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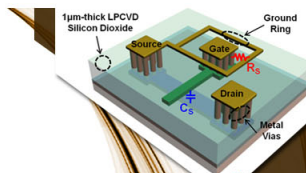
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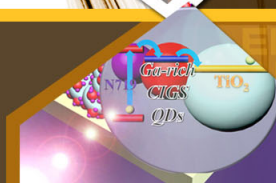


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# Hamiltonian magnetohydrodynamics: Lagrangian, Eulerian, and dynamically accessible stability—Theory

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Stability conditions of magnetized plasma flows are obtained by exploiting the Hamiltonian structure of the magnetohydrodynamics (MHD) equations and, in particular, by using three kinds of energy principles. First, the Lagrangian variable energy principle is described and sufficient stability conditions are presented. Next, plasma flows are described in terms of Eulerian variables and the noncanonical Hamiltonian formulation of MHD is exploited. For symmetric equilibria, the energy-Casimir principle is expanded to second order and sufficient conditions for stability to symmetric perturbation are obtained. Then, dynamically accessible variations, i.e., variations that explicitly preserve invariants of the system, are introduced and the respective energy principle is considered. General criteria for stability are obtained, along with comparisons between the three different approaches. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4819779>]

## I. INTRODUCTION

In this paper, a companion to Ref. 2 and its predecessor Ref. 1, we explore further ramifications of the Hamiltonian nature of ideal magnetohydrodynamics (MHD). Whereas in Refs. 1 and 2, the subject matter concerned the construction and origin of variational principles for equilibria, here we present the comprehensive approach to stability of MHD equilibria that is a direct consequence of the Hamiltonian nature of this system. The presentation organizes scattered approaches into the cohesive Hamiltonian framework, which will be seen to be useful for obtaining, interpreting, and comparing stability results.

Ultimately, the stability results we consider, which are a consequence of the Hamiltonian form, have their origin in two energy theorems of mechanics: Lagrange's theorem and Dirichlet's theorem (see Ref. 3 for review). The former is the root of the necessary and sufficient hydromagnetic energy principle of Refs. 4–6 for static equilibria, while the latter is the root of various Eulerian sufficient conditions for stability (see, e.g., Ref. 7). Since MHD, being a set of partial differential equations, is an infinite-dimensional Hamiltonian system, there are technical aspects not present in the theorems of mechanics. For example, MHD can be expressed in terms of the Lagrangian or Eulerian variable descriptions, each of which enforces constraints in particular but nonequivalent ways. A main goal of this paper is to explore the consequences for stability for such different ways of enforcing constraints. The results of Refs. 1 and 2 will be used in Secs. II, III, and IV to construct three kinds of energy principles for stability of both static and stationary MHD equilibria. In these sections, results of a more general nature are obtained, while more specific examples and comparisons will be made in a companion paper.<sup>8</sup>

More specifically, in Sec. II, energy stability in the purely Lagrangian variable framework, as considered in

Ref. 9, will be treated. This form extends the classical hydromagnetic energy principle of Ref. 4, obtained for static configurations, to stationary flows. En route to our results we briefly do the following: (i) review the Hamiltonian description in terms of Lagrangian variables and describe a time-dependent relabeling transformation, which to our knowledge has not heretofore been given, a transformation that will be needed for later development, (ii) review the map from Lagrangian to Eulerian variables, so as to understand how the time dependence of stationary equilibria in the Lagrangian picture relates to time-independent Eulerian equilibria and how such time dependence can be removed, and (iii) expand Lagrangian particle trajectories to obtain energy expressions for perturbations of general equilibria and use these expressions for obtaining sufficient conditions for stability of equilibria with stationary flow.

In Sec. III, the second kind of energy principle is described, one that has purely Eulerian form in terms of the usual MHD variables. This form has been called the energy-Casimir method (see, e.g., Refs. 3, 7, 10, and 11), although the method predates the name and, in fact, it appeared in the early plasma literature in several contexts, the earliest of which appears to be Ref. 12. This energy principle gives sufficient conditions for stability by expanding a functional  $F$  composed of the sum of the Eulerian energy plus Casimir invariants, an example being the cross helicity  $\int d^3x \mathbf{v} \cdot \mathbf{B}$ , to second order. If this second variation is sign definite, then  $F$  serves as a Lyapunov functional for stability. With this energy principle, we can assess the stability of equilibria within the assumed symmetry class. In this context, very general and new stability conditions are obtained.

