Calculation of vortex states using dissipative structures with Dirac constraint theory

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Summary

Brackets for generating a variety of hybrid Hamiltonian and gradient flows are described in general terms. Such brackets are adapted to construct numerical algorithms for calculating vortex states. A variety of examples are given, including barotropic and baroclinic cases.

Dynamical systems, finite or infinite, that describe physical phenomena typically have parts that are in some sense Hamiltonian and parts that can be recognized as dissipative, with the Hamiltonian part being generated by a Poisson bracket and the dissipative part being some kind of gradient flow. The description of Hamiltonian systems has received much attention over nearly two centuries and, although some forms of dissipation have received general attention, the understanding and classification of dissipative dynamics is a much broader topic and consequently less well developed.

Early formalisms for dissipation include that due to Rayleigh and the Cahn-Hilliard type of system, but formalisms of greater complexity and interest are those that emerge from a Hamiltonian structure. Examples of the latter include double bracket dynamics due to Brockett and Young et al. and metriplectic flows introduced in [1]. (See [2, 3] and references therein for general discussion.) Double bracket flows dissipate energy while preserving Casimir invariants, while metriplectic flows embody the first and second laws of thermodynamics and, thus, conserve energy and produce entropy.

Both double bracket and metriplectic flows have interesting algebraic, geometric, and functional analytic properties (see [3]), depending on the context, but of interest here is how they can be used for practical computations. For illustration purposes, consider the simple case of a finite-dimensional manifold $\mathcal{Z}_s$, that is both symplectic and Reimannian. Because $\mathcal{Z}_s$ is symplectic, given any smooth function $f : \mathcal{Z}_s \to \mathbb{R}$ there naturally corresponds the Hamiltonian vector field $X_f := [z, f]$, where $[\,,\,]$ is the Poisson bracket and z denotes coordinates of $\mathcal{Z}_s$. Because $\mathcal{Z}_s$ is Riemannian it has a metric $\mathbf{g}(X, Y)$ defined on vector fields X, Y. With this machinery there is a symmetric bracket on pairs of functions given in terms of two Hamiltonian vector fields

$$(f, g) := \mathbf{g}(X_f, X_g)\,. \tag{1}$$

Given (1), one can further define gradient flows on $\mathcal{Z}_s$ as follows:

$$\dot{z} = (z, S)\,, \tag{2}$$

where $S : \mathcal{Z}_s \to \mathbb{R}$ is a single function that generates the flow. If $\mathcal{Z}_s$ is also Kähler, then there is a natural hybrid Hamiltonian and dissipative flow on $\mathcal{Z}_s$

$$\dot{z} = (z, S) + [z, H] \tag{3}$$

where the Hamiltonian, H, and entropy, S, could be identical.

If the manifold of interest is a Poisson manifold, $\mathcal{Z}_p$ then a similar construction follows, but now the symmetric backet $(\,,\,)$ has degeneracy,

$$(C, g) \equiv 0 \qquad \forall\; g\,.$$

where C denotes any Casimir invariant and g is any function on $\mathcal{Z}_p$. Dynamics generated by such a bracket will relax to equilibrium points while conserving the invariants C, with the generating function, say S, serving as a Lyapunov function.

Additional constraints can be added by building the symmetric bracket on $\mathcal{Z}_p$ from a Dirac bracket, which for two constraints, $\phi_{1,2}$, is given by

$$[f,g]_D := \frac{1}{[\phi_1,\phi_2]}\bigg([\phi_1,\phi_2][f,g] - [f,\phi_1][g,\phi_2] + [g,\phi_1][f,\phi_2]\bigg)\,, \tag{4}$$

which is easily seen to satisfy $[\phi_{1,2}, g]_D \equiv 0$, for all g, and can be shown to satisfy the Jacobi identity. Inserting two Dirac vector fields $X_f^D := [z, f]_D$ into $\mathbf{g}$ thus produces a gradient flow that preserves $\phi_{1,2}$ as well as the Casimir invariants.

We have used this latter Dirac construction to calculate a variety of vortex states [5]. This requires lifting the ideas above to infinite dimensions, in which case the symmetric bracket has the general form

$$(F,G)_D = \int_{\mathcal{D}} d^N x' \int_{\mathcal{D}} d^N x''\, [F, \chi^i(x')]_D\, \mathcal{G}ij(x',x'')[\chi^j(x''), G]_D\,, \tag{5}$$

where F, G are functionals of the fields χ. For Euler's equation, the sole field is the vorticity and the basic bracket from which $[\,,\,]_D$ is constructed is the Lie-Poisson bracket for Euler's equation (see [6]). The Casimir invariants are functions like the enstrophy, which follow from the solution being a rearrangement. Results from two calculations are shown in Fig. 1.

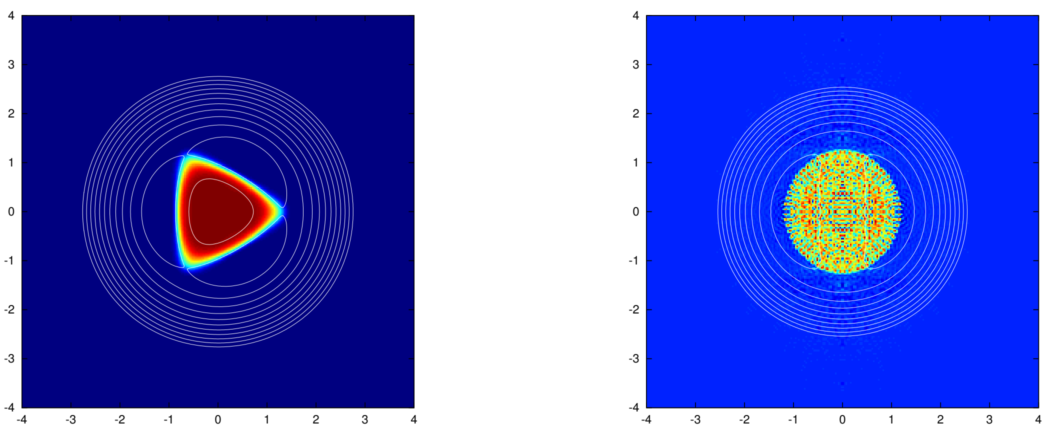

Fig. 1: (*left*) Casimir (rearrangement) preserving relaxation to three-fold symmetric vortex. (*right*) Constrained Kelvin sponge obtained by minimizing energy at fixed rearrangement and angular momentum. For both plots shading represents vorticity while contours represent streamfunction.

References

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