# Hamiltonian magnetohydrodynamics: Helically symmetric formulation, Casimir invariants, and equilibrium variational principles

T. Andreussi, P. J. Morrison, and F. Pegoraro

## Related Articles

## Additional information on Phys. Plasmas

# Hamiltonian magnetohydrodynamics: Helically symmetric formulation, Casimir invariants, and equilibrium variational principles

T. Andreussi,[1,a)] P. J. Morrison,[2,b)] and F. Pegoraro[3,c)]

[1]*Alta S.p.A., Pisa 56121, Italy*
[2]*Institute for Fusion Studies and Department of Physics, The University of Texas at Austin, Austin, Texas 78712-1060, USA*
[3]*Dipartimento di Fisica E. Fermi, Pisa 56127, Italy*

The noncanonical Hamiltonian formulation of magnetohydrodynamics (MHD) is used to construct variational principles for continuously symmetric equilibrium configurations of magnetized plasma, including flow. In particular, helical symmetry is considered, and results on axial and translational symmetries are retrieved as special cases of the helical configurations. The symmetry condition, which allows the description in terms of a magnetic flux function, is exploited to deduce a symmetric form of the noncanonical Poisson bracket of MHD. Casimir invariants are then obtained directly from the Poisson bracket. Equilibria are obtained from an energy-Casimir principle and reduced forms of this variational principle are obtained by the elimination of algebraic constraints. © *2012 American Institute of Physics.* [http://dx.doi.org/10.1063/1.4714761]

## I. INTRODUCTION

Ideal magnetohydrodynamics (MHD) has served as a most important tool for assessing the design and interpretation of laboratory plasma experiments and for understanding phenomena in naturally occurring plasmas (e.g., Refs. 1 and 2). Variational principles for equilibria, or as it is sometimes argued for preferred states, for a wide variety of geometrical configurations have been discovered over a period of many years (e.g., Refs. 3–11). In addition, $\delta W$ energy principles,[12,13] which provide necessary and sufficient conditions for stability of static equilibria, and other energy-like principles, which provide only sufficient conditions for stability in terms of the Lagrangian displacement variable[14] or in terms of purely Eulerian quantities,[7,10,11] have been discovered and effectively utilized.

All of the above variational principles for equilibria, which were for the most part discovered in an *ad hoc* manner, and all of the energy principles, both Lagrangian and Eulerian, are a consequence of the fact that ideal MHD is a Hamiltonian field theory. That MHD is Hamiltonian was first shown in terms of the Lagrangian variable description in Ref. 15 and in terms of the Eulerian variable description in Refs. 16–18 where the noncanonical Poisson bracket was introduced. In the Hamiltonian context, it is seen that existence of variational principles for equilibrium states is merely the result of the general fact that equilibria are extremal points of Hamiltonian functionals. Similarly, the existence of the $\delta W$ energy principle for static equilibria is an infinite-dimensional version of Lagrange's stability condition of mechanics (e.g., Refs. 19 and 20), a consequence of which is that the operator appearing in $\delta W$ is formally self-adjoint because it is a second variation and no further proof is required. Also, all of the sufficient conditions for stability of equilibria are infinite-dimensional versions of Dirichlet's stability condition[20–22] and these can be directly derived from the Hamiltonian formulation. (For discussion of these ideas in the ideal fluid context, see Ref. 22.)

The purpose of the present paper and its companion[23] is to continue with the approach of Ref. 24, which starts from the noncanonical Poisson bracket of Refs. 16–18 and then reduces to obtain the Hamiltonian formulations for translational and rotational symmetry. Here, we generalize and obtain an inclusive Hamiltonian description for any metric symmetry. From the noncanonical Poisson bracket, we derive large families of Casimir invariants that are then used to obtain general variational principles for equilibria, including equilibria with helical symmetry and flow. This prepares the way for our companion paper,[23] where we consider stability via several approaches.

Specifically, in Sec. II we briefly review the Hamiltonian description of MHD as given in Refs. 16–18. This is followed in Sec. III by the symmetry reduction, which is done by effecting the chain rule for functional derivatives. Then in Sec. IV, Casimir invariants are obtained directly from the noncanonical Poisson bracket, and this allows us to construct the equilibrium variational principles in Sec. V. These variational principles are then reduced by the elimination of algebraic constraints to obtain variation principles for special cases. In Sec. VI, several applications of helical equilibria both with and without flow are discussed.

## II. NONCANONICAL HAMILTONIAN DESCRIPTION OF MHD

Following Morrison and Greene,[16] the ideal dynamics of MHD plasma is described in terms of the Eulerian variables $Z := (\rho, \mathbf{v}, s, \mathbf{B})$, i.e., the plasma density $\rho$, the flow velocity $\mathbf{v}$, the magnetic field $\mathbf{B}$, the entropy per unit mass, $s$

a)Author to whom correspondence should be addressed. Electronic mail: t.andreussi@alta-space.com. URL: http://www.alta-space.com.
b)Electronic address: morrison@physics.utexas.edu.
c)Electronic address: pegoraro@df.unipi.it.

(or alternatively the plasma temperature or pressure), in the Hamiltonian form:

$$\frac{\partial Z}{\partial t} = \{Z, H\}_Z, \tag{1}$$

where $H$ is the Hamiltonian for MHD corresponding to the energy,

$$H = \int_V \left[ \frac{1}{2} \rho v^2 + \rho U + \frac{1}{8\pi} B^2 \right] d^3 r, \tag{2}$$

and $\{\cdot, \cdot\}$ represents the noncanonical Poisson bracket of MHD. In Eq. (2), the function $U = U(\rho, s)$ represents the internal energy of the plasma, which is related to the plasma pressure and temperature by the relationships $p = \rho^2 \partial U / \partial \rho$ and $T = \partial U / \partial s$; we note that gravitational effects could be included by adding a term $\rho \varphi$ to the integrand where $\varphi$ is an external potential. The bracket of Eq. (1), which follows from the canonical Hamiltonian formulation of Newcomb[15] through the transformation from canonical Lagrangian to noncanonical Eulerian variables, is given by

$$\{F, G\}_Z = -\int_V \left\{ F_\rho \nabla \cdot G_{\mathbf{v}} - G_\rho \nabla \cdot F_{\mathbf{v}} + \frac{\nabla \times \mathbf{v}}{\rho} \cdot (G_{\mathbf{v}} \times F_{\mathbf{v}}) \right.$$
$$+ \frac{\nabla s}{\rho} \cdot (F_s G_{\mathbf{v}} - G_s F_{\mathbf{v}})$$
$$+ \mathbf{B} \cdot \left[ \left( \frac{1}{\rho} F_{\mathbf{v}} \cdot \nabla \right) G_{\mathbf{B}} - \left( \frac{1}{\rho} G_{\mathbf{v}} \cdot \nabla \right) F_{\mathbf{B}} \right]$$
$$\left. + \mathbf{B} \cdot \left[ \left( \nabla \frac{1}{\rho} F_{\mathbf{v}} \right) \cdot G_{\mathbf{B}} - \left( \nabla \frac{1}{\rho} G_{\mathbf{v}} \right) \cdot F_{\mathbf{B}} \right] \right\} d^3 r, \tag{3}$$

where $F$ and $G$ are two generic functionals and subscripts indicate functional derivatives.