Next, in Sec. IV, the third kind of energy principle, one that uses dynamically accessible variations, terminology for a concept introduced in Ref. 13 for a general class of variations generated from the noncanonical Poisson bracket and consequently explicitly preserves invariants of the system, is

described. (See Ref. 3 for review.) Dynamically accessible variations do not rely on any symmetry and thus give general criteria for stability. Therefore, they provide information about the generality of our second class of energy principles.

Finally, in Sec. V we summarize and conclude. Here, we discuss our results, state implications about nonlinear stability, and make comparisons between the various kinds of stability. Comparisons are made on a general level, which is somewhat complicated, but this will pave the way for the companion paper of Ref. 8, where a collection of more specific examples will be treated and explicit comparisons made.

## II. LAGRANGIAN STABILITY

### A. General Hamiltonian form, relabeling, and conservation laws

Consider a general Hamiltonian field description in terms of a configuration field  $\mathbf{q}(\mathbf{a}, t) \in \mathbb{R}^3$  for each fixed  $\mathbf{a} \in \mathbb{R}^3$ . This field  $\mathbf{q}$  will be the Lagrangian variable that determines the position of a fluid element labeled by  $\mathbf{a}$ . The usual three-dimensional spatial domain is assumed, but the treatment of this subsection applies to any number of dimensions. Suppose  $\mathbf{q}$  has a canonical conjugate  $\boldsymbol{\pi}$  and both are labeled by the continuum variable  $\mathbf{a} \in \mathbb{R}^3$ , i.e., the dynamical variables of the Hamiltonian description are the pair of vector valued functions  $(\mathbf{q}(\mathbf{a}, t), \boldsymbol{\pi}(\mathbf{a}, t))$ . It is common to assume that the fluid element described by  $\mathbf{q}$  is labeled by its initial condition,  $\mathbf{q}(0, t) = \mathbf{a}$ , but as will soon become evident this is not necessary. The phase space of this setup is sometimes interpreted as an infinite-dimensional symplectic manifold, or for “natural” Hamiltonians by  $T^*Q$ , where  $Q$  is the set of smooth invertible mappings of the spatial domain, indicated here by  $\mathbf{q}$ , and  $T^*Q$  denotes the space (cotangent bundle) with base coordinate  $\mathbf{q}$  together with fiber coordinate  $\boldsymbol{\pi}$ . Because the infinite-dimensional geometry of this description is backed by meager mathematical rigor, the language of Lagrange<sup>14</sup> and Newcomb<sup>15</sup> will be used here, except because general curvilinear coordinates may be employed indices will be placed as in Refs. 3, 16, and 17 indicating their tensorial character, viz.  $\mathbf{q} \rightarrow q^i$  and  $\boldsymbol{\pi} \rightarrow \pi_i$ , where  $i = 1, 2, 3$ .

In terms of the canonical coordinates,  $q^i$ , and momentum densities,  $\pi_i$ , the dynamics can be written as

$$\dot{\pi}_i = \{\pi_i, H\} = -\frac{\delta H}{\delta q^i} \quad \text{and} \quad \dot{q}^i = \{q^i, H\} = \frac{\delta H}{\delta \pi_i}, \quad (1)$$

where ‘ $\dot{\cdot}$ ’ means derivative with respect to  $t$  at fixed label  $\mathbf{a}$  and the Poisson bracket  $\{\cdot, \cdot\}$  is canonical and given by

$$\{F, G\} = \int d^3a \left( \frac{\delta F}{\delta q^i} \frac{\delta G}{\delta \pi_i} - \frac{\delta G}{\delta q^i} \frac{\delta F}{\delta \pi_i} \right). \quad (2)$$

In Eq. (2),  $F$  and  $G$  are functionals, repeated indices are to be summed, and  $\delta F / \delta q^i$  denotes the functional derivative of  $F$  with respect to  $q^i$  (see, e.g., Ref. 3). Given a Hamiltonian functional of the form

$$H[\mathbf{q}, \boldsymbol{\pi}] = \int d^3a \mathcal{H}(\mathbf{q}, \boldsymbol{\pi}, \nabla_a \mathbf{q}, \nabla_a \boldsymbol{\pi}, \dots, \mathbf{a}, t), \quad (3)$$

where  $\nabla_a := \partial / \partial \mathbf{a}$ , Eqs. (1) and (2) imply a set of partial differential equations.

