Caldeira-Leggett Model, Landau Damping, and the Vlasov-Poisson System[1]

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Abstract

The Caldeira-Leggett Hamiltonian (Eq. (1) below) describes the interaction of a discrete harmonic oscillator with a continuous bath of harmonic oscillators. This system is a standard model of dissipation in macroscopic low temperature physics, and has applications to superconductors, quantum computing, and macroscopic quantum tunneling. The similarities between the Caldeira-Leggett model and the linearized Vlasov-Poisson equation are analyzed, and it is shown that the damping in the Caldeira-Leggett model is analogous to that of Landau damping in plasmas [1]. An invertible linear transformation [2, 3] is presented that converts solutions of the Caldeira-Leggett model into solutions of the linearized Vlasov-Poisson system.

Key words: Landau damping, continuum damping, quantum dissipation, Lie-Poisson bracket, Caldeira-Leggett, Hamiltonian

1. Introduction

In 1946 Landau [1] theoretically predicted the collisionless damping of the electric field in a plasma governed by the Vlasov-Poisson system. This result has been of great importance in the field of plasma physics, and indeed collisionless or continuum damping, as it is sometimes called, occurs in a wide variety of kinetic and fluid plasma models that possesses a continuous spectrum. For example, such damping occurs in the context of Alfven waves

[1]Dedicated to Steve Childress whose unique combination of mathematical and physical insight has been both an inspiration and a guiding light.

in magnetohydrodynamics (see e.g. Chap. 10 of [4]) and has been proposed as a mechanism for plasma heating in response to electromagnetic waves.

Many other systems also undergo Landau damping, both inside and outside of plasma physics. It is not surprising that Landau damping exists in stellar dynamics governed by the Jeans equation [5] because this equation is of Vlasov type but with an attractive interaction potential. In fact, Landau damping occurs in collisionless kinetic theories with a rather large class of potentials, and recently has been proven rigorously to exist in the nonlinear case [6, 7]. Landau damping exists in the context of the fluid mechanics of shear flow (see e.g. [8, 9] which contains a list of original sources over a period of more than 50 years) and the description of wind driven water waves. It also appears in multiphase media [10] and has been established for systems containing large numbers of coupled oscillators, most notably the Kuramoto model. This has implications for biological models describing the synchronization or decoherence of the flashing of fireflies and chirping of crickets as well as other phenomenon in mathematical biology [11].

Another class of continuum systems involves the interaction of a discrete oscillator with a continuous bath of oscillators. In these systems the oscillator can be a particle or one mode of some field, and the bath often represents thermal fluctuations or radiation. One of the first detailed treatments of such a system is due to Dirac [12], but early on Van Kampen also used such a model to describe the emission and absorption of light by an atom [13]. The single wave model of plasma physics, which describes both beam plasma and laser plasma interaction physics [14, 15, 16], is also an example. The example of interest in this paper is the Caldeira-Leggett model [17].

The Caldeira-Leggett model was invented in order to study quantum tunneling in the presence of dissipation and the quantum limit of Brownian motion [18]. A model of this type was deemed necessary because quantum mechanics is incompatible with frictional forces. However, the Caldeira-Leggett model is a Hamiltonian system that exhibits dissipation by coupling to a continuum, i.e., it has Landau damping. The Caldeira-Leggett Hamiltonian is the sum of the Hamiltonian of a classical harmonic oscillator, the Hamiltonian of continuous bath of harmonic oscillators, and a linear coupling term between the discrete and continuous degrees of freedom. The discrete degree of freedom corresponds to a macroscopic system and the bath of oscillators represent the environment. The coupling causes the discrete oscillator to damp by transference of energy to the continuum. This system has become a standard model for studying the physics of low temperature quantum systems, and it has numerous applications ranging from the understanding of superconducting circuit elements to qubits in quantum computers [19].

We analyze the classical Caldeira-Leggett model using a procedure analogous to that used by Landau to analyze the Vlasov-Poisson system of plasma physics. Following Landau, the initial value problem can be solved using the Laplace transform and the rate of decay can be derived in the weak damping limit. This paralleling of Landau's original calculation suggests a connection between this system and the Vlasov-Poisson system. In fact, we will show that both systems can be mapped into a normal form that is common to a large class of infinite-dimensional Hamiltonian systems that have a continuous spectrum [2, 3, 9, 20].

The Caldeira-Leggett model, like all Hamiltonian systems in the class, has a continuous spectrum that is responsible for the damping through phase mixing (filamentation) and the Riemann-Lebesgue lemma. Because this structure is shared by a number of important physical systems, it is interesting to determine the nature of their similarities. It is well-known that the properties of linear ordinary differential equations are closely tied to the spectra of their time evolution operators. In fact, for given spectra there are a number of normal forms. Any linear finite-dimensional Hamiltonian system can be reduced to one of these normal forms (ODEs) through an appropriate transformation, and in this sense the behavior of such systems is completely understood. The theory of normal forms for infinite-dimensional Hamiltonian systems is not nearly as well-developed as that for finite-dimensional systems, but for some systems much is known. For systems with continuous spectra, the analog of diagonalization is conversion into a multiplication operator. If the original system is $\dot{f} = \mathcal{L}f$, then a transformation T such that $T\mathcal{L}T^{-1}$ is a multiplication operator would diagonalize the system. Any two systems that have the same normal form would thus be equivalent through some linear transformation.

This procedure has been performed for the linearized Vlasov-Poisson equation [3], and when the spectrum is purely continuous the time evolution operator is equivalent to the multiplication operator x. This discovery led to the discovery of an entire class of transformations diagonalizing linear infinite-dimensional Hamiltonian systems of a certain form [20]. In fact, it is always possible to perform such a transform in the special case of a bounded, self-adjoint operator [21]. The operators dealt with here are usually unbounded and non-normal (even if they did exist in a Hilbert space), as is often the case when dealing with continuous Hamiltonian matter models. A precursor to the discovery of such transformations is existence of a complete basis of singular eigenfunctions of the original equation, a treatment that is common for systems with continuous spectra that dates back to Dirac [12]. In fact these methods have been developed in parallel within the field of plasma physics beginning with the work of Van Kampen [22] and within condensed matter physics through the work of Dirac and later Fano [23]. Caldeira and his collaborators developed a diagonalization method for the Caldeira-Leggett model [24], although they were primarily interested in the time evolution of the discrete degree of freedom and thus did not write down the full inverse of their transformation. In this paper we complete the treatment of the Caldeira-Leggett system. Then, we note that the normal form is the same as that of the Vlasov-Poisson system and that the models are thus equivalent through the use of an integral transform.

Specifically, in Sec. 2 we review the Caldeira-Leggett model and then, in the spirit of Landau [1], present its Laplace transform solution in Sec. 3. This is followed by obtaining the singular eigenfunctions, in the spirit of Van Kampen [22] and Dirac [12], and the invertible integral transform akin to that of [3] for transforming to normal form. In Sec. 5 we show explicitly how the Caldeira-Leggett model is equivalent to a case of the linearized Vlasov-Poisson system. Finally, in Sec. 6 we conclude.