Given a generic functional $F$, the functional derivative is defined by $\delta F = \int_V F_Z \cdot \delta Z \, d^3 r$ (cf., e.g., Ref. 22) and, in particular, the functional derivatives of the Hamiltonian (2) with respect to the variables $Z$ are

$$H_\rho = \frac{1}{2} v^2 + U + \frac{p}{\rho}, \quad H_{\mathbf{v}} = \rho \mathbf{v}, \quad H_s = \rho T, \quad H_{\mathbf{B}} = \frac{1}{4\pi} \mathbf{B}. \tag{4}$$

The functional derivatives of the variables $Z$ can be calculated by making use of the identity

$$Z(\mathbf{x}) = \int_V Z(\mathbf{x}') \delta(\mathbf{x}' - \mathbf{x}) d^3 r, \tag{5}$$

giving, for example, $\delta \rho(\mathbf{x}) / \delta \rho(\mathbf{x}') = \delta(\mathbf{x}' - \mathbf{x})$, which removes the integral of the Poisson bracket when evaluating (1).

Substituting expressions (4) and (5) into Eq. (1), we obtain the equations of MHD,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}), \tag{6}$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla \left( \frac{v^2}{2} + U + \frac{p}{\rho} \right) - (\nabla \times \mathbf{v}) \times \mathbf{v} + T \nabla s$$
$$+ \frac{1}{4\pi\rho} (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{7}$$

$$\frac{\partial s}{\partial t} = -\mathbf{v} \cdot \nabla s, \tag{8}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{B} \times \mathbf{v}), \tag{9}$$

where Eq. (6) represents mass conservation equation, Eq. (7) represents momentum balance, Eq. (8) represents entropy advection, and Eq. (9) is Faraday's law for a perfectly conductive medium. In actuality, the Poisson bracket of (3) gives MHD in conservation form, in which Eqs. (7) and (9) differ by terms involving $\nabla \cdot \mathbf{B}$, but this will not bear on our development. (In Ref. 18, it was shown that $\nabla \cdot \mathbf{B} = 0$ is not needed for MHD to be Hamiltonian and the results of Ref. 25 indicate that the conservation form is superior for numerical computation.)

The Poisson bracket of (3) can be rewritten in terms of any complete set of variables—switching from one set to another amounts to a change of coordinates. A convenient form of the MHD Poisson bracket is obtained by using, instead of the variables $\mathbf{v}$ and $s$, the density variables $\mathbf{M} = \rho \mathbf{v}$ and $\sigma = \rho s$. We let $\mathcal{Z} := (\rho, \mathbf{M}, \sigma, \mathbf{B})$ denote the new set. To transform from $Z$ to $\mathcal{Z}$, we use the functional chain rule identities,

$$F_\rho|_{\mathbf{v}, s} = F_\rho|_{\mathbf{M}, \sigma} + \mathbf{v} \cdot F_{\mathbf{M}} + s F_\sigma, \quad F_{\mathbf{v}} = \rho F_{\mathbf{M}}, \quad F_s = \rho F_\sigma, \tag{10}$$

with $F_{\mathbf{B}}$ unchanged, to transform the Poisson bracket of (3) into

$$\{F, G\}_{\mathcal{Z}} = -\int_V \{\rho(F_{\mathbf{M}} \cdot \nabla G_\rho - G_{\mathbf{M}} \cdot \nabla F_\rho)$$
$$+ \mathbf{M} \cdot [(F_{\mathbf{M}} \cdot \nabla) G_{\mathbf{M}} - (G_{\mathbf{M}} \cdot \nabla) F_{\mathbf{M}}]$$
$$+ \sigma(F_{\mathbf{M}} \cdot \nabla G_\sigma - G_{\mathbf{M}} \cdot \nabla F_\sigma)$$
$$+ \mathbf{B} \cdot [(F_{\mathbf{M}} \cdot \nabla) G_{\mathbf{B}} - (G_{\mathbf{M}} \cdot \nabla) F_{\mathbf{B}}]$$
$$+ \mathbf{B} \cdot (\nabla F_{\mathbf{M}} \cdot G_{\mathbf{B}} - \nabla G_{\mathbf{M}} \cdot F_{\mathbf{B}})\} d^3 r. \tag{11}$$

The bracket of (11) is the Lie-Poisson bracket (see Ref. 22), i.e., a bracket linear in each variable, obtained in Ref. 16.

## III. SYMMETRIC MHD

All geometric symmetries can be described as a combination of axial and translational symmetry, a decomposition of helical symmetry. Given a cylindrical coordinate system $(r, \phi, z)$, we define a helical coordinate $u = \phi[l]\sin\alpha + z\cos\alpha$, where $[l]$ is a scale length and $\alpha$ defines the helical angle. The unit vector in the direction of the coordinate $u$ can be written as

$$\mathbf{u} = kr\nabla u = \hat{\phi} k[l] \sin\alpha + \hat{\mathbf{z}} kr \cos\alpha, \tag{12}$$

where $k = ([l]^2 \sin^2\alpha + r^2 \cos^2\alpha)^{-1/2}$ represents a metric factor. The second helical direction is given in terms of the following unit vector:

$$\mathbf{h} = kr\nabla r \times \nabla u = -\hat{\phi} kr \cos\alpha + \hat{\mathbf{z}} k[l] \sin\alpha, \tag{13}$$

and the helical symmetry is expressed by the fact that $\mathbf{h} \cdot \nabla f = 0$, where $f$ is a generic scalar function. The direction $\mathbf{h}$, called the symmetry direction, can be chosen to obtain axial ($\alpha = 0$), translational ($\alpha = \pi/2$), or true helical ($0 < \alpha < \pi/2$) symmetry, with the metric factor $k$ changing accordingly. In the following, we use the identities,

$$\nabla \cdot \mathbf{h} = 0 \quad \text{and} \quad \nabla \times (k\mathbf{h}) = -\mathbf{h}k^3[l]\sin 2\alpha, \qquad (14)$$

which imply for $\sin 2\alpha = 0$ the existence of the coordinate $\nabla h = k\mathbf{h}$ in the symmetry direction.

Using the notation described before, the magnetic field and the mass flow can be rewritten as

$$\begin{aligned} \mathbf{B}(r,u) &= B_h(r,u)\mathbf{h} + \mathbf{B}_\perp(r,u), \\ \mathbf{M}(r,u) &= M_h(r,u)\mathbf{h} + \mathbf{M}_\perp(r,u) \end{aligned} \qquad (15)$$

and, since $\nabla \cdot \mathbf{B} = 0$, the magnetic field perpendicular to the symmetry direction can be expressed in terms of a magnetic flux function $\psi = \psi(r,u)$ as $\mathbf{B}_\perp(r,u) = \nabla\psi \times k\mathbf{h}$.