Consider general transformations of such Hamiltonian systems under an arbitrary time-dependent, invertible relabeling

$$\mathbf{a} = \mathfrak{A}(\mathbf{b}, t) \leftrightarrow \mathbf{b} = \mathfrak{B}(\mathbf{a}, t), \quad (4)$$

i.e.,  $\mathfrak{A} = \mathfrak{B}^{-1}$ . It should be emphasized that the transformation of  $\mathfrak{B}$  and its inverse is not connected at this point to the dynamics in any way, nor is it related to symmetry as in Refs. 16–18. This kind of label change does not usually appear in traditional finite-dimensional Hamiltonian theory, since it would amount to a time-dependent change of the label  $i$  of, e.g., a canonical coordinate  $q^i(t)$ . However, this relabeling transformation is in fact a time-dependent canonical transformation induced by  $\mathbf{Q}(\mathbf{b}, t) = \mathbf{q}(\mathfrak{A}(\mathbf{b}, t), t)$ , the transformation to the new coordinate. To understand how the associated momentum transforms, the following type-2 time-dependent generating functional is used:

$$F_2[\mathbf{q}, \boldsymbol{\Pi}, t] = \int d^3a \int d^3b \mathbf{q}(\mathbf{a}, t) \cdot \boldsymbol{\Pi}(\mathbf{b}, t) \delta(\mathbf{a} - \mathfrak{A}(\mathbf{b}, t)), \quad (5)$$

where  $\delta$  is the Dirac delta function. (In tensorial form,  $\mathbf{q} \cdot \boldsymbol{\Pi} = q^i \Pi_i$ .) The direct transformation from the Hamiltonian theory in terms of  $(\mathbf{q}(\mathbf{a}, t), \boldsymbol{\pi}(\mathbf{a}, t))$  to that in terms of  $(\mathbf{Q}(\mathbf{b}, t), \boldsymbol{\Pi}(\mathbf{b}, t))$  is given by

$$\boldsymbol{\pi} = \frac{\delta F_2}{\delta \mathbf{q}}, \quad \mathbf{Q} = \frac{\delta F_2}{\delta \boldsymbol{\Pi}}, \quad \text{and} \quad \tilde{H} = H + \frac{\partial F_2}{\partial t}. \quad (6)$$

From Eq. (5), it follows that

$$\boldsymbol{\pi}(\mathbf{a}, t) = \frac{\boldsymbol{\Pi}(\mathbf{b}, t)}{\mathfrak{J}}, \quad \mathbf{Q}(\mathbf{b}, t) = \mathbf{q}(\mathbf{a}, t), \quad (7)$$

where  $\delta(\mathbf{a} - \mathfrak{A}) = \delta(\mathbf{b} - \mathfrak{A}^{-1}) / \mathfrak{J}$  has been used, and

$$\frac{\partial F_2}{\partial t} = - \int d^3b \boldsymbol{\Pi} \cdot (\mathbf{V} \cdot \nabla_b \mathbf{Q}) = - \int d^3b \Pi_i V^j \frac{\partial Q^i}{\partial b^j}, \quad (8)$$

where the determinant  $\mathfrak{J} := |\partial a^i / \partial b^j|$ , which means  $d^3a = \mathfrak{J} d^3b$ , and

$$\mathbf{V}(\mathbf{b}, t) := \dot{\mathfrak{B}} \circ \mathfrak{B}^{-1} = \dot{\mathfrak{B}}(\mathfrak{A}(\mathbf{b}, t), t). \quad (9)$$