2. Caldeira-Leggett Hamiltonian

As noted above, the Caldeira-Leggett model is an infinite-dimensional Hamiltonian system describing the interaction of a discrete degree of freedom with an infinite continuum of modes [18]. The continuum is typically referred to as the environment. The Caldeira-Leggett model has the following Hamiltonian:

$$
\begin{aligned}
H_{CL}[q,p;Q,P] = \frac{\Omega}{2}P^2 + \frac{1}{2}\left(\Omega + \int_{\mathbb{R}_+} dx\, \frac{f(x)^2}{2x}\right) Q^2 \\
+ \int_{\mathbb{R}_+} dx\, \left[\frac{x}{2}(p(x)^2 + q(x)^2) + Qq(x)f(x)\right] ,
\end{aligned} \tag{1}
$$

which together with the Poisson bracket

$$
\{A,B\} = \left(\frac{\partial A}{\partial Q}\frac{\partial B}{\partial P} - \frac{\partial A}{\partial P}\frac{\partial B}{\partial Q}\right) + \int_{\mathbb{R}_+} dx\, \left(\frac{\delta A}{\delta q}\frac{\delta B}{\delta p} - \frac{\delta A}{\delta p}\frac{\delta B}{\delta q}\right) \tag{2}
$$

produces the equation of motion for observables in the form $\dot{F} = \{F, \mathcal{H}\}$, where F is any functional of the discrete, (Q,P), and continuum, (q,p), coordinates and momenta. Note, it is assumed that $f(x)$ is chosen so that the integrals of (1) exist. The coefficient of Q^2 includes a frequency shift term that is used to make the Hamiltonian positive definite. We take p and q to be functions on the positive real line, $\mathbb{R}_+$, and P and Q to be real numbers. Hamilton's equations for the Caldeira-Leggett system are thus,

$$
\dot{q}(x) = xp(x) \tag{3}
$$

$$
\dot{p}(x) = -xq(x) - Qf(x) \tag{4}
$$

$$
\dot{Q} = \Omega P \tag{5}
$$

$$
\dot{P} = -\left(\Omega + \int_{\mathbb{R}_+} dx\, \frac{f(x)^2}{2x}\right) Q - \int_{\mathbb{R}_+} dx\, q(x)f(x)\,. \tag{6}
$$

This system was originally introduced by Caldeira and Leggett in 1981 [17]. They initially considered a very massive harmonic oscillator coupled to a large number of light harmonic oscillators with varying frequencies, and then studied the limit of the light oscillators becoming a continuous spectrum. The coupling causes Q to decay to zero with time, and therefore the system can be used to model dissipation. This makes it an ideal system to model the effects of dissipation in quantum mechanics and especially quantum tunneling. It has been extensively studied and is frequently mentioned in the condensed matter literature. There have been some controversies about the physics of the damping and the physicality of the initial conditions [19]. Connecting this system with plasma physics, where much intuition has been developed over the years about wave-particle interaction, can help to improve the understanding of its behavior. For example, a clear picture of filamentation can be viewed in the numerical work of [25].

Systems with continuous spectra exhibit phase mixing or filamentation. A wide variety of systems have this property, and of course most famously the Vlasov-Poisson system for which Landau first discovered his damping. We will demonstrate explicitly that the damping mechanism of the Caldeira-Leggett model is Landau damping. Furthermore we will derive a transformation that converts the Caldeira-Leggett model into the equation describing the time evolution of a single mode of the linearized Vlasov-Poisson system. Two copies of the Caldeira-Leggett model will be canonically equivalent to the time evolution of a pair of opposite k Fourier modes of the linearized Vlasov-Poisson equation.

3. The Landau Solution and Landau Damping

One of the classical calculations in plasma physics is the solution of the linearized Vlasov-Poisson equation using the Laplace transform. This yields a formula for the solution of the initial value problem and also facilitates the derivation of the damping rate for the electric field. It is possible to do the same thing for the Caldeira-Leggett model. We begin with the set of Hamilton's equations that were written down in the previous section and eliminate the two momenta to derive a pair of second order equations for the coordinates,

$$\ddot{q}(x) = -x^2 q(x) - Qxf(x) \tag{7}$$

$$\ddot{Q} = -\Omega_c^2 Q - \Omega \int_{\mathbb{R}_+} dx\, f(x)q(x)\,, \tag{8}$$

where for convenience we use the corrected frequency,

$$\Omega_c^2 := \Omega^2 + \Omega \int_{\mathbb{R}_+} dx\, f(x)^2/2x\,. \tag{9}$$

Defining the Laplace transform of the coordinates by

$$\tilde{q}(x,s) = \int_{\mathbb{R}_+} dt\, q(x,t)e^{-st} \tag{10}$$

$$\tilde{Q}(s) = \int_{\mathbb{R}_+} dt\, Q(t)e^{-st} \tag{11}$$

results in the following set of algebraic equations:

$$s^2\tilde{q}(x,s) = -x^2\tilde{q}(x,s) - \tilde{Q}(s)xf(x) + sq(0,x) + \dot{q}(0,x) \tag{12}$$

$$s^2\tilde{Q}(s) = -\Omega_c^2\tilde{Q}(s) - \Omega\int_{\mathbb{R}_+} dx\,\tilde{q}(x,s)f(x) + sQ(0) + \dot{Q}(0)\,, \tag{13}$$

which can be easily solved for $\tilde{Q}(s)$,

$$\tilde{Q}(s) = \left[-\Omega\int_{\mathbb{R}_+} dx\,\frac{f(x)(sq(0,x)+\dot{q}(0,x))}{s^2+x^2} + sQ(0) + \dot{Q}(0)\right] \div \left[s^2 + \Omega_c^2 - \Omega\int_{\mathbb{R}_+} dx\,\frac{xf(x)^2}{s^2+x^2}\right]\,. \tag{14}$$

The Laplace transform is inverted using the Mellin inversion formula,

$$Q(t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} ds\,\tilde{Q}(s)e^{st}\,, \tag{15}$$

where β is any real number that ensures $\tilde{Q}(s)$ is analytic for $Re(s) > \beta$. This integral is usually evaluted using Cauchy's integral formula, whence asymptotically in the long-time limit the behavior of the solution is given by the poles of $\tilde{Q}(s)$. Thus, the solution will be dominated by an exponentially decaying term arising from the pole of $\tilde{Q}(s)$ closest to the real axis. We assume the closest pole is indeed close to the real axis and that there are no poles with a positive real part, i.e., that the solutions are stable. This is the weak damping limit. As long as $f(x)$ is Hölder continuous, the poles of $\tilde{Q}(s)$ come from the zeros of the denominator, so we are interested in the roots of the equation

$$0 = s^2 + \Omega_c^2 - \Omega\int_{\mathbb{R}_+} dx\,\frac{xf(x)^2}{s^2+x^2} = s^2 + \Omega_c^2 - \Omega\int_{\mathbb{R}} dx\,\frac{f(|x|)_-^2}{2(x-is)}\,. \tag{16}$$