Given a generic functional $F$ and using the chain rule, the following functional derivative relations result

$$F_{B_h} = F_\mathbf{B} \cdot \mathbf{h}, \quad F_\psi = \nabla \cdot (F_\mathbf{B} \times k\mathbf{h}), \quad \text{and}$$
$$F_\mathbf{M} = F_{M_h}\mathbf{h} + F_{\mathbf{M}_\perp}. \qquad (16)$$

In term of the variables $\mathcal{Z}_S := (\rho, \mathbf{M}_\perp, M_h, \sigma, \psi, B_h)$, the Poisson bracket of Eq. (3) transforms into the "symmetric" MHD bracket given by

$$\begin{aligned} \{F,G\}_{\mathcal{SYM}} = -\int_V &\{\rho(F_{\mathbf{M}_\perp} \cdot \nabla G_\rho - G_{\mathbf{M}_\perp} \cdot \nabla F_\rho) \\ &+ M_h[F_{\mathbf{M}_\perp} \cdot \nabla(kG_{M_h}) - G_{\mathbf{M}_\perp} \cdot \nabla(kF_{M_h})]/k \\ &+ (k^2[l]\sin 2\alpha)M_h\mathbf{h} \cdot (F_{\mathbf{M}_\perp} \times G_{\mathbf{M}_\perp}) + \mathbf{M}_\perp \\ &\cdot [(F_{\mathbf{M}_\perp} \cdot \nabla)G_{\mathbf{M}_\perp} - (G_{\mathbf{M}_\perp} \cdot \nabla)F_{\mathbf{M}_\perp}] \\ &+ \sigma(F_{\mathbf{M}_\perp} \cdot \nabla G_\sigma - G_{\mathbf{M}_\perp} \cdot \nabla F_\sigma) \\ &+ kB_h[F_{\mathbf{M}_\perp} \cdot \nabla(G_{B_h}/k) - G_{\mathbf{M}_\perp} \cdot \nabla(F_{B_h}/k)] \\ &+ \psi(F_{\mathbf{M}_\perp} \cdot \nabla G_\psi - G_{\mathbf{M}_\perp} \cdot \nabla F_\psi) \\ &- \psi(F_\psi \nabla \cdot G_{\mathbf{M}_\perp} - G_\psi \nabla \cdot F_{\mathbf{M}_\perp}) \\ &- (k^3[l]\sin 2\alpha)\nabla\psi \cdot (F_{B_h}G_{\mathbf{M}_\perp} - G_{B_h}F_{\mathbf{M}_\perp}) \\ &+ \psi([G_{B_h}/k, kF_{M_h}] - [F_{B_h}/k, kG_{M_h}])\}d^3r, \end{aligned} \qquad (17)$$

where $[F,G] := (\nabla F \times \nabla G) \cdot k\mathbf{h}$. Because this calculation is similar to one of Ref. 24, we forgo the details.

Using (17), the equations for symmetric MHD dynamics are obtained:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{M}_\perp, \qquad (18)$$

$$\frac{\partial M_h}{\partial t} = -k\nabla \cdot \left(\mathbf{M}_\perp \frac{M_h}{k\rho}\right) + k\left[\psi, \frac{1}{4\pi k}B_h\right], \qquad (19)$$

$$\begin{aligned} \frac{\partial \mathbf{M}_\perp}{\partial t} = &-\rho\nabla\left(\frac{M^2}{2\rho^2} + U + \frac{p}{\rho}\right) - \left(\nabla \times \frac{\mathbf{M}_\perp}{\rho}\right) \times \mathbf{M}_\perp \\ &-(\nabla \cdot \mathbf{M}_\perp)\frac{\mathbf{M}_\perp}{\rho} + kM_h\nabla\frac{M_h}{k\rho} \\ &+ (k^2[l]\sin 2\alpha)\left(\frac{M_h}{\rho}\mathbf{h} \times \mathbf{M}_\perp\right) + \rho T\nabla\frac{\sigma}{\rho} \\ &- \nabla \cdot \left(\frac{k^2}{4\pi}\nabla\psi\right)\nabla\psi - kB_h\nabla\frac{B_h}{4\pi k} \\ &- (k^3[l]\sin 2\alpha)\frac{B_h}{4\pi}\nabla\psi, \end{aligned} \qquad (20)$$

$$\frac{\partial \sigma}{\partial t} = -\nabla \cdot \left(\mathbf{M}_\perp \frac{\sigma}{\rho}\right), \qquad (21)$$

$$\begin{aligned} \frac{\partial B_h}{\partial t} = &-\frac{1}{k}\nabla \cdot \left(\mathbf{M}_\perp \frac{kB_h}{\rho}\right) + (k^3[l]\sin 2\alpha)\nabla\psi \\ &\cdot \frac{\mathbf{M}_\perp}{\rho} + \frac{1}{k}\left[\psi, \frac{kM_h}{\rho}\right], \end{aligned} \qquad (22)$$

$$\frac{\partial \psi}{\partial t} = -\nabla\psi \cdot \frac{\mathbf{M}_\perp}{\rho}. \qquad (23)$$

In comparison to Eqs. (6)–(9), the number of equations needed to describe the symmetric dynamics is reduced because of the introduction of $\psi$. Moreover, the differential operator $\nabla$ in Eqs. (18)–(23) only depends on $u$ and $r$.

## IV. SYMMETRIC CASIMIRS

Now, we seek the Casimir invariants associated with the helically symmetric MHD bracket (17), i.e., functionals $C$ that satisfy $\{F,C\}_{\mathcal{SYM}} = 0$ for all functionals $F$. With (17), we see that $\{F,C\}_{\mathcal{SYM}} = 0$ implies

$$\begin{aligned} \int_V &\left[F_\rho\mathfrak{C}_1 + kF_{M_h}\mathfrak{C}_2 + F_\sigma\mathfrak{C}_3 + \frac{1}{k}F_{B_h}\mathfrak{C}_4 \right. \\ &\left. + F_\psi\mathfrak{C}_5 + F_{\mathbf{M}_\perp} \cdot \mathfrak{C}_6\right]d^3r = 0, \end{aligned} \qquad (24)$$

where the functions $\mathfrak{C}_i$ are given by

$$\mathfrak{C}_1 = -\nabla \cdot (\rho C_{\mathbf{M}_\perp}), \qquad (25)$$

$$\mathfrak{C}_2 = -\nabla \cdot \left(\frac{1}{k}M_h C_{\mathbf{M}_\perp}\right) - \left[\psi, \frac{1}{k}C_{B_h}\right], \qquad (26)$$

$$\mathfrak{C}_3 = -\nabla \cdot (\sigma C_{\mathbf{M}_\perp}), \qquad (27)$$

$$\begin{aligned} \mathfrak{C}_4 = &-\nabla \cdot (kB_h C_{\mathbf{M}_\perp}) - [\psi, kC_{M_h}] \\ &+ (k^4[l]\sin 2\alpha)(\nabla\psi \cdot C_{\mathbf{M}_\perp}), \end{aligned} \qquad (28)$$

$$\mathfrak{C}_5 = -\nabla\psi \cdot C_{\mathbf{M}_\perp}, \qquad (29)$$

$$\mathfrak{C}_6 = -\rho \nabla C_\rho - \frac{M_h}{k} \nabla(k C_{M_h}) - (k^2[l]\sin 2\alpha) M_h (C_{\mathbf{M}_\perp} \times \mathbf{h})$$

$$-[(\nabla \times \mathbf{M}_\perp) \times C_{\mathbf{M}_\perp} + \nabla(\mathbf{M}_\perp \cdot C_{\mathbf{M}_\perp})$$

$$+(\nabla \cdot C_{\mathbf{M}_\perp})\mathbf{M}_\perp] - \sigma \nabla C_\sigma - k B_h \nabla \left( \frac{1}{k} C_{B_h} \right)$$

$$+C_\psi \nabla \psi - (k^3[l]\sin 2\alpha) C_{B_h} \nabla \psi. \tag{30}$$

Since each term in the bracket must vanish separately, this implies the Casimir conditions $\mathfrak{C}_i = 0$ for $i = 1, \ldots, 6$.