Equation (8) follows from  $(\partial \mathfrak{A}^i / \partial b^k)(\partial \mathfrak{B}^k / \partial a^j) = \delta_j^i$  and

$$\frac{d}{dt} \mathfrak{B}^k(\mathfrak{A}(\mathbf{b}, t), t) = \dot{\mathfrak{B}}^k + \frac{\partial \mathfrak{B}^k}{\partial a^j} \partial_t \mathfrak{A}^j = \frac{db^k}{dt} = 0. \quad (10)$$

Recall ‘ $\dot{\cdot}$ ’ always means time differentiation at fixed  $\mathbf{a}$ , while  $\partial_t$  will mean at fixed  $\mathbf{b}$ . The formulas of (7) and (8) are valid with substitution of either  $\mathbf{a}$  or  $\mathbf{b}$ , using Eq. (4).

Because the transformation is generated by  $F_2$ , it is a canonical transformation, i.e., the Poisson bracket becomes

$$\{F, G\} = \int d^3b \left( \frac{\delta F}{\delta Q^i} \frac{\delta G}{\delta \Pi_i} - \frac{\delta G}{\delta Q^i} \frac{\delta F}{\delta \Pi_i} \right), \quad (11)$$

the Hamiltonian in the new variables becomes

$$\tilde{H}[\mathbf{Q}, \mathbf{\Pi}] = H - \int d^3b \mathbf{\Pi} \cdot (\mathbf{V} \cdot \nabla_b \mathbf{Q}). \quad (12)$$

The second term of Eq. (12), the one that comes from  $\partial F_2/\partial t$ , will be referred to as the *fictitious* term. The transformed equations of motion are given by

$$\partial_t \Pi_i = \{\Pi_i, \tilde{H}\} \quad \text{and} \quad \partial_t Q^i = \{Q^i, \tilde{H}\}. \quad (13)$$

The relabeling transformation of Eq. (4) can also be interpreted as transformation to a moving noninertial frame of reference. With this interpretation,  $\mathbf{q}$  describes motion relative to states in the inertial frame with coordinates  $\mathbf{a}$ , and the relabeling transformation amounts to transformation to a noninertial frame with  $\mathbf{Q}$  describing motion relative to the frame with coordinates  $\mathbf{b}$ . This explains why relabeling gives rise to the presence of the fictitious (noninertial) term in the Hamiltonian. It is important to reiterate that  $\mathbf{q}$ ,  $\mathbf{Q}$ ,  $\mathfrak{B}$ , and  $\mathfrak{A}$  are all invertible maps (parameterized by time) defined on the same configuration space.

In the case where  $H$  is time-independent, energy is conserved, i.e.,  $\dot{H} = 0$ . If  $H$  has no explicit dependence on  $\mathbf{q}$ , but depends on  $\nabla_a \mathbf{q}$  and possibly higher derivatives, the momentum  $\mathbf{P} := \int d^3a \boldsymbol{\pi}$  is conserved. This momentum expression inserted into the Poisson bracket generates an operator for space translations. Conservation of  $\mathbf{P}$  follows from

$$\dot{\mathbf{P}} = \{\mathbf{P}, H\} = - \int d^3a \frac{\delta H}{\delta \mathbf{q}} = \int d^3a \nabla_a(\dots) = 0, \quad (14)$$

where the last equality is true for any functional that depends on first and higher order derivatives of  $\mathbf{q}$ . Similarly, for isotropic Hamiltonians the angular momentum,  $\mathbf{L} = \int d^3a \mathbf{q} \times \boldsymbol{\pi}$ , can be shown to be conserved, which is the case for the MHD Hamiltonian. When  $\mathbf{L}$  is inserted into the Poisson bracket, an expression for the operator that generates rotations is obtained. The Hamiltonian with these invariants and another, the position of the center of mass that generates Galilean boosts, together with the Poisson bracket, are a realization of the algebra of the ten parameter Galilean group (see Ref. 19).