Here $f(|x|)_-^2$ is the antisymmetric extension of $f(x^2)$ defined by $f(|x|)_-^2 = sgn(x)f(|x|)^2$. Making the substitution $\omega = is$, yields the dispersion relation, which in the limit ω tends to the real axis becomes

$$\omega^2 - \Omega_c^2 + \frac{\Omega}{2}⨍_{\mathbb{R}} dx\,\frac{f(|\omega|)_-^2}{x-\omega} + \frac{i\pi\Omega}{2}f(|x|)_-^2 = 0\,, \tag{17}$$

where ⨍ denotes the Cauchy principal value integral. For quantities not yet integrated, we will denote this by **PV**. This equation can be viewed as the dispersion relation in the weak damping limit. Let ω_c be a real solution to the real part of the above equation. Then let ω be a root of the previous equation, assume $\gamma = Im(\omega)$ is small, and solve for γ to first order; i.e. $0 = 2i\omega_c\gamma + i\pi\Omega f(|\omega_c|)^2_-/2$ or

$$\gamma = -\frac{\pi\Omega}{4|\omega_c|} f(|\omega_c|)^2 \,. \tag{18}$$

There are a large number of methods used to derive damping of Q for this model. The standard approach is to attempt to prove that after suitable approximations Q satisfies the equation of motion of a damped harmonic oscillator. The treatment here is almost identical to the method that was used to treat the Vlasov-Poisson equation by Landau, and agrees with other derivations of the damping rate in the weak damping limit[17].

4. Van Kampen modes: diagonalization of the Caldeira-Leggett model

The Laplace transform method is just one way to treat the Vlasov equation. Another way is to write the solution as a superposition of a continuous spectrum of normal modes, a method attributed to Van Kampen [22]. Such modes of the Vlasov equation are called the Van Kampen modes, and we will see that they exist for the Caldeira-Leggett model as well. We formally calculate the Van Kampen modes for this system and use them to motivate the definition of an invertible integral transform, akin to those of [2, 3, 9, 20], that maps the Caldeira-Leggett model to action-angle variables, the normal form for this Hamiltonian model. The nature of the transformation depends on the coupling function $f(x)$. In the present treatment we will assume that $f(0) = 0$, but that f does not vanish otherwise. We will also assume that the dispersion relation does not vanish anywhere. This excludes the possibility of discrete modes embedded in the continuous spectrum. The case where the Caldeira-Leggett model possesses such modes will be treated in future work. As stated above, the normal form of the Caldeira-Leggett Hamiltonian will be seen to be equivalent to that for the Vlasov-Poisson system through the integral transformation introduced in [2, 3].

The first step is to obtain a solution with time dependence $\exp(-iut)$ and derive equations for the amplitudes of a single mode (q_u, p_u, Q_u, P_u). To this end consider

$$\begin{aligned}
iuq_u(x) &= -xp_u(x) \\
iup_u(x) &= xq_u(x) + Q_u f(x) \\
iuQ_u &= -\Omega P_u \\
iuP_u &= \left(\Omega + \int_{\mathbb{R}_+} dx\, \frac{f(x)^2}{2x}\right) Q_u + \int_{\mathbb{R}_+} dx\, q_u(x) f(x)\,.
\end{aligned} \tag{19}$$

Note, although we use the subscript, $u \in \mathbb{R}$ is a continuum label. Eliminating the momenta from Eqs. (19) yields

$$(u^2 - x^2) q_u(x) = Q_u x f(x) \tag{20}$$

$$(u^2 - \Omega_c^2) Q_u = \Omega \int_{\mathbb{R}_+} dx\, q_u(x) f(x)\,, \tag{21}$$

where recall Ω_c is defined by (9). Of these, (20) is solved following Van Kampen (a generalized function solution that dates to Dirac [12]) giving the general form for q_u

$$q_u(x) = \mathbf{PV}\, \frac{Q_u x f(x)}{u^2 - x^2} + C_u Q_u \delta(|u| - x)\,. \tag{22}$$

Substitution of (22) into (21) determines C_u,

$$u^2 - \Omega_c^2 = \Omega \fint_{\mathbb{R}_+} dx\, \frac{x f(x)^2}{u^2 - x^2} + \Omega C_u f(|u|) \tag{23}$$

$$C_u = \frac{u^2 - \Omega_c^2}{\Omega f(|u|)} - \fint_{\mathbb{R}} dx\, \frac{f(|x|)_-^2}{2(u - x) f(|u|)}\,. \tag{24}$$

Therefore we can specify an initial condition on the amplitudes Q_u and compute the corresponding coordinates and momenta by an integral over the real line. Each mode oscillates with a different real frequency, with the expression for the solution given by

$$q(x,t) = \fint_{\mathbb{R}} du\, \frac{Q_u x f(x)}{u^2 - x^2} e^{-iut} + \int_{\mathbb{R}} du\, C_u Q_u \delta(|u| - x) e^{-iut} \tag{25}$$

$$Q(t) = \int_{\mathbb{R}} du\, Q_u e^{-iut}\,, \tag{26}$$

were Q_u acts an amplitude function that determines which Van Kampen modes are excited.

The Caldeira-Leggett model can be diagonalized and solved by making use of the integral transform alluded to above. Previously, Caldeira et al. [24] derived a transformation that diagonalizes the Caldeira-Leggett model. However, they were interested in solving for the evolution of the variable Q and therefore did not attempt to write down the full inverse of the operator (except in a special case where they made use of the evolution of the reservoir). We will extend their results by deriving the inverse map that we will use to establish the equivalence with the Vlasov-Poisson system.

In order to define the transform, we introduce a number of other important maps and introduce our notation. Extensive use will be made of the Hilbert transform, which is defined for a function $g(x)$ on $\mathbb{R}$ by

$$H[g](v) = \frac{1}{\pi} ⨍_{\mathbb{R}} dx\, \frac{g(x)}{x - v}\,.$$

We also need some Hilbert transform identities [26, 3]. Let g, g_1, and g_2 be functions of $x \in \mathbb{R}$ and suppose that all the expressions we write down are well defined, then the following hold:

$$H[H[g]] = -g \tag{27}$$

$$H[g_1 H[g_2] + g_2 H[g_1]] = H[g_1]H[g_2] - g_1 g_2 \tag{28}$$

$$H[vg] = vH[g] + \frac{1}{\pi}\int_{\mathbb{R}} dx\, g\,. \tag{29}$$

Next we define two functions, ϵ_R and ϵ_I by

$$\epsilon_I = \pi f(x)^2 \qquad \text{and} \qquad \epsilon_R = 2\frac{x^2 - \Omega_c^2}{\Omega} + \pi H[f(|x|)_-^2]\,. \tag{30}$$

These together with $|\epsilon|^2 := \epsilon_I^2 + \epsilon_R^2$ are used to define the following integral transforms:

Definition For functions $h(x)$ on $\mathbb{R}_+$, the transform

$$T_+[h](u) := \epsilon_R h(|u|) + \epsilon_I H[h(|x|)](u)\,,$$

while

$$\widehat{T_+}[h](u) := \frac{\epsilon_R}{|\epsilon|^2} h(u) - \frac{\epsilon_I}{|\epsilon|^2} H[h(|x|)](u)\,,$$

