a velocity field depending linearly on the cartesian coordinates. A full description is in [4], chapter 4.

Relative to a rotating reference frame in which the cartesian coordinates x are aligned with the principal axes of the ellipsoid, fluid motions are allowed that have the form

$$u(x) = \left(\dot{A} + A\Lambda\right)A^{-1}x \tag{61}$$

where $A = \mathrm{diag}(a_1, a_2, a_3)$ and Λ is an antisymmetric matrix. A and Λ are in general time-dependent, but the full spatial dependence of u is that of linearity in x, as explicitly expressed in this equation. The rotation rate of this rotating frame is expressed via a second antisymmetric matrix Ω, and the dynamical equations governing the time evolution of the variables A, Ω, Λ may be written as (cf. [4], chapter 4, Eq. (57))

$$\left[\frac{d^2A}{dt^2} + \frac{d}{dt}(A\Lambda - \Omega A) + \dot{A}\lambda - \Omega\dot{A} + A\Lambda^2 + \Omega^2 A - 2\Omega A\Lambda\right]A^{-1} = -\frac{3}{2}mG\mathcal{A} + \frac{2p_c}{\rho}A^{-2}, \tag{62}$$

where $\mathcal{A} = \mathcal{A}(A) = \mathrm{diag}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ represents the coefficients in the self-gravitational potential

$$V(x) = \frac{3}{4}mG\left(\mathcal{I} - \sum_{i=1}^{3}\mathcal{A}_i x_i^2\right), \tag{63}$$

which is valid inside the ellipsoid. These coefficients are determined by the semiaxes via the formulas.[7]

$$\mathcal{I} = \int_0^\infty \frac{du}{\Delta(u)},\ \mathcal{A}_i = \int_0^\infty \frac{du}{(a_i^2 + u)\Delta(u)}, \quad \text{where} \quad \Delta(u) = \sqrt{(a_1^2+u)(a_2^2+u)(a_3^2+u)}. \tag{64}$$

The scalar p_c is the pressure at the center $x = 0$.

The system (62) consists of 12 first-order equations in the 12 unknowns of $A, \dot{A}, \Omega$andΛ in which ρ and p_c appear as parameters. It arises from Eq. (1) only, i.e., from the imposition of the law of conservation of momentum only. It must be augmented by further information in order to render it determinate. For incompressible flow, two conditions are imposed that are consistent with Eq. (2) of mass conservation: the density[8] is set equal to a constant (which is therefore excluded from the list of variables) and the solenoidal condition $\sum \dot{a}_i/a_i = 0$ is imposed. One can then express p_c in terms of the dynamical variables $A, \dot{A}, \Omega, \Lambda$ by taking the trace of each side of Eq. (62) and putting $\frac{d}{dt}\sum \dot{a}_i/a_i = 0$; then one has 12 equations in 12 unknowns in which the solenoidal condition $\sum \dot{a}_i/a_i = 0$ is preserved by virtue of the choice of p_c together with the initial data.[9]

Appendix B. The hybrid coordinate systems

The transformation of Eqs. (53) and (54) to the equations governing the dynamics of $(Q, T, \mathcal{N})$ was demonstrated in Section 5, and their equivalence to Riemann's equations of (62) with (56) was demonstrated in Section 6. Thus Riemann's equations are simply the moment equations as we have derived them with velocities and coordinates resolved in a reference frame rotating with the body of the ellipsoid.

[7] The definitions given here differ by a factor $a_1a_2a_3$ from those given in [4].

[8] Or alternatively the product $a_1a_2a_3$.

[9] Alternatively one can eliminate p_c from the system and achieve a system of 10 equations in 10 unknowns.

In fact, there are four reference frames of interest. The *first*, with variables $(\Sigma, \mathcal{M})$, uses velocities measured in the inertial frame and resolved along axes in the inertial frame while the *fourth*, with variables $(Q, \mathcal{N})$, suppressing the dependence on the rotation matrix, measures and resolves velocities in the rotating frame (where Riemann's equations live). There is also a *second*, hybrid frame, with variables $(\Sigma, \tilde{\mathcal{M}})$, where velocities are measured in the inertial frame but resolved along axes in the rotating frame and a *third*, hybrid frame, with variables $(Q, \tilde{\mathcal{N}})$,where velocities are measured in the rotating frame but resolved along axes in the inertial frame. We present the transformation to these frames here.