In terms of the relabeled coordinates, the same transpires. Although a time-independent  $H$  may obtain explicit time dependence when written in terms of  $(\mathbf{Q}, \mathbf{\Pi})$  and likewise  $\tilde{H}$ , constants of motion remain constants of motion. For example, the momentum  $\mathbf{P}$  written in terms of the relabeled coordinates becomes  $\mathbf{P} = \int d^3b \mathbf{\Pi}$ , and because  $\delta/\delta \mathbf{Q}$  of the fictitious term of Eq. (12) is still of the form  $\nabla_b(\dots)$ , it follows that  $\{\mathbf{P}, \tilde{H}\} = 0$ . Similarly, the angular momentum remains conserved. Thus, upon relabeling the set of invariants, with the new Poisson bracket, remain a realization of the algebra of the Galelian group. This is to be expected since the Eulerian description does not see the labels and the Eulerian constants of motion with the noncanonical Poisson bracket are a realization of the Galelian group.

When systems have symmetry one can transform into a new frame of reference. When doing so, the Hamiltonian generally changes because the transformation is a time-

dependent canonical transformation. For example, using momentum conservation the old Hamiltonian  $H$  becomes  $H_\lambda = H + \boldsymbol{\lambda} \cdot \mathbf{P}$ , where the parameter  $\boldsymbol{\lambda}$  determines the speed of the translating frame. Extremals of  $H_\lambda$  are equilibria in the translating frame and, thus, correspond to uniformly translating states in the original frame. Similarly, equilibria in rotating frames are extremals of  $H_\zeta = H + \boldsymbol{\zeta} \cdot \mathbf{L}$ , where  $\boldsymbol{\zeta}$  determines the magnitude and direction of the rotation. Such Hamiltonian shifts can be used to obtain stability results for a larger class of states.

## B. MHD and the Lagrange-Euler map

The Hamiltonian for MHD lies in the class of so-called “natural” Hamiltonians of the form

$$H[\mathbf{q}, \boldsymbol{\pi}] = \int d^3a \left[ \frac{|\boldsymbol{\pi}|^2}{2\rho_0} + \rho_0 \mathcal{W}(\mathbf{q}, \nabla_a \mathbf{q}, \dots, \mathbf{a}) \right], \quad (15)$$

where  $\mathcal{W}$  is some potential energy density and  $\rho_0 = \rho_0(\mathbf{a})$  is a given function that denotes the mass density of a Lagrangian fluid element. In a general coordinate system  $|\boldsymbol{\pi}|^2 = g^{ij}(\mathbf{q}) \pi_i \pi_j =: \pi^i \pi_i$ , while for Cartesian coordinates the metric is  $\eta_{ij} = \eta^{ij} = \delta_{ij}$ , the usual Kronecker symbol.

The specific form of the Hamiltonian for MHD must satisfy the Eulerian closure principle described in Ref. 20; that is, it must be expressible in terms of the Eulerian variables of the theory. Using the notation of Ref. 2, the set of Eulerian variables for MHD is denoted by  $Z := (\rho, \mathbf{v}, s, \mathbf{B})$ , or alternatively  $\mathcal{Z} := (\rho, \mathbf{M} := \rho \mathbf{v}, \sigma := \rho s, \mathbf{B})$ , with the map from the Lagrangian variables  $(\mathbf{q}, \boldsymbol{\pi})$  to Eulerian variables  $Z$  given by

$$\rho(\mathbf{x}, t) = \frac{\rho_0(\mathbf{a})}{J(\mathbf{a}, t)} \Big|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x}, t)}, \quad (16)$$

$$v_i(\mathbf{x}, t) = \frac{\pi_i(\mathbf{a}, t)}{\rho_0(\mathbf{a})} \Big|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x}, t)}, \quad (17)$$

$$s(\mathbf{x}, t) = s_0(\mathbf{a}) \Big|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x}, t)}, \quad (18)$$

$$B^i(\mathbf{x}, t) = \frac{\partial q^i(\mathbf{a}, t)}{\partial a^i} \frac{B_0^i(\mathbf{a})}{J(\mathbf{a}, t)} \Big|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x}, t)}, \quad (19)$$