Related to the above transforms are two more transforms,

$$T_-[h](u) := \epsilon_R h(|u|) + \epsilon_I H[sgn(x) h(|x|)](u)\,,$$

and

$$\widehat{T_-}[h](u) := \frac{\epsilon_R}{|\epsilon|^2} h(u) - \frac{\epsilon_I}{|\epsilon|^2} H[sgn(x)h(|x|)](u)\,.$$

Using the transform T_+ it is possible to write the map from the amplitudes of the Van Kampen modes Q_u to the functions $(q(x), Q)$. To see this consider the expression for $q(x)$ in terms of the amplitude function Q_u, and simplify it using the Hilbert transform as follows:

$$\begin{aligned} q(x) &= \fint_{\mathbb{R}} du \, \frac{Q_u x f(x)}{u^2 - x^2} + \int_{\mathbb{R}} du \, C_u Q_u \delta(|u| - x) \\ &= \fint_{\mathbb{R}} du \, \frac{x f(x) Q_u}{2u} \left(\frac{1}{u - x} + \frac{1}{u + x} \right) + C_x Q_x + C_{-x} Q_{-x} \\ &= \pi x f(x) \left(H \left[\frac{Q_u}{2u} \right] (x) + H \left[\frac{Q_u}{2u} \right] (-x) \right) + 2C_x (Q_x + Q_{-x})\,. \end{aligned} \tag{31}$$

Next, decompose Q_u into its symmetric and antisymmetric parts: $Q_u = Q_{+u} + Q_{-u}$ and observe that the antisymmetric parts vanish from both sides of (31),

$$\begin{aligned} q(x) &= \pi x f(x) H[Q_{+u}/u](x) + 2C_x Q_{+x} \\ &= \pi f(x) H[Q_{+u}] + \left(2\,\frac{x^2 - \Omega_c^2}{\Omega f(x)} - \fint_{\mathbb{R}} dx' \, \frac{f(|x'|)_-^2}{(x - x') f(x)} \right) Q_{+x}\,, \end{aligned} \tag{32}$$

where the second line follows from the third Hilbert transform identity combined with the fact that Q_{u+}/u is antisymmetric and thus has a vanishing integral. Now multiply both sides of (32) by $f(x)$ and find

$$\begin{aligned} f(x) q(x) &= \pi f(x)^2 H[Q_{+u}] + \left(2\,\frac{x^2 - \Omega_c^2}{\Omega} - \fint_{\mathbb{R}} dx' \, \frac{f(|x'|)_-^2}{(x - x')} \right) Q_{+x} \\ &= \epsilon_I H[Q_{+u}] + \epsilon_R Q_{+x} \\ &= T_+[Q_{+u}]\,. \end{aligned} \tag{33}$$

Now we are set to define a transformation.

Definition Let Q_{+u} be a function on $\mathbb{R}_+$, then the map

$$I_c[Q_+] := \left(\frac{1}{f(x)} T_+[Q_{u+}], 2 \int_{\mathbb{R}_+} du \, Q_{u+} \right)\,.$$

The map $I_c[Q_+]$, a map from the Van Kampen mode amplitudes to the original dynamical variables, has an inverse. To see this note that $\epsilon_R = S + H[\epsilon_I]$, where $S = 2(x^2 - \Omega_c^2)/\Omega$, and let g be a function on $\mathbb{R}_+$. Then,

$$H[Sg(|x|)] = SH[g(|x|)] + \frac{4u}{\pi\Omega}\int_{\mathbb{R}_+} dx\, g\,,$$

where we have used our Hilbert transform identities to move the x^2 outside of the Hilbert transform of g. Using this, consider the following sequence of identities:

$$\begin{aligned}
\widehat{T_+}[T_+[g]] &= \frac{\epsilon_R}{|\epsilon|^2}(\epsilon_R g + \epsilon_I H[g]) - \frac{\epsilon_I}{|\epsilon|^2}H[\epsilon_R g + \epsilon_I H[g]] \\
&= \frac{\epsilon_R^2}{|\epsilon|^2}g + \frac{\epsilon_R\epsilon_I}{|\epsilon|^2}H[g] - \frac{\epsilon_I S}{|\epsilon|^2}H[g] - \frac{\epsilon_I}{|\epsilon|^2}H[H[\epsilon_I]g + \epsilon_I H[g]] - \frac{\epsilon_I}{|\epsilon|^2}\frac{4u}{\pi\Omega}\int_{\mathbb{R}_+} dx\, g \\
&= \frac{\epsilon_R^2}{|\epsilon|^2}g + \frac{\epsilon_R\epsilon_I}{|\epsilon|^2}H[g] - \frac{\epsilon_I S}{|\epsilon|^2}H[g] - \frac{\epsilon_I}{|\epsilon|^2}(H[\epsilon_I]H[g] - g\epsilon_I) - \frac{\epsilon_I}{|\epsilon|^2}\frac{4u}{\pi\Omega}\int_{\mathbb{R}_+} dx\, g \\
&= g + \frac{\epsilon_R\epsilon_I}{|\epsilon|^2}H[g] - \frac{\epsilon_I S}{|\epsilon|^2}H[g] - \frac{\epsilon_I}{|\epsilon|^2}H[\epsilon_I]H[g] - \frac{\epsilon_I}{|\epsilon|^2}\frac{4u}{\pi\Omega}\int_{\mathbb{R}_+} dx\, g \\
&= g + \frac{\epsilon_R\epsilon_I}{|\epsilon|^2}H[g] - \frac{\epsilon_I}{|\epsilon|^2}H[g](S + H[\epsilon_I]) - \frac{\epsilon_I}{|\epsilon|^2}\frac{4u}{\pi\Omega}\int_{\mathbb{R}_+} dx\, g \\
&= g + \frac{\epsilon_R\epsilon_I}{|\epsilon|^2}H[g] - \frac{\epsilon_R\epsilon_I}{|\epsilon|^2}H[g] - \frac{\epsilon_I}{|\epsilon|^2}\frac{4u}{\pi\Omega}\int_{\mathbb{R}_+} dx\, g \\
&= g - \frac{\epsilon_I}{|\epsilon|^2}\frac{4u}{\pi\Omega}\int_{\mathbb{R}_+} dx\, g\,,
\end{aligned} \tag{34}$$

where in each step use has been made of the various identities above. Because the integral of Q_+ is equal to $Q/2$, we can define the inverse of I as follows:

$$\widehat{I_c}[q(x), Q] = \widehat{T_+}[f(x)q(x)] + \frac{2u}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2}Q\,.$$