We first investigate the case where $C_{\mathbf{M}_\perp} = 0$, which implies the reduced set of conditions

$$\mathfrak{C}_2 = -\left[ \psi, \frac{1}{k} C_{B_h} \right] = 0, \tag{31}$$

$$\mathfrak{C}_4 = -[\psi, k C_{M_h}] = 0, \tag{32}$$

$$\mathfrak{C}_6 = -\rho \nabla C_\rho - \frac{M_h}{k} \nabla(k C_{M_h}) - \sigma \nabla C_\sigma - k B_h \nabla \left( \frac{1}{k} C_{B_h} \right)$$

$$+C_\psi \nabla \psi - (k^3[l]\sin 2\alpha) C_{B_h} \nabla \psi = 0. \tag{33}$$

Upon substituting the functional

$$C_1 = \int_V \rho \mathcal{J} \left( \frac{\sigma}{\rho}, \psi, \frac{1}{\rho} \left[ \frac{\sigma}{\rho}, \psi \right], \frac{1}{\rho} \left[ \frac{1}{\rho} \left[ \frac{\sigma}{\rho}, \psi \right], \psi \right],$$

$$\times \frac{1}{\rho} \left[ \frac{\sigma}{\rho}, \frac{1}{\rho} \left[ \frac{\sigma}{\rho}, \psi \right] \right], \ldots \right) d^3 r \tag{34}$$

into Eqs. (31)–(33), it is straightforward to prove that $C_1$ defines a family of Casimir invariants. In fact, since

$$\frac{1}{\rho} \left[ \frac{\sigma}{\rho}, \psi \right] = \frac{\mathbf{B}}{\rho} \cdot \nabla \frac{\sigma}{\rho} \quad \text{and} \quad \frac{1}{\rho} \left[ \frac{1}{\rho} \left[ \frac{\sigma}{\rho}, \psi \right], \psi \right]$$

$$= \frac{\mathbf{B}}{\rho} \cdot \nabla \left( \frac{\mathbf{B}}{\rho} \cdot \nabla \frac{\sigma}{\rho} \right), \tag{35}$$

the Casimir invariants (34) are similar, but not equivalent, to those of Refs. 26 and 27, which are more general than those described in Ref. 24. This Casimir is akin to Ertel's potential vorticity of geophysical fluid dynamics, since both can be traced to Nöther's second theorem (see Refs. 26 and 27).

Next, from conditions (31) and (33) we deduce that

$$C_2 = \int_V [k B_h \mathcal{H}(\psi) + (k^4[l]\sin 2\alpha) \mathcal{H}^-(\psi)] d^3 r, \tag{36}$$

where $\mathcal{H}^-(\psi) := \int^\psi \mathcal{H}(\psi') d\psi'$, defines a second family of Casimir invariants. It can be shown that, for $\mathcal{H} = 2\psi$, the Casimir $C_2$ reduces to the well-known magnetic helicity. Indeed, the family of Casimir invariants (36) represents a generalization of the magnetic helicity analogous to that found in Ref. 28 for axisymmetric equilibria. Analogously, from condition (32) we obtain the third family of Casimirs

$$C_3 = \int_V \frac{1}{k} M_h \mathcal{G}(\psi) d^3 r. \tag{37}$$

If we suppose $C_{\mathbf{M}_\perp} \neq 0$, then, upon setting expression (29) to zero, it follows that

$$C_{\mathbf{M}_\perp} = \nabla \psi \times A k \mathbf{h}, \tag{38}$$

where $A$ is a generic function. Therefore, from the expressions (25)–(28) we obtain the following Casimir conditions:

$$\mathfrak{C}_1 = [\psi, \rho A] = 0, \tag{39}$$

$$\mathfrak{C}_2 = \left[ \psi, \frac{M_h}{k} A - \frac{1}{k} C_{B_h} \right] = 0, \tag{40}$$

$$\mathfrak{C}_3 = \rho A \left[ \psi, \frac{\sigma}{\rho} \right] = 0, \tag{41}$$

$$\mathfrak{C}_4 = [\psi, k B_h A - k C_{M_h}] = 0, \tag{42}$$

which imply that, unless (see Eq. (41))

$$\left[ \psi, \frac{\sigma}{\rho} \right] = 0, \tag{43}$$

no further Casimir functionals can be found. It can be easily shown that condition (43) holds for stationary flows and vice versa (from $\nabla \cdot \mathbf{M} = 0$, we deduce $\mathbf{M} = \nabla \chi \times k \mathbf{h}$, and using the perfect conductivity equation, we obtain $[\psi, \chi] = 0$. Analogously, the entropy equation becomes $[\sigma/\rho, \chi] = 0$ and, except where $\nabla \chi = 0$ and $\nabla \psi \neq 0$, $[\sigma/\rho, \psi] = 0$). If Eq. (43) holds, from condition (39) we obtain $A = \mathcal{F}/\rho$, where $\mathcal{F}$ is a generic function of $\psi$ or $\sigma/\rho$, and conditions (40) and (42) imply

$$C_{B_h} = \frac{M_h}{\rho} \mathcal{F} \quad \text{and} \quad C_{M_h} = \frac{B_h}{\rho} \mathcal{F}, \tag{44}$$

plus solutions in the form of (36) and (37). By integrating conditions (38) and (44), we obtain

$$C_4 = \int_V (\mathbf{M}_\perp \cdot \mathbf{B}_\perp + M_h B_h) \mathcal{F}/\rho d^3 r = \int_V \mathbf{v} \cdot \mathbf{B} \mathcal{F} d^3 r, \tag{45}$$

which also satisfies the condition given by Eq. (30), and thus defines the last family of Casimir invariants. For $\mathcal{F} = 1$, the family of Casimir invariants $C_4$ reduces to the well-known cross-helicity invariant.

For flows that satisfy condition (43), the family of invariants (34) can be rewritten in the simpler form

$$C_1 = \int_V \rho \mathcal{J} d^3 r, \tag{46}$$

where $\mathcal{J}$ is a generic function of $\psi$ or $\sigma/\rho$.