where on the right-hand side  $\rho_0$ ,  $s_0$ , and  $B_0^i$  are, respectively, the plasma density, the entropy per unit mass, and the  $i$ -th component of the magnetic field, and the subscript zero indicates that these functions are attributes of the Lagrangian fluid elements and thus depend on the label  $\mathbf{a}$ . The left-hand side gives the set of usual Eulerian variables with  $\rho$  the plasma density,  $\mathbf{v}$  the flow velocity,  $\mathbf{B}$  the magnetic field, and  $s$  the entropy per unit mass that are functions of the Eulerian observation position  $\mathbf{x}$ . The Lagrange-Euler map is effected using  $\mathbf{a} = \mathbf{q}^{-1}(\mathbf{x}, t)$  (the function  $\mathbf{q}^{-1}(\mathbf{x}, t)$ , the inverse of  $\mathbf{q}(\mathbf{a}, t)$ , indicates the label of the particle that, at time  $t$ , is located at the observation point  $\mathbf{x}$ ). Here, the determinant  $J := |\partial q^i/\partial a^i|$  should not be confused with  $\mathfrak{J} = |\partial a^i/\partial b^j|$  introduced in Sec. II A. See Refs. 3, 15–17, and 21 for more details.

The Lagrange-Euler map can also be used to express the variables  $Z$  in terms of the relabeled canonical coordinates



( $\mathbf{Q}, \mathbf{\Pi}$ ). For example, the entropy per unit mass that will be observed at point  $\mathbf{x}$  at time  $t$  will be that attached to the fluid element there then; hence, it is gotten by solving  $\mathbf{x} = \mathbf{Q}(\mathbf{b}, t)$  for  $\mathbf{b}$ , giving  $\mathbf{b} = \mathbf{Q}^{-1}(\mathbf{x}, t)$  and

$$\begin{aligned} s(\mathbf{x}, t) &= s_0(\mathbf{a}, t)|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x}, t)} = s_0(\mathfrak{Q}(\mathbf{b}, t), t)|_{\mathbf{b}=\mathbf{Q}^{-1}(\mathbf{x}, t)} \\ &=: \tilde{s}_0(\mathbf{b}, t)|_{\mathbf{b}=\mathbf{Q}^{-1}(\mathbf{x}, t)}. \end{aligned} \quad (20)$$

Similarly, the Eulerian velocity  $\mathbf{v}(\mathbf{x}, t)$ , when represented in terms of the new variables, is still the velocity of the fluid element that is at the observation point  $\mathbf{x}$  at time  $t$ , but now given in terms of  $\partial_t \mathbf{Q}(\mathbf{b}, t)$ ,

$$\begin{aligned} \mathbf{v}(\mathbf{x}, t) &= \dot{\mathbf{q}}(\mathbf{a}, t)|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x}, t)} \\ &= (\partial_t \mathbf{Q}(\mathbf{b}, t) + \mathfrak{B}(\mathfrak{Q}(\mathbf{b}, t), t) \cdot \nabla_b \mathbf{Q}(\mathbf{b}, t))|_{\mathbf{b}=\mathbf{Q}^{-1}(\mathbf{x}, t)} \\ &= (\partial_t \mathbf{Q} + \mathbf{V} \cdot \nabla_b \mathbf{Q})|_{\mathbf{b}=\mathbf{Q}^{-1}(\mathbf{x}, t)}. \end{aligned}$$

Here, the second term comes from label advection.

Mass conservation implies  $\rho_0 d^3 a = \tilde{\rho}_0 d^3 b$ , and thus  $\tilde{\rho}_0 = \mathfrak{J} \rho_0$ . Evidently,  $\tilde{J} := |\partial Q^i / \partial b^j| = J \mathfrak{J}$  and  $\tilde{\rho}_0 / \tilde{J} = \rho_0 / J$ . In terms of the relabeled coordinate, the Eulerian density is

$$\rho(\mathbf{x}, t) = \frac{\rho_0}{J} \Big|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x}, t)} = \frac{\tilde{\rho}_0}{\tilde{J}} \Big|_{\mathbf{b}=\mathbf{Q}^{-1}(\mathbf{x}, t)}, \quad (21)$$

where composition of arguments is as in Eq. (20).