For the fourth frame, we showed in Section 5 that with $\Sigma = T^t Q T$ and $\mathcal{N} = T\mathcal{M}T^t$ the equations of motion for $(\Sigma, \mathcal{M})$, (53) and (54), become

$$\dot{\mathcal{Q}} = [\Omega, Q] + \mathcal{N} + \mathcal{N}^t \quad \text{and} \quad \dot{\mathcal{N}} = [\Omega, \mathcal{N}] + \mathcal{N}^t Q^{-1}\mathcal{N} + \mathcal{F}. \tag{65}$$

where $\mathcal{F}$ represent pressure and forcing terms which transform in a straightforward fashion.

In the third frame the velocities are resolved along the inertial frame coordinates but are measured along some rotating frame. At the outset there is no need to bias this frame by requiring it to be the frame rotating with the body so we can consider an arbitrary angular velocity vector ω such that $\mathbf{u}^{Rot} = \mathbf{u}^{Inert} - \omega \times \mathbf{x}$. So, defining $\tilde{\mathcal{M}}_{ij} = \int \rho x_i u_j^{Rot}\, d^3x$ we find

$$\tilde{\mathcal{M}}_{ij} = \mathcal{M}_{ij} - \int \rho x_i \epsilon_{jkl}\omega_k x_l\, d^3x = \mathcal{M}_{ij} - \Sigma_{il}\tilde{\Omega}_{lj}, \tag{66}$$

where $\tilde{\Omega}_{lj} = \epsilon_{ljk}\omega_k$. Therefore the dynamics of Σ and $\tilde{\mathcal{M}}$ are governed by

$$\dot{\Sigma} = \tilde{\mathcal{M}} + \tilde{\mathcal{M}}^t + \Sigma\tilde{\Omega} - \tilde{\Omega}\Sigma \tag{67}$$

$$\dot{\tilde{\mathcal{M}}} = \tilde{\mathcal{M}}^t\Sigma^{-1}\tilde{\mathcal{M}} - \tilde{\Omega}\tilde{\mathcal{M}} - \tilde{\mathcal{M}}\tilde{\Omega} - \Sigma\tilde{\Omega}\tilde{\Omega} - \Sigma\dot{\tilde{\Omega}} + \mathcal{F}. \tag{68}$$

So far, $\tilde{\Omega}$ can be a completely arbitrary, prespecified function of time. The terms on the right had side of (68) represent advection, Coriolis, centripetal, Euler and external forces, respectively. If we choose a frame to coincide with the body of the ellipsoid, then Σ must be diagonal and, in this manner, $\tilde{\Omega}$ is determined.

The equations for moments completely specified in the rotating reference frame can be arrived at by either conjugating (67) and (68) with an orthogonal matrix or by shifting the velocity in (65). We shall perform both. Defining $\tilde{\mathcal{N}} = \mathcal{N} - Q\Omega$ and inserting into (65) gives easily

$$\dot{Q} = \tilde{\mathcal{N}}^t + \tilde{\mathcal{N}} \tag{69}$$

$$\dot{\tilde{\mathcal{N}}} = \tilde{\mathcal{N}}^t Q^{-1}\tilde{\mathcal{N}} - 2\tilde{\mathcal{N}}\Omega - Q\Omega\Omega - Q\dot{\Omega} + \mathcal{F}. \tag{70}$$

Alternatively, using $Q = T\Sigma T^t$ and $\tilde{\mathcal{N}} = T\tilde{\mathcal{M}}T^t$, substituting in (67) and (68), and identifying $\Omega = T\tilde{\Omega}T^t$ gives again (69) and (70). Note, with the above definition of $\tilde{\Omega}$, defining $\tilde{T}$ by $\dot{\tilde{T}} = -\tilde{\Omega}\tilde{T}$, results in $\tilde{T}^t = T$.

Appendix C. The Jacobi identity for general Dirac brackets

It is known (cf. [10]) that a Dirac bracket based on a canonical bracket satisfies the Jacobi identity and therefore provides a valid bracket. To our knowledge there is no explicit corresponding proof in the literature for the case when the original bracket is more general, i.e., not necessarily canonical. We provide that proof here.