Since Casimirs are conserved quantities, their integrands, say $\mathcal{C}_i$, are densities associated with the "currents" $\mathbf{J}_i$ that satisfy conservation equations of the form $\partial \mathcal{C}_i / \partial t + \nabla \cdot \mathbf{J}_i = 0$, where $i = 1, \ldots, 4$. These Casimir currents are given by

$$\mathbf{J}_1 = \mathbf{M}_\perp \mathcal{J},$$

$$\mathbf{J}_2 = \left(\mathbf{M}_\perp \frac{kB_h}{\rho} + \frac{kM_h}{\rho}\mathbf{B}_\perp\right)\mathcal{H},$$

$$\mathbf{J}_3 = \left(\mathbf{M}_\perp \frac{M_h}{k\rho} + \frac{B_h}{4\pi k}\mathbf{B}_\perp\right)\mathcal{G}, \qquad (47)$$

$$\mathbf{J}_4 = \mathbf{M} \times (\mathbf{B} \times \mathbf{M})\frac{\mathcal{F}}{\rho^2} - \mathbf{B}\left(\frac{M^2}{2\rho^2} + U + \frac{p}{\rho}\right)\mathcal{F}.$$

If we assume the bounding surface is a fixed magnetic surface, i.e., $\mathbf{n} \cdot \mathbf{B} = 0$ and $\mathbf{n} \cdot \mathbf{M} = 0$, this surface respects the symmetry, and the unit surface normal $\mathbf{n}$ satisfies $\mathbf{n} \cdot \mathbf{h} = 0$. Consequently, $\mathbf{n} \cdot \mathbf{B}_\perp = 0$ and $\mathbf{n} \cdot \mathbf{M}_\perp = 0$. Thus, for this kind of fixed boundary condition, the Casimirs are conserved. However, the possibility of Casimir injection exists and in a future publication we will consider more general boundary conditions.

## V. VARIATIONAL PRINCIPLE AND EQUILIBRIA

Now, we proceed to construct the energy-Casimir variational principle for symmetric MHD equilibria. With the knowledge that extrema of the energy-Casimir functional must correspond to equilibria, we consider

$$\mathfrak{F} = H - \int_V \rho\mathcal{J}d^3r - \int_V [kB_h\mathcal{H} + (k^4[l]\sin 2\alpha)\mathcal{H}^-]d^3r$$

$$- \int_V \frac{1}{k}M_h\mathcal{G}d^3r - \int_V \mathbf{v}\cdot\mathbf{B}\mathcal{F}d^3r, \qquad (48)$$

where the Hamiltonian (2) is expressed in terms of symmetric variables $\mathcal{Z}_S$ as

$$H = \int_V \left(\frac{M_h^2}{2\rho} + \frac{M_\perp^2}{2\rho} + \rho U + \frac{k^2|\nabla\psi|^2}{8\pi} + \frac{B_h^2}{8\pi}\right)d^3r, \qquad (49)$$

and $\mathcal{F}, \mathcal{G}, \mathcal{H}(\mathcal{H}^-)$, and $\mathcal{J}$ are four arbitrary functions of $\psi$. Moreover, in order to satisfy Eq. (43) we consider $\sigma/\rho = \mathcal{S}(\psi)$. Thus, the constrained energy in terms of the variables $\mathcal{Z}_S$ is given by

$$\mathfrak{F}[\mathcal{Z}_S] = \int_V \left(\frac{M_\perp^2}{2\rho} + \frac{M_h^2}{2\rho} + \rho U + \frac{k^2|\nabla\psi|^2}{8\pi} + \frac{B_h^2}{8\pi}\right.$$

$$\left. -\rho\mathcal{J} - kB_h\mathcal{H} - k^4[l]\sin 2\alpha \,\mathcal{H}^- - \frac{1}{k}M_h\mathcal{G} - \frac{\mathbf{M}}{\rho}\cdot\mathbf{B}\mathcal{F}\right)d^3r, \qquad (50)$$

or in terms of the variables $Z_S := (\rho, \mathbf{v}_\perp, v_h, \psi, B_h)$ is given by

$$\mathfrak{F}[Z_S] = \int_V \left(\frac{\rho v_\perp^2}{2} + \frac{\rho v_h^2}{2} + \rho U + \frac{k^2|\nabla\psi|^2}{8\pi} + \frac{B_h^2}{8\pi} - \rho\mathcal{J}\right.$$

$$\left. -kB_h\mathcal{H} - k^4[l]\sin 2\alpha \,\mathcal{H}^- - \frac{1}{k}\rho v_h\mathcal{G} - \mathbf{v}\cdot\mathbf{B}\mathcal{F}\right)d^3r. \qquad (51)$$

The first variation of the latter expression is given by

$$\delta\mathfrak{F} = \int_V \left[(\rho\mathbf{v}_\perp - \mathbf{B}_\perp\mathcal{F})\cdot\delta\mathbf{v}_\perp + \left(\rho v_h - B_h\mathcal{F} - \frac{1}{k}\rho\mathcal{G}\right)\delta v_h\right.$$

$$+ \left(\frac{v^2}{2} + U + \frac{p}{\rho} - \mathcal{J} - \frac{1}{k}v_h\mathcal{G}\right)\delta\rho$$

$$+ \left(\frac{B_h}{4\pi} - k\mathcal{H} - v_h\mathcal{F}\right)\delta B_h + \left(-\nabla\cdot\left(\frac{k^2}{4\pi}\nabla\psi\right)\right.$$

$$+ \rho T\mathcal{S}' - \rho\mathcal{J}' - kB_h\mathcal{H}' - k^4[l]\sin 2\alpha\,\mathcal{H}$$

$$\left.\left. - \frac{1}{k}\rho v_h\mathcal{G}' - \mathbf{v}\cdot\mathbf{B}\mathcal{F}' + \nabla\cdot(\mathcal{F}k\mathbf{h}\times\mathbf{v}_\perp)\right)\delta\psi\right]d^3r. \qquad (52)$$

Here, we have integrated by parts and neglected surface terms consistent with assumed boundary conditions. Symmetric equilibria thus satisfy the set of equations

$$\rho\mathbf{v}_\perp - \mathbf{B}_\perp\mathcal{F} = 0, \qquad (53)$$

$$\rho v_h - B_h\mathcal{F} - \frac{1}{k}\rho\mathcal{G} = 0, \qquad (54)$$

$$\frac{v^2}{2} + U + \frac{p}{\rho} - \mathcal{J} - \frac{1}{k}v_h\mathcal{G} = 0, \qquad (55)$$

$$\frac{B_h}{4\pi} - k\mathcal{H} - v_h\mathcal{F} = 0, \qquad (56)$$

$$-\nabla\cdot\left(\frac{k^2}{4\pi}\nabla\psi\right) + \rho T\mathcal{S}' - \rho\mathcal{J}' - kB_h\mathcal{H}'$$

$$- k^4[l]\sin 2\alpha\,\mathcal{H} - \frac{1}{k}\rho v_h\mathcal{G}' - \mathbf{v}\cdot\mathbf{B}\mathcal{F}'$$