Finally, the magnetic field is similarly expressed as

$$B^i(\mathbf{x}, t) = \frac{\partial Q^i \tilde{B}_0^j}{\partial b^j \tilde{J}} \Big|_{\mathbf{b}=\mathbf{Q}^{-1}(\mathbf{x}, t)}. \quad (22)$$

The usual equations of motion for  $Z$  follow from either of the expressions in terms of  $\mathbf{q}$  or  $\mathbf{Q}$  (see Ref. 2). The notation  $\partial/\partial t$  will be used to denote differentiation of Eulerian quantities at fixed  $\mathbf{x}$ .

The Hamiltonian for MHD is

$$H[\mathbf{q}, \boldsymbol{\pi}] = \int d^3 a \left[ \frac{\boldsymbol{\pi}_i \boldsymbol{\pi}^i}{2\rho_0} + \rho_0 U(s_0, \rho_0/J) + \frac{\partial q_i \partial q^j B_0^k B_0^\ell}{\partial a^k \partial a^\ell 8\pi J} \right]. \quad (23)$$

The function  $U$  of Eq. (23) is the internal energy per unit mass of the plasma. As it is written, it can be expressed as a function of  $\rho$  and  $s$ , i.e.,  $U = U(s, \rho)$ ; this is necessary for this Hamiltonian to satisfy the Eulerian closure principle,<sup>20</sup> which in this case means that upon substitution of Eq. (19) Eq. (23) becomes

$$H = \int d^3 x \left[ \frac{\rho}{2} |\mathbf{v}|^2 + \rho U(s, \rho) + \frac{|\mathbf{B}|^2}{8\pi} \right], \quad (24)$$

an expression entirely in terms of the variables  $Z$ . With the usual thermodynamic relations, the pressure is given  $p = \rho^2 \partial U / \partial \rho$  and the temperature by  $T = \partial U / \partial s$ . For polytropic equations of state,  $p = \kappa(s) \rho^\gamma$ ,  $U = \kappa(s) \rho^{\gamma-1} / (\gamma-1)$  and with this choice the internal energy integrand of Eq. (24) is  $\rho U = p / (\gamma-1)$ . Isothermal processes ( $\gamma = 1$ ) have  $U = \kappa \ln(\rho)$ .

The MHD model can be generalized by altering the Hamiltonian in many physically meaningful ways: for

example, an anisotropic pressure tensor can be treated as in Refs. 19 and 22 by assuming  $U$  depends on  $B = |\mathbf{B}|$  with

$$p_{\parallel} = \rho^2 \frac{\partial U}{\partial \rho} \quad \text{and} \quad p_{\perp} = \rho^2 \frac{\partial U}{\partial \rho} + \rho B \frac{\partial U}{\partial B}, \quad (25)$$

which gives the CGL equations,<sup>23</sup> and the effects of a gravitational force can be modeled by adding to the integrand of Eq. (23) a term  $\rho_0 \phi$ , where  $\phi$  is an external potential.

Now consider explicitly the effect of the relabeling transformation of Eq. (4) on the MHD Hamiltonian, which we write out in tensorial form

$$\begin{aligned} \tilde{H}[\mathbf{Q}, \mathbf{\Pi}] &= \int d^3 b \left[ \frac{\Pi_i \Pi^i}{2\tilde{\rho}_0} - \Pi_i V^j \frac{\partial Q^i}{\partial b^j} \right. \\ &\quad \left. + \tilde{\rho}_0 U(\tilde{s}_0, \tilde{\rho}_0/\tilde{J}) + \frac{\partial Q_i \partial Q^j \tilde{B}_0^k \tilde{B}_0^\ell}{\partial b^k \partial b^\ell 8\pi \tilde{J}} \right], \\ &= K + H_f + W, \end{aligned} \quad (26)$$

where  $K$  is the kinetic energy,  $H_f$  is the fictitious term of Eq. (8), and  $W$  represents the sum of the internal and magnetic field energies. The Hamiltonians of Eqs. (23) and (26) and the brackets of Eqs. (2) and (11) are the starting point for the equilibrium and stability analysis of Sec. II C.

### C. Lagrangian description of equilibrium