The above transform ignores the (p, P) variables and only produces the symmetric part of the Van Kampen modes. We derive the other half of the transformation from a mixed variable generating functional. To this end, define $Q_+ = \bar{Q}$ and $Q_- = \bar{P}$ and rescale the coordinate part of the transformation by choosing $Q = 2\int_{\mathbb{R}_+} du\, \bar{Q}\sqrt{\epsilon_I/(\pi|\epsilon|^2)}$:

$$\bar{Q} = \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\hat{T_+}[f(x)q(x)] + \frac{2u}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2}Q\right) = \widehat{I}[q(x), Q] \tag{35}$$

$$(Q, q(x)) = \left(2\int_{\mathbb{R}_+} du\, \bar{Q}\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}}, \frac{1}{f(x)}T_+\left[\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}}\bar{Q}\right]\right) = I[\bar{Q}]\,. \tag{36}$$

Then we introduce the mixed variable type-2 generating functional

$$\mathcal{F}[q, Q, \bar{P}] = \int_{\mathbb{R}_+} du\, \bar{P}\, \widehat{I}[q(x), Q]\,,$$

which produces the transformations in the usual way:

$$p(x) = \frac{\delta \mathcal{F}}{\delta q} = f(x)\, \widehat{T_+}^{\dagger} \left[\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \bar{P} \right] \tag{37}$$

$$P = \frac{\delta \mathcal{F}}{\delta Q} = \int_{\mathbb{R}_+} du\, \frac{2u\bar{P}}{\Omega} \sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}}\,. \tag{38}$$

Calculating the adjoint of $\widehat{T}$ simplifies the resulting expression for $p(x)$, viz.,

$$p(x) = \frac{1}{f(x)} T_- \left[\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} \bar{P} \right]\,. \tag{39}$$

Now, analogous to I we define the operator $J[\bar{P}] = (p(x), P)$ by

$$J[\bar{P}] = \left(\frac{1}{f(x)} T_- \left[\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} \bar{P} \right], \int_{\mathbb{R}_+} du \frac{2u\bar{P}}{\Omega} \sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} \right)\,, \tag{40}$$

which can be inverted through the use of the Hilbert transform identities. Define $\bar{P}_c = \bar{P}\sqrt{\pi|\epsilon|^2/\epsilon_I}$, and consider the expression

$$\begin{aligned} &\frac{\epsilon_R}{|\epsilon|^2} p(x) - \frac{\epsilon_I}{|\epsilon|^2} H\left[sgn(x) p(|x|)\right] \\ &\quad = \frac{\epsilon_R}{|\epsilon|^2}(\epsilon_I H[sgn(u)\bar{P}_c] + \epsilon_R \bar{P}_c) - \frac{\epsilon_I}{|\epsilon|^2} H[sgn(x)\epsilon_I H[sgn(u)\bar{P}_c] + sgn(x)\epsilon_R \bar{P}_c]\,. \end{aligned} \tag{41}$$

Paralleling the method used to invert the map from $\bar{Q}$ to (q, Q), we see a difference occurs when evaluating the term $\epsilon_I H[sgn(x)\epsilon_R \bar{P}]/|\epsilon|^2$, i.e.

$$\begin{aligned} \frac{\epsilon_I}{|\epsilon|^2} H[sgn(u) u^2 \bar{P}_c] &= x^2 \frac{\epsilon_I}{|\epsilon|^2} H[sgn(u)\bar{P}_c] + \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2} \int_{\mathbb{R}_+} du\, u\bar{P}_c \\ &= x^2 \frac{\epsilon_I}{|\epsilon|^2} H[sgn(x)\bar{P}_c] + \frac{\Omega}{\pi}\frac{\epsilon_I}{|\epsilon|^2} P\,. \end{aligned} \tag{42}$$

With this expression we can directly use the inversion calculation for the (q, Q) case to obtain the following expression for the full transformation:

$$\bar{P} = \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \left(\widehat{T_-}[f(x)p(x)] + \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2} P \right) \tag{43}$$

$$\bar{Q} = \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \left(\widehat{T_+}[f(x)q(x)] + \frac{2u}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2} Q \right)\,. \tag{44}$$

Applying this transformation to Hamilton's equations yields the equations for a continuum of harmonic oscillators. This can be seen directly for both $\bar{P}$ and $\bar{Q}$.

$$
\begin{aligned}
\dot{\bar{Q}} &= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T_+}[f(x)\dot{q}(x)] + \frac{2u}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2}\dot{Q}\right) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T_+}[xf(x)p(x)] + \frac{2u}{\pi}\frac{\epsilon_I}{|\epsilon|^2}P\right) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\frac{\epsilon_R}{|\epsilon|^2}\,uf(u)p(u) - \frac{\epsilon_I}{|\epsilon|^2}\,H[|x|f(|x|)p(|x|)] + \frac{2u}{\pi}\frac{\epsilon_I}{|\epsilon|^2}P\right) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\frac{\epsilon_R}{|\epsilon|^2}\,uf(u)p(u) - \frac{\epsilon_I}{|\epsilon|^2}\,uH[sgn(x)f(|x|)p(|x|)] + \frac{2u}{\pi}\frac{\epsilon_I}{|\epsilon|^2}P\right) \\
&= u\bar{P}\,.
\end{aligned}
\tag{45}
$$

Similarly, for $\bar{P}$,

$$
\begin{aligned}
\dot{\bar{P}} &= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T_-}[f(x)\dot{p}(x)] + \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}\dot{P}\right) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T_-}[-xf(x)q(x) - f(x)^2Q] - \frac{2\Omega_s}{\pi}\frac{\epsilon_I}{|\epsilon|^2}Q - \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}\int_{\mathbb{R}_+} dx\, f(x)q(x)\right) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(-u\widehat{T_+}[f(x)q(x)] + \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}\int_{\mathbb{R}_+} dx\, f(x)q(x) - \widehat{T_-}[f(x)^2]\,Q\right) \\
&\qquad\qquad - \frac{2\Omega_s}{\pi}\frac{\epsilon_I}{|\epsilon^2|}\,Q - \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}\int_{\mathbb{R}_+} dx\, f(x)q(x) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(-u\widehat{T_+}[f(x)q(x)] - \frac{\epsilon_R}{|\epsilon|^2}\,f(x)^2\,Q + \frac{\epsilon_I}{|\epsilon|^2}\,H[sgn(x)f(|x|)^2]\,Q - \frac{2\Omega_s}{\pi}\frac{\epsilon_I}{|\epsilon|^2}\,Q\right) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(-u\widehat{T_+}[f(x)q(x)] - \frac{2u^2}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2}\,Q\right) \\
&= -u\,\bar{Q}\,.
\end{aligned}
\tag{46}
$$

Let this full map be called I_f and consider I_f as a map from the Banach space $L_p\times L_p\times\mathbb{R}^2$, $p>1$, to the Banach space $L_p\times L_p$. The operator I_f is a bounded linear functional between these two spaces, because each term is either a multiplication operator that is bounded on L_p, an L_p function, or a bounded function multiplied by the Hilbert transform, which is another bounded operator. In order to establish the equivalence with the normal mode it is important to specify the phase space of the dynamical variables. Using this map we can simply choose each functional space to be L_p and have a well defined map in each case. This map demonstrates how the Caldeira-Leggett model can be written as a superposition of a continuous spectrum of singular eigenmodes.