$$+ \nabla\cdot(\mathcal{F}k\mathbf{h}\times\mathbf{v}_\perp) = 0. \qquad (57)$$

Equations (54) and (56) can be combined to obtain

$$v_h = \left(4\pi k\mathcal{H}\frac{\mathcal{F}}{\rho} + \frac{\mathcal{G}}{k}\right)(1 - \mathrm{M}^2)^{-1} \quad \text{and}$$

$$B_h = \left(4\pi k\mathcal{H} + 4\pi\mathcal{F}\frac{\mathcal{G}}{k}\right)(1 - \mathrm{M}^2)^{-1}, \qquad (58)$$

which are two explicit relationships for $v_h$ and $B_h$ that make it possible to express these two variables in terms of the flux function, the cylindrical radius (which appears in $k$), and the plasma density. The dimensionless parameter $\mathrm{M}^2 = 4\pi\mathcal{F}^2/\rho$ that appears in the first of Eq. (58) is the square of the Alfvén Mach number. Notice that on Alfvén surfaces, i.e., points where $\mathrm{M} = 1$, the regularity condition (see, e.g., Ref. 29),

$$4\pi k\mathcal{H}\frac{\mathcal{F}}{\rho} + \frac{\mathcal{G}}{k} = 0 \Longleftrightarrow 4\pi k\mathcal{H} + 4\pi\mathcal{F}\frac{\mathcal{G}}{k} = 0, \qquad (59)$$

needs to be satisfied. In general, given the flux functions $\mathcal{F}, \mathcal{G},$ and $\mathcal{H}$ and the boundary conditions, we can only check *a posteriori* whether the regularity condition is satisfied or not (of course, compatibility of the flux functions can be assessed *a priori*; for example, if $\mathcal{F} > 0$ and $\mathcal{G} > 0$, then $\mathcal{H} < 0$).

Equation (55) gives a relationship between the plasma density, the magnetic flux function and its gradient, and $k$,

$$\frac{k^2}{2}|\nabla\psi|^2\left(\frac{\mathcal{F}}{\rho}\right)^2 + \frac{v_h^2}{2} + U + p/\rho - \frac{v_h}{k}\mathcal{G} = \mathcal{J}, \qquad (60)$$

where $U + p/\rho$ is the enthalpy. Equation (60), a generalization of the Bernoulli equation of hydrodynamics, can be viewed as an equation for the density $\rho$ given $\psi$, making use of the second of Eq. (58) and a particular choice of the Casimir functions $\mathcal{F}, \mathcal{G}, \mathcal{H}$, and $\mathcal{J}$; however, in general it is not possible to obtain an explicit form for $\rho$.

The first term in Eq. (57) can be rewritten in terms of the variables $r$ and $u$ as

$$\nabla\cdot\left(\frac{k^2}{4\pi}\nabla\psi\right) = \frac{1}{4\pi r^2}\left[\frac{\partial^2\psi}{\partial u^2} + r\frac{\partial}{\partial r}\left(rk^2\frac{\partial\psi}{\partial r}\right)\right], \qquad (61)$$

which corresponds to the differential operator of the so-called JOKF equation[30–32] (named after the authors of Ref. 30). Moreover, by using Eq. (53), the last two terms of Eq. (57) can be manipulated to obtain the following expressions:

$$\mathbf{v}\cdot\mathbf{B}\mathcal{F}' = v_h B_h \mathcal{F}' + k^2|\nabla\psi|^2\frac{\mathcal{F}\mathcal{F}'}{\rho} \quad \text{and}$$

$$\nabla\cdot(\mathcal{F}k\mathbf{h}\times\mathbf{v}_\perp) = \nabla\cdot\left(\frac{k^2\mathcal{F}^2}{\rho}\nabla\psi\right). \qquad (62)$$

Then, Eq. (57) becomes

$$\nabla\cdot\left[(1-\mathrm{M}^2)\frac{k^2}{4\pi}\nabla\psi\right] + k^2|\nabla\psi|^2\frac{\mathcal{F}\mathcal{F}'}{\rho}$$

$$= \rho\left(T\mathcal{S}' - \mathcal{J}' - v_h\frac{\mathcal{G}'}{k}\right) - B_h(k\mathcal{H}' + v_h\mathcal{F}')$$

$$- (k^4[l]\sin 2\alpha)\mathcal{H}, \qquad (63)$$

which is a generalization of the JOKF equation that includes flow.

The above equations were previously presented in Ref. 33 and various special solutions were obtained by several authors.[34–40] However, the general variational principle $\delta\mathfrak{F} = 0$ for helical equilibria with flow appears to be new, as well as reduced variational principles that we subsequently obtain by eliminating the algebraic constraints.

Upon choosing $k = 1/r$ and $\alpha = 0$, Eq. (63) reduces to the azimuthally symmetric case and one obtains the generalized Grad Shafranov equation with flow discussed in Ref. 24. Similarly, upon choosing $k = 1$ and $\alpha = \pi/2$, this equation reduces to the translationally symmetric case discussed in Ref. 41. As discussed in Refs. 2 and 29, the equation for the generalized equilibria is hyperbolic for $\mathrm{M}_c^2 \leq \mathrm{M}^2 \leq \mathrm{M}_s^2$ and for $\mathrm{M}^2 \geq \mathrm{M}_f^2$, where $\mathrm{M}_c^2 \equiv \gamma p/(\gamma p + B^2/4\pi)$ is the square Alfvén Mach number corresponding to the "cusp velocity" and

$$\mathrm{M}_{f,s}^2 \equiv \frac{4\pi\gamma p + B^2}{2B_\perp^2}\left\{1 \pm \left[1 - \frac{16\pi\gamma p B_\perp^2}{(4\pi\gamma p + B^2)^2}\right]^{1/2}\right\} \qquad (64)$$

is that relative to the fast and slow magnetosonic velocities, respectively, $\mathrm{M}_f^2$ and $\mathrm{M}_s^2$.

The variational principle of Eq. (52) can be reduced in several steps by "back-substituting" various algebraic relations. First, by substituting the expression for the perpendicular velocity given by Eq. (53) into the functional $\mathfrak{F}$ we obtain a variational principle that depends on the reduced set of independent variables, $\psi, \rho, v_h$, and $B_h$, viz.

$$\mathfrak{F}[\psi,\rho,v_h,B_h] = \int_V \left(\frac{\rho v_h^2}{2} + \rho U + (1-\mathrm{M}^2)\frac{k^2|\nabla\psi|^2}{8\pi} + \frac{B_h^2}{8\pi}\right.$$

$$\left. -\rho\mathcal{J} - kB_h\mathcal{H} - k^4[l]\sin 2\alpha\,\mathcal{H}^- - \frac{1}{k}\rho v_h\mathcal{G} - v_h B_h\mathcal{F}\right)d^3r.$$

$$(65)$$

Similarly, we can reduce further by using Eq. (56) to eliminate $B_h$, yielding,

$$\mathfrak{F}[\psi,\rho,v_h] = \int_V \left(\frac{\rho v_h^2}{2} + \rho U + (1-\mathrm{M}^2)\frac{k^2|\nabla\psi|^2}{8\pi}\right.$$

$$-\frac{1}{8\pi}(4\pi k\mathcal{H} + 4\pi v_h\mathcal{F})^2 - \rho\mathcal{J}$$

$$\left. -k^4[l]\sin 2\alpha\,\mathcal{H}^- - \frac{1}{k}\rho v_h\mathcal{G}\right)d^3r. \qquad (66)$$