We must show that

$$\{\{F, G\}_D, H\}_D + \text{ cyclic permutations} = 0 \tag{71}$$

for all F, G, H and any invertible ω. Therefore

$$\begin{aligned}\{\{F, G\}_D, H\}_D &= \{\{F, G\}, H\}_D - \{\{F, C_\mu\}\,\omega^{-1}_{\mu\nu}\,\{C_\nu, G\}, H\}_D \\ &= \{\{F, G\}, H\} - \{\{F, G\}, C^\mu\}\,\omega^{-1}_{\mu\nu}\,\{C^\nu, H\} - \{\{F, C^\mu\}\,\omega^{-1}_{\mu\nu}\,\{C^\nu, G\}, H\} \\ &\quad + \{\{F, C^\alpha\}\,\omega^{-1}_{\alpha\beta}\,\{C^\beta, G\}, C^\mu\}\,\omega^{-1}_{\mu\nu}\,\{C^\nu, H\}\end{aligned}$$

where the subscripts on the right hand side have all been dropped in the second line since it is unambiuously written in terms of the Lie–Poisson bracket. Upon cyclic permutations, the first term will cancel due to the Jacobi identity which holds for the Lie–Poisson bracket, so we can dispose of it immediately. Using the Leibnitz rule, the left hand side of (71) becomes

$$\begin{aligned} &= -\{\{F,G\},C^\mu\}\,\omega^{-1}_{\mu\nu}\,\{C^\nu,H\} - \{\{F,C^\mu\},H\}\,\omega^{-1}_{\mu\nu}\,\{C^\nu,G\} \\ &\quad - \{F,C^\mu\}\,\omega^{-1}_{\mu\nu}\,\{\{C^\nu,G\},H\} - \{F,C^\mu\}\{\,\omega^{-1}_{\mu\nu}\,,H\}\{C^\nu,G\} \\ &\quad + \{\{F,C^\alpha\},C^\mu\}\,\omega^{-1}_{\alpha\beta}\,\{C^\beta,G\}\,\omega^{-1}_{\mu\nu}\,\{C^\nu,H\} \\ &\quad + \{F,C^\alpha\}\,\omega^{-1}_{\alpha\beta}\,\{\{C^\beta,G\},C^\mu\}\,\omega^{-1}_{\mu\nu}\,\{C^\nu,H\} \\ &\quad + \{F,C^\alpha\}\{\,\omega^{-1}_{\alpha\beta}\,,C^\mu\}\{C^\beta,G\}\,\omega^{-1}_{\mu\nu}\,\{C^\nu,H\} + \text{ c.p.'s.} \end{aligned}$$

The ω^{-1} term can be pulled out of the bracket in all of the terms by recognizing the relation

$$\begin{aligned} \{\omega^{-1}_{\mu\nu},F\} &= -\omega^{-1}_{\mu\alpha}\omega^{-1}_{\beta\nu}\{\omega_{\alpha\beta},F\} && (72)\\ &= -\omega^{-1}_{\mu\alpha}\omega^{-1}_{\beta\nu}\{\{C^\alpha,C^\beta\},F\}. && (73) \end{aligned}$$

The first three terms and their permutations cancel due to the Jacobi Identity as do the second three terms and their permutations. Finally, the last term and its permutations cancel amongst themselves due to the Jacobi identity. In this way, it can be shown that the Dirac bracket defines a Lie algebra with an even number of Casimirs more than the original algebra for *any* bracket.

Appendix D. Non-persistence of invariants

The Dirac bracket construction ensures that the existence of the Lie–Dirac invariants. However, if there exist other dynamical invariants of the unconstrained system, i.e. invariants that commute with the Hamiltonian under the unconstrained bracket, canonical or Lie–Poisson, then there is no reason that these invariants will remain invariants under the Dirac bracket dynamics. Here we give an example where dynamical invariance is lost.

Consider an N-body type of system with a Hamiltonian of the form

$$H(p,q) = \sum_{i=1}^{N} \frac{p_i^2}{2} + \mathcal{V} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i,j=1}^{N} V(x_i - x_j), \tag{74}$$

where $V(x_i - x_j) = V(x_j - x_i)$, and dynamics generated under the canonical Poisson bracket,

$$\{f,g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial p_i} \right). \tag{75}$$

This system conserves the total momentum $P = \sum_{k=1}^{N} p_i$, as is easily shown.