Because the transformation to the normal form was a canonical one, the normal form Hamiltonian should be the original Hamiltonian of the Caldeira-Leggett model written in the new coordinates. We will verify this by direct substitution. For convenience we introduce the quantities

$$\begin{aligned} A &= \frac{\Omega}{2}P^2 + \frac{1}{2}\int_{\mathbb{R}_+} dx\, xp(x)^2 \\ B &= \frac{\Omega_s}{2}Q^2 + \int_{\mathbb{R}_+} \left(\frac{x}{2}q(x)^2 + f(x)q(x)Q\right) dx\,, \end{aligned}$$

where $\Omega_s = \Omega_c^2/\Omega$. Evidently, $H_{CL} = A + B$. Then,

$$\begin{aligned} A &= \frac{\Omega}{2}P\int_{\mathbb{R}_+} du\, \frac{2u\bar{P}}{\Omega}\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} + \frac{1}{2}\int_{\mathbb{R}_+} du\, xp(x)f(x)\widehat{T_+}^{\dagger}\left[\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\bar{P}\right] \\ &= P\int_{\mathbb{R}_+} du\, u\bar{P}\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} + \frac{1}{2}\int_{\mathbb{R}_+} du\, \widehat{T_+}\left[xp(x)f(x)\right]\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\bar{P} \\ &= P\int_{\mathbb{R}_+} du\, u\bar{P}\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} + \frac{1}{2}\int_{\mathbb{R}_+} du\, u\widehat{T_-}\left[f(x)p(x)\right]\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\bar{P} \\ &= \frac{1}{2}\int_{\mathbb{R}_+} du\, u\bar{P}\left(\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T_-}[f(x)p(x)] + \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}P\right)\right) \\ &= \frac{1}{2}\int_{\mathbb{R}_+} du\, u\bar{P}^2\,. \end{aligned} \tag{47}$$

Similarly,

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}_+} dx\, xq(x)^2 &= \frac{1}{2}\int_{\mathbb{R}_+} du\, \sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}}\widehat{T_+}^{\dagger}\left[\frac{xq(x)}{f(x)}\right]\bar{Q} \\ &= \frac{1}{2}\int_{\mathbb{R}_+} du\, \bar{Q}\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}}\left(u\frac{\epsilon_R q(u)}{f(u)} - uH\left[\frac{\epsilon_I(|x|)q(|x|)}{f(|x|)}\right] - 2\int_{\mathbb{R}_+} dx\, f(x)q(x)\right) \\ &= -\frac{1}{2}\int_{\mathbb{R}_+} dx\, Qf(x)q(x) + \frac{1}{2}\int_{\mathbb{R}_+} du\, \bar{Q}u\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\widehat{T_+}\left[f(x)q(x)\right]\,. \end{aligned} \tag{48}$$

Now, analyzing the entire expression for B in a sequence of steps,

$$
\begin{aligned}
B &= \frac{1}{2}\int_{\mathbb{R}_+} dx\, Q f(x) q(x) + \frac{1}{2}\int_{\mathbb{R}_+} du\, \bar{Q} u \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\widehat{T_+}\left[f(x)q(x)\right] + \frac{\Omega_s}{2} Q^2 \\
&= \frac{1}{2}\int_{\mathbb{R}_+} du\, \bar{Q} u \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\widehat{T_+}\left[f(x)q(x)\right] + \Omega_s Q \int_{\mathbb{R}_+} du\, \bar{Q}\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} + \frac{Q}{2}\int_{\mathbb{R}_+} dx\, T_+\left[\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}}\bar{Q}\right] \\
&= \frac{1}{2}\int_{\mathbb{R}_+} du\, \bar{Q} u \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\widehat{T_+}\left[f(x)q(x)\right] \\
&\qquad + \frac{Q}{2}\int_{\mathbb{R}_+} dx\, \left(\left(\frac{2u^2}{\Omega}\bar{Q} + \pi H[sgn(x) f(|x|)^2]\bar{Q}\right)\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} + \epsilon_I H[\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}}\bar{Q}]\right) \\
&= \frac{1}{2}\int_{\mathbb{R}_+} du\, \bar{Q} u \left(\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\widehat{T_+}\left[f(x)q(x)\right] + \frac{2u}{\Omega}\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} Q\right) \\
&= \frac{1}{2}\int_{\mathbb{R}_+} du\, u \bar{Q}^2\,.
\end{aligned}
\tag{49}
$$

With (47) and (49) we obtain $H_{CL} = \int_{\mathbb{R}_+} duu\left(\bar{Q}^2 + \bar{P}^2\right)/2$ – the Hamiltonian for a continuous spectrum of harmonic oscillators and the normal form for the Caldeira-Leggett model.

5. Equivalence to the Linearized Vlasov-Poisson Equation

The treatment of the Caldeira-Leggett model of Sec. 4 is similar to an analysis of the linearized Vlasov-Poisson equation performed in [3, 27]. In those papers an integral transform was presented that transforms the Vlasov equation into a continuous spectrum of harmonic oscillators. The two systems are identical except the spectrum of the Caldeira-Leggett model only covers the positive real line. Now we explicitly produce a transformation that takes one system into the other.

The Vlasov equation describes the kinetic theory of a collisionless plasma. Spatially homogeneous distribution functions are equilibria, and linearization about such states are often studied in plasma physics. In the case of one spatial dimension and an equilibrium distribution function $f_0(v)$, the linearized Vlasov-Poisson equation around f_0 is given by

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \phi}{\partial x} f_0' = 0 \tag{50}$$

$$\frac{\partial^2 \phi}{\partial x^2} = -4\pi e \int_{\mathbb{R}} dv\, f \tag{51}$$

where $f_0' = df_0/dv$. These equations inherit the noncanonical Hamiltonian structure of the full Vlasov-Poisson system [28] and have a Poisson bracket given by

$$\{F, G\}_L = \int\!\!\int dx dv\, f_0 \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right] . \tag{52}$$

This bracket is of a form that is typical for Hamiltonian systems describing continuous media (cf. e.g. [29, 30]). The Hamiltonian is given by

$$H_L = -\frac{m}{2}\int\!\!\int dv dx\, v\, \frac{f^2}{f_0'} + \frac{1}{8\pi}\int dx \left(\frac{\partial \phi}{\partial x}\right)^2 , \tag{53}$$

and the Vlasov-Poisson equation can be written as $\dot{f} = \{f, H_L\}_L$.

The spatial dependence of the Vlasov-Poisson equation can be removed by performing a Fourier transform. This allows the potential to be explicitly eliminated from the equation

$$\frac{\partial f_k}{\partial t} - ikvf_k - \frac{4\pi i e^2}{mk} f_0'(v) \int_{\mathbb{R}} dv\, f_k = 0 . \tag{54}$$

The Hamiltonian structure in terms of the Fourier modes has the new bracket

$$\{F, G\}_L = \sum_{k=1}^{\infty} \frac{ik}{m} \int_{\mathbb{R}} dv\, f_0' \left(\frac{\delta F}{\delta f_k}\frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k}\frac{\delta F}{\delta f_{-k}}\right) . \tag{55}$$

and the Hamiltonian functional is simply (53) written in terms of the Fourier modes.