Next, we can use the first expression of Eq. (58) to eliminate the dependence on $v_h$, obtaining the functional

$$\mathfrak{F}[\psi,\rho] = \int_V \left[\rho U + (1-\mathrm{M}^2)\frac{k^2|\nabla\psi|^2}{8\pi} - \rho\mathcal{J}\right.$$

$$-k^4[l]\sin 2\alpha\,\mathcal{H}^- - \left(\frac{\rho\mathcal{G}^2}{2k^2} + 2\pi k^2\mathcal{H}^2 + 4\pi\mathcal{H}\mathcal{G}\mathcal{F}\right)$$

$$\left. \times(1-\mathrm{M}^2)^{-1}\right]d^3r. \qquad (67)$$

One could attempt to reduce further, but because of the form of Eq. (60), the density cannot be explicitly eliminated without making further assumptions. However, the density can be viewed as an implicit functional of $\psi$ through Eq. (60). Thus, in a sense, we have a minimal variational principle in terms of the variable $\psi$.

Although the variational principle of Eq. (67) is minimal, it may not be the most efficacious to use. Observe, the last substitution introduced a potential singularity at $\mathrm{M} = 1$. If we seek extrema of Eq. (67) by considering a sequence of $L^2$ functions, the principle (67) in general leads to singularities on $\mathrm{M} = 1$. However, if we expand $v_h, \rho$, and $\psi$ and insert into the variational principle (66), the quantity $v_h$ will always be regular and this also follows for the integrand. Nevertheless, the principle of Eq. (67) may be useful. For example, suppose $M$ depends only on $\psi$, which is the case for incompressible equilibria (cf. Ref. 40). Then, the term $\mathcal{E}_p := \int_V (1-\mathrm{M}^2)k^2|\nabla\psi|^2 d^3r/8\pi$ can be simplified by a simple variable change from $\psi$ to a new variable $\chi$. If we suppose $\psi = \Psi(\chi)$, substitute into $\mathcal{E}_p$, and set $\Psi'^2(1 - $

$M^2) = 1$ we obtain $\mathcal{E}_p = \int_V k^2 |\nabla \chi|^2 d^3 r / 8\pi$. Therefore, the transformation

$$\chi = \int^\psi \sqrt{1 - M^2(\psi')} d\psi' \tag{68}$$

eliminates the $|\nabla \psi|$ term from Eq. (63) and yields an equation in terms of $\chi$ that is identical to that without "poloidal" flow. Thus one can use Eq. (68) to map equilibria without flow into those with flow profiles determined by $M(\psi)$. This transformation was first noted in Ref. 42 for two-dimensional axisymmetric equilibria and generalized, including the helical case, in Ref. 40.

## VI. SUMMARY AND DISCUSSION

In this paper, we have written the noncanonical Hamiltonian structure of MHD in a general form that includes translational, azimuthal, and helical symmetry. From the noncanonical Poisson bracket, we obtained Casimir invariants for all symmetries, including a new ones that did not appear in Ref. 24. From these invariants, we constructed variational principles for equilibria, including helical symmetry, and showed how to reduce these variational principles to fewer numbers of variables. A general equilibrium equation that includes general flow was presented.

The variational principles we obtained are useful for constructing solutions by the direct method of the calculus of variations.[43] One can insert sequences of functions and reduce the extremization to the solution of algebraic equations. Approximate solutions for the case of axisymmetric and fully 3D equilibria have been obtained in this way in Refs. 44–48. Similarly, axisymmetric equilibria with flow have been obtained for application to laboratory and astrophysical plasmas[8] and plasma thrusters.[9,49] Likewise, the variational principle of Eq. (67) can be used to construct helical equilibria with and without flow that are of importance for both laboratory and naturally occurring plasmas. We list several possibilities.

First, the plasma thruster problem treated in Refs. 9 and 49 can be generalized to include the helical structures that have been observed to arise from the saturation of kink modes.[50,51] Ascertaining the nature of these structures is important for determining the effectiveness of these thrusters. This will be the subject of a future publication.

Another potential application would be to analyze helical structures called "snakes" that were detected in the JET experiment at Culham.[52] These structures, detected by soft x-ray emission, are formed by local plasma cooling caused by the ablation of a pellet injected into the tokamak. They have been interpreted as a persistent local modification along a closed magnetic field line of the global toroidal axisymmetric equilibrium. This structure in the plasma and its persistence might be described as a helical static equilibrium along the closed magnetic flux tube crossed by the pellet, and thus would be accessible by our variational principle.

Helical configurations that appear in reverse field pinch configurations, the so-called quasi single helicity states (e.g., Ref. 50) present another application. These states result from

plasma self-organization, where a dominant mode tends to suppress modes with different helicity, and have reduced magnetic turbulence and better energy confinement. Since all these helical states have a large aspect ratio, toroidal curvature effects may be neglected to first order and their equilibrium configuration can be described by our variational principles. Helical structures (flux ropes) are also found to arise in numerical simulations of three-dimensional magnetic reconnection processes.[53]

Similarly, helical equilibria can be used to model straight (large aspect ratio) stellarator configurations (e.g., Ref. 54). The Helically Symmetric Experiment at Madison Wisconsin[55] has a quasi-helically symmetric magnetic field structure and thus avoids the consequences of the lack of symmetry in the magnetic fields in conventional stellarators, which results in large deviations of particle orbits from magnetic surfaces and direct loss orbits.

Helical equilibria are of special importance for space configurations where they arise naturally as the result of the plasma streaming and kinking. In Refs. 32 and 56, the explicit construction of globally regular helical solutions for helical equilibria was carried out. These solutions are of mathematical interest since they show that helical equilibrium solutions can be found as continuous deformations of cylindrically symmetric equilibria. At the same time, they provide useful models of plasma jets in space. The extension from static to stationary helical equilibria (i.e., equilibria with flow) is of major interest for the description of plasma jets in space. In this case, exact solutions of our generalized JOKF equation (57) can be searched for by means of our reduced variational principle (67), in a manner similar to that used to obtain the axisymmetric thruster equilibria of Refs. 9 and 49.

Obtaining equilibria that are extrema of the variational principle (67) allows us to consider their stability by effecting the second variation. We will consider a variety of such energy stability calculations for a variety of equilibrium states in Ref. 23.