Now, suppose we constrain away one of the degrees of freedom, by choosing

$$C^1 = x_1 \quad \text{and} \quad C^2 = p_1 \tag{76}$$

which results in the following Dirac bracket:

$$\{f,g\}_D = \sum_{i=2}^{N} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial p_i} \right) \tag{77}$$

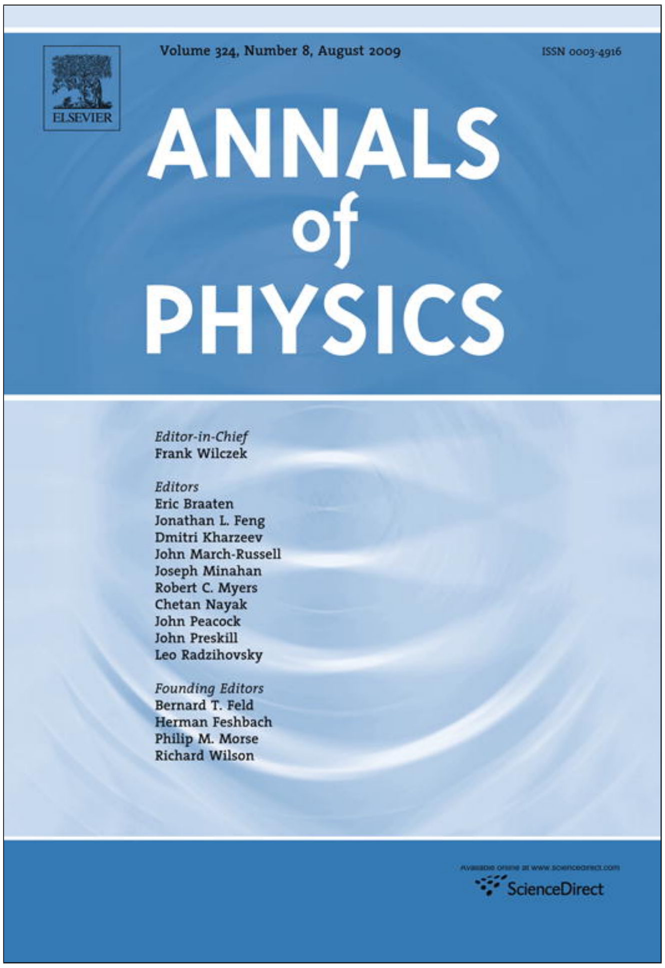
Thus under the constrained dynamics

$$\dot{P} = \frac{\partial \mathcal{V}}{\partial x_1} \neq 0. \tag{78}$$

We lose Newton's third law because reaction forces are nulled out by the constraint.

References

[1] P.J. Morrison, Rev. Mod. Phys. 70 (2) (1998) 467–521.
[2] G.L. Dirichlet, J. Reine Angew. Math. 58 (1860) 181–216.
[3] B. Riemann, Abh. d. Königl. Gesell. der Wis. zu Göttingen 9 (1860) 3–36.
[4] S. Chandrasekhar, Ellipsoidal Figures of Equilibrium, Dover, New York, 1987.
[5] G. Rosensteel, Ann. Phys. 186 (1988) 230–291.
[6] P.J. Morrison, J.M. Greene, Phys. Rev. Lett. 45 (1980) 790–793.
[7] S. Nguyen, L.A. Turski, Physica A 272 (1999) 48–55.
[8] D. Lewis, J. Marsden, T. Ratiu, J. Math. Phys. 28 (1987) 2508–2515.
[9] F. Fasso, D. Lewis, J. Rat. Mech. Anal. 158 (2001) 259–292.
[10] E.C.G. Sudarshan, N. Makunda, Classical Dynamics: A Modern Perspective, John Wiley & Sons, New York, 1974.


Volume 324, Number 8, August 2009
ISSN 0003-4916
ELSEVIER
ANNALS
of
PHYSICS
Editor-in-Chief
Frank Wilczek
Editors
Eric Braaten
Jonathan L. Feng
Dmitri Kharzeev
John March-Russell
Joseph Minahan
Robert C. Myers
Chetan Nayak
John Peacock
John Preskill
Leo Radzihovsky
Founding Editors
Bernard T. Feld
Herman Feshbach
Philip M. Morse
Richard Wilson
ScienceDirect