One way to canonize this bracket is with the following scalings:

$$q_k(v, t) = f_k \qquad \text{and} \qquad p_k(v, t) = \frac{mf_{-k}}{ikf_0'} , \tag{56}$$

where $k > 0$. In terms of these variables the Poisson bracket has canonical form, i.e.

$$\{F, G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} dv \left(\frac{\delta F}{\delta q_k}\frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k}\frac{\delta F}{\delta p_k}\right) . \tag{57}$$

From this point it is possible to derive a canonical transformation that diagonalizes the Hamiltonian. We make the following definitions:

$$\varepsilon_I(v) = -\frac{4\pi^2 e^2 f_0'}{mk^2 \int_{\mathbb{R}} dv\, f_0} \qquad \varepsilon_R(v) = 1 + H[\varepsilon_I]\,, \tag{58}$$

$$G_k[f] = \varepsilon_R f + \varepsilon_I H[f] \qquad \widehat{G_k}[f] = \frac{\varepsilon_R}{|\varepsilon|^2} f - \frac{\varepsilon_I}{|\varepsilon|^2} H[f]\,. \tag{59}$$

It was proven in [3] that $G_k = \widehat{G_k}^{-1}$.

A transformation to the new set of variables $(\mathcal{Q}_k, \mathcal{P}_k)$ that diagonalizes the system will be given in terms of the variables (q_k, p_k). To this end we first introduce the intermediate variables $(\mathcal{Q}_k', \mathcal{P}_k')$ defined by

$$q_k = G_k[\mathcal{Q}_k'] \qquad \text{and} \qquad \mathcal{Q}_k' = \widehat{G_k}[q_k]\,. \tag{60}$$

The corresponding momentum portion of the canonical transformation is induced by the following mixed variable generating functional:

$$\mathcal{F}[q_k, \mathcal{P}_k'] = \sum_{k=1}^{\infty} \int_{\mathbb{R}} du\, \mathcal{P}_k' \widehat{G_k}[q_k]\,, \tag{61}$$

whence we obtain via $\mathcal{Q}_k' = \delta\mathcal{F}/\delta\mathcal{P}_k'$ and $p_k = \delta\mathcal{F}/\delta q_k$,

$$\mathcal{Q}_k' = \widehat{G_k}[q_k] \qquad \text{and} \qquad \mathcal{P}_k' = \widehat{G_k}^{\dagger}[p_k] \tag{62}$$

Then, the variables $(\mathcal{Q}_k, \mathcal{P}_k)$ are defined as

$$\mathcal{Q}_k = (\mathcal{Q}_k' - i\mathcal{P}_k')/\sqrt{2} \qquad \text{and} \qquad \mathcal{P}_k = (\mathcal{P}_k' - i\mathcal{Q}_k')/\sqrt{2}\,, \tag{63}$$

in terms of which the Vlasov-Poisson Hamiltonian has the form of a continuum of harmonic oscillators (see [27] for an explicit calculation),

$$H_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} du \left(\mathcal{Q}_k^2 + \mathcal{P}_k^2\right)/2\,. \tag{64}$$

Thus, for a single value of k, this is the same normal form as that of the Caldeira-Leggett model, with the exception that the integral here is over the entire real line instead of just the half line. If we consider two copies of the Caldeira-Leggett model the normal form would be the same as that for a single k value of the linearized Vlasov-Poisson system. By composing the transformation that diagonalizes the Caldeira-Leggett model with the inverse of the transformation that diagonalizes the Vlasov-Poisson system we obtain a map that converts solutions of one system into solutions of the other system. Explicitly suppose that we have two copies of the Caldeira-Leggett Hamiltonian, with the same coupling function $f(x)$. Then set the normal form of the second copy equal to the normal form of the Vlasov equation on the negative real line. Let $(q_1(x), p_1(x), Q_1, P_1)$ be one set of solutions to the Caldeira-Leggett model and let $(q_2(x), p_2(x), Q_2, P_2)$ be another and let $\Theta(x)$ be the Heaviside function. Then we can write a solution to the linearized Vlasov-Poisson equation using the following map:

$$f_k(v,t) = G_k\left[\frac{1}{\sqrt{2}}\left(\Theta(u)\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T_+}[f(x)q_1(x)] + \frac{2u}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2}Q_1\right)\right.\right. \tag{65}$$

$$+ \Theta(-u)\sqrt{\frac{\pi|\epsilon|(-u)^2}{\epsilon_I(-u)}}\left(\widehat{T}_+[f(x)q_2(x)](-u) + \frac{-2u}{\pi\Omega}\frac{\epsilon_I(-u)}{|\epsilon(-u)|^2}Q_2\right) \tag{66}$$

$$+ i\Theta(u)\sqrt{\frac{\pi|\epsilon^2|}{\epsilon_I}}\left(\widehat{T_-}[f(x)p_1(x)] + \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}P_1\right) \tag{67}$$

$$\left.\left. + i\Theta(-u)\sqrt{\frac{\pi|\epsilon(-u)|^2}{\epsilon_I(-u)}}\left(\widehat{T_-}[f(x)p_2(x)] + \frac{2}{\pi}\frac{\epsilon_I(-u)}{|\epsilon(-u)|^2}P_2\right)\right)\right] \tag{68}$$

$$f_{-k}(v,t) = \frac{kf_0'}{m}G_k^\dagger\left[\frac{1}{\sqrt{2}}\left(\Theta(u)\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T}_+[f(x)q_1(x)] + \frac{2u}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2}Q_1\right)\right.\right. \tag{69}$$

$$+ \Theta(-u)\sqrt{\frac{\pi|\epsilon|(-u)^2}{\epsilon_I(-u)}}\left(\widehat{T}_+[f(x)q_2(x)](-u) + \frac{-2u}{\pi\Omega}\frac{\epsilon_I(-u)}{|\epsilon(-u)|^2}Q_2\right) \tag{70}$$

$$- i\Theta(u)\sqrt{\frac{\pi|\epsilon^2|}{\epsilon_I}}\left(\widehat{T_-}[f(x)p_1(x)] + \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}P_1\right) \tag{71}$$

$$\left.\left. - i\Theta(-u)\sqrt{\frac{\pi|\epsilon(-u)|^2}{\epsilon_I(-u)}}\left(\widehat{T_-}[f(x)p_2(x)] + \frac{2}{\pi}\frac{\epsilon_I(-u)}{|\epsilon(-u)|^2}P_2\right)\right)\right] \tag{72}$$

This map is invertible using the formulas presented earlier in the paper. Given a single mode of the linearized Vlasov-Poisson system, $f_k(v,t)$, we can write two solutions to the

Caldeira-Leggett model as follows:

$$(q_1(x,t), Q_1(t)) = I[\frac{1}{2}\Re(\hat{G}[f_k](u,t) + \hat{G}[f_k](-u,t))] \tag{73}$$

$$(p_1(x,t), P_1(t)) = J[\frac{1}{2}\Im(\hat{G}[f_k](u,t) - \hat{G}[f_k](-u,t))] \tag{74}$$

$$(q_2(x,t), Q_2(t)) = I[\frac{1}{2}\Im(\hat{G}[f_k](u,t) + \hat{G}[f_k](-u,t))] \tag{75}$$

$$(p_2(x,t), P_2(t)) = J[\frac{1}{2}\Re(\hat{G}[f_k](-u,t) - \hat{G}[f_k](u,t))] \tag{76}$$

Therefore one would expect the solutions of the Caldeira-Leggett model to share the same properties as the solutions of the Vlasov-Poisson system.