[1] J. P. Freidberg, *Ideal Magnetohydrodynamics,* Modern Perspectives in Energy (Springer, Berlin, 1987).
[2] J. P. Goedbloed and S. Poedts, *Principles of Magnetohydrodynamics: With Applications to Laboratory and Astrophysical Plasmas* (Cambridge University Press, Cambridge, 2004).
[3] L. Woltjer, Proc. Natl. Acad. Sci. U.S.A. **44**, 833–841 (1958); **45**, 769–771 (1959).
[4] L. Woltjer, Astrophys. J. **130**, 400–4 (1959); *ibid.* 405–13.
[5] J. B. Taylor, Phys. Rev. Lett. **33**, 1139–1141 (1974).
[6] J. B. Taylor, Rev. Mod. Phys. **58**, 741–763 (1986).
[7] E. Hameiri, Phys. Plasmas **5**, 3270–3281 (1998).
[8] J. P. Goedbloed, Phys. Plasmas **11**, L81–L83 (2004).
[9] T. Andreussi and F. Pegoraro, Phys. Plasmas **15**, 092108 (2008).
[10] K. I. Ilin and V. A. Vladimirov, Phys. Plasmas **11**, 3586–3593 (2004).
[11] G. N. Throumoulopoulos and H. Tasso, Phys. Plasmas **14**, 122104 (2007).
[12] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. R. Soc. London, Ser. A **244**, 17 (1958).
[13] K. Hain, R. Lüst, and A. Schlüter, Z. Naturforsch. **12a**, 833 (1957).

[14]E. A. Frieman and M. Rotenberg, Rev. Mod. Phys. **32**, 898 (1960).

[15]W. A. Newcomb, Nucl. Fusion Suppl. Part 2, 451–463 (1962).

[16]P. J. Morrison and J. M. Greene, Phys. Rev. Lett. **45**, 790–794 (1980).

[17]P. J. Morrison and J. M. Greene, Phys. Rev. Lett. **48**, 569 (1982).

[18]P. J. Morrison, AIP Conf. Proc. **88**, 13–45 (1982).

[19]J. L. Lagrange, *Mécanique Analytique (Analytical Mechanics)*, translated and edited by A. Boissonnade and V. N. Vagliente (Kluwer Academic, Boston, MA, Imprint: Dordrecht, 1997).

[20]V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Dynamics III: Mathematical Aspects of Classical and Celestial Mechanics* (Springer, Berlin, 1990).

[21]P. G. L. Dirichlet, Crelle **32**, 3 (1846); See Note II of J. L. Lagrange, *Mécanique Analytique*, Quatrième Édition, Tome Premier (Gauthier-Villars et Fils, Paris, 1888), p. 457.

[22]P. J. Morrison, Rev. Mod. Phys. **70**, 467 (1998).

[23]T. Andreussi, P. J. Morrison, and F. Pegoraro, "Hamiltonian Magnetohydrodynamics: Lagrangian, Eulerian, and Dynamically Accessible Stability" (unpublished).

[24]T. Andreussi, P. J. Morrison, and F. Pegoraro, Plasma Phys. Controlled Fusion **52**, 055001 (2010).

[25]S. K. Gudunov, Dokl. Akad. Nauk **139**, 521 (1961).

[26]N. Padhye and P. J. Morrison, Phys. Lett. A **219**, 287–292 (1996).

[27]N. Padhye and P. J. Morrison, Plasma Phys. Rep. **22**, 869–877 (1996).

[28]A. Bhattacharjee and R. L. Dewar, Phys. Fluids **25**, 887 (1982).

[29]R. V. E. Lovelace, C. Mehanian, C. M. Mobarry, and M. E. Sulkanen, Astrophys. J., Suppl. Ser. **62**, 1–37 (1986).

[30]J. L. Johnson, C. R. Oberman, R. M. Kulsrud, and E. A. Frieman, Phys. Fluids **1**, 281–296 (1958).

[31]B. B. Kadomtsev, Sov. Phys. JETP **10**, 962–963 (1960).

[32]O. I. Bogoyavlenskij, Lett. Math. Phys. **51**, 235–247 (2000).

[33]K. C. Tsinganos, Astrophys. J. **259**, 820–831 (1982).

[34]L. J. Palumbo and A. M. Platzeck, Astrophys. J. **416**, 656 (1993).

[35]M. Villata and K. Tsinganos, Phys. Fluids B **5**, 2153 (1993).

[36]M. Villata and A. Ferrari, Astron. Astrophys. **284**, 663 (1994).

[37]M. Villata and A. Ferrari, Phys. Plasmas **1**, 2200 (1994).

[38]L. Del Zanna and C. Chiuderi, Astron. Astrophys. **310**, 341 (1996).

[39]L. J. Palumbo and A. M. Platzeck, J. Plasma Phys. **60**, 449 (1998).

[40]G. N. Throumoulopoulos and H. Tasso, J. Plasma Phys. **62**, 449 (1999).

[41]J. P. Goedbloed and A. Lifschitz, Phys. Plasmas **4**, 3544 (1997).

[42]P. J. Morrison, Bull. Am. Phys. Soc. **31**, 1609 (1986).

[43]R. Courant and D. Hilbert, *Methods of Mathematical Physics I* (Wiley Interscience, New York, 1953).

[44]O. Betancourt and P. Garabedian, Proc. Natl. Acad. Sci. U.S.A. **73**, 984 (1976).

[45]S. P. Hirshman and J. C. Whitson, Phys. Fluids **26**, 3554 (1983).

[46]J. D. Hanson, S. P. Hirshman, S. F. Knowlton, S. F. Knowlton, L. L. Lao, E. A. Lazarus, and J. M. Shields, Nucl. Fusion **49**, 075031 (2009).

[47]P. R. Garabedian, Proc. Natl. Acad. Sci. U.S.A. **99**, 10257 (2002).

[48]P. R. Garabedian, Proc. Natl. Acad. Sci. U.S.A. **100**, 13741 (2003).

[49]T. Andreussi and F. Pegoraro, Phys. Plasmas **17**, 063507 (2010).

[50]M. Zuin, R. Cavazzana, E. Martines, G. Serianni, V. Antoni, M. Bagatin, M. Andrenucci, F. Paganucci, and P. Rossetti, Phys. Rev. Lett. **92**, 5003 (2004).

[51]W. F. Bergerson, F. Auriemma, B. E. Chapman, W. X. Ding, P. Zanca, D. L. Brower, P. Innocente, L. Lin, R. Lorenzini, E. Martines, B. Momo, J. S. Sarff, and D. Terranova, Phys. Rev. Lett. **107**, 5003 (2011).

[52]J. A. Wesson, Plasma Phys. Controlled Fusion **37**, A337–A346 (1995).

[53]W. Daughton, V. Roytershteyn, H. Karimabadi, L. Yin, B. J. Albright, B. Bergen, and K. J. Bowers, Nature Phys. **7**, 539 (2011).

[54]L. P. Ku and A. H. Boozer, Nucl. Fusion **51**, 013004 (2011).

[55]J. M. Canik, D. T. Anderson, F. S. B. Anderson, K. M. Likin, J. N. Talmadge, and K. Zhai, Phys. Rev. Lett. **98**, 085002 (2007).

[56]O. I. Bogoyavlenskij, Phys. Rev. E **62**, 8616–8627 (2000).