It was remarked earlier that both systems exhibit damping. In the Vlasov-Poisson case the electric field decays, and in the Caldeira-Leggett model it is the discrete coordinate Q. The existence of the transformation between the two systems gives us a way to understand what determines the damping rate in each case. In the standard calculation of the Landau damping rate for the Vlasov equation, it is clear that the rate depends only on the location of the closest zero in the lower half complex plane of the dispersion relation, which only depends on the equilibrium f_0. The same is true for the Caldeira-Leggett model, where the damping of Q depends on the coupling function f. It is clear that integral transformations change the rate of damping, as all the instances of the Vlasov equation and the Caldeira-Leggett model share the same normal form but generally have different damping rates.

It is possible to interpret Landau damping using the normal forms and canonical transformation. The dynamical variables of the normal form have a time evolution $\sim \exp(-iut)$. The observables can then be expressed as some operator on this oscillatory dynamical variable. The result will be an oscillatory integral over the real line, and by the Riemann-Lebesgue lemma we know that that such an integrated quantity will decay to zero in the long-time limit. For the systems at hand, this integral can be deformed into the lower half complex plane, and Cauchy's theorem can be used to see that the behavior is governed by the locations of the poles of the analytic continuation of the oscillatory integrand. These poles determine the exponential damping rate. In these systems the poles are clearly introduced by the continuation (following the Landau prescription) of the dispersion relation in the integral transformations, which is therefore the origin of Landau damping. We will demonstrate this explicitly for the damping of the coordinate Q in the Caldeira-Leggett model.

Starting from the solution,

$$\begin{aligned} Q(t) &= \int_{\mathbb{R}} du\, (\bar{Q}(|u|)\cos(ut) + sgn(u)\bar{P}(|u|)\sin(ut))\frac{f(|u|)}{|\epsilon|} \\ &= \int_{\mathbb{R}} du\, (\hat{I}[\mathring{q}(x), \mathring{Q}]\cos(ut) + \hat{J}[\mathring{p}(x), \mathring{P}]sgn(u)\cos(ut))\frac{f(|u|)}{|\epsilon|}\,, \end{aligned} \tag{77}$$

we see that each term in the integrand of (77) has an oscillatory part and has poles at the zeros of $|\epsilon|^2$. The damping rate will be based on the closest zero of $|\epsilon|$, the dispersion relation for the Caldeira-Leggett model. Likewise, for the Vlasov-Poisson system we can write a similar expression for the density $\rho_k(t)$,

$$\begin{aligned} \rho_k(t) &= \int_{\mathbb{R}} dv\, G_k[\hat{G}_k[\mathring{f}]e^{-iut}] \\ &= \int_{\mathbb{R}} du\, \left(\varepsilon_R(\hat{G}_k[\mathring{f}]e^{-iut}) - H[\varepsilon_I]\hat{G}_k[\mathring{f}]e^{-iut}\right) \\ &= \int_{\mathbb{R}} du\, \hat{G}_k[\mathring{f}]e^{-iut}\,. \end{aligned} \tag{78}$$

The damping rates are given by the poles of $\hat{G}_k$, and the observed rate will be due to the closest zero of $|\varepsilon|^2$ to the real axis. Therefore, mathematically the source of the damping in the Vlasov-Poisson and Caldeira-Leggett models are identical, it being the nearest pole introduced by the integral transformation that diagonalizes the system.

6. Conclusion

To summarize, we have shown how the Caldeira-Leggett model can be analyzed the same way as the Vlasov-Poisson system. We wrote down the solution using the Laplace transform, an expression for the time evolution as an integral in the complex plane over the initial conditions. It was then indicated how Cauchy's theorem can be used to derive the time asymptotic behavior of the solution, and it was described how the long-time damping rate is equal to the distance from the real axis of the closest zero of the dispersion relation (when analytically continued into the lower half complex plane). Thus, the damping of the

Caldeira-Leggett model can be seen to be a rediscovery of Landau (or continuum) damping. Caldeira and Leggett introduced their system to study damping in quantum mechanical systems, and it is now seen to be one of many interesting physical examples of Hamiltonian systems that exhibit such behavior.

Next we described how to analyze the Caldeira-Leggett model by means of singular eigenmodes, paralleling Van Kampen's well-known treatment of the Vlasov-Poisson system. Here the solution was written as an integral over a distribution of such modes, each of which is itself a solution that oscillates with some real frequency. We described how Hamiltonian systems with continuous spectra generally have a solution formula in terms of such an integral over singular eigenmodes. This type of formal expansion led to a an explicit integral transformation that transforms the original Caldeira-Leggett system into a pure advection problem, just as is the case for the Vlasov-Poisson system. It was noted that a general class of such transformations was written down in [3] and was subsequently extended to a larger class of Hamiltonian systems [20]. The existence of these transformations amounts to a theory of normal forms for systems with a continuous spectrum, analogous to the theory of normal forms for finite degree-of-freedom Hamiltonian systems. This enabled us to write down an explicit transformation that converts the time evolution operator for the Caldeira-Leggett Hamiltonian into a multiplication operator and we found the inverse of this map. In this way we showed that the Caldeira-Leggett model shares the same normal form as the Vlasov-Poisson system, along with a number of other Hamiltonian systems that occur in different physical contexts.

One reason for investigating Hamiltonian structure is the existence of universal behavior shared by such systems. For example, linear Hamiltonian system with the same normal form are equivalent. This suggests some further avenues for research. Here we only treated the case where the dispersion relation of the Caldeira-Leggett model has no roots with real frequency; i.e. spectrum was purely continuous. When there are roots, the spectrum is no longer purely continuous and there are embedded eigenvalues, as is known to be the case for the Vlasov-Poisson system [31]. Consequently, one obtains a different normal form, one with a discrete component, and this and more complicated normal forms could be explicated.

We expect that there is a transformation that takes Vlasov-Poisson system with embedded modes into the Caldeira-Leggett model with embedded modes. Also, finite degree-of-freedom Hamiltonian systems are known to have only certain bifurcations of spectra, for example, as governed by Krein's theorem. Since there is a generalization of this theorem for Vlasov-like systems [32], one could investigate bifurcations in the context of the Caldeira-Leggett model. Another possibility would be to use the tools developed [27] to do statistical mechanics over the continuum bath. Lastly, the integral transform we presented is intimately related to the Hilbert transform, which is known to be an important tool in signal processing. In the same vein the integral transform for the Vlasov-Poisson system of Ref. [3] has been shown to be a useful experimental tool [33, 34] and one could explore experimental ramifications in the context of the Caldeira-Leggett model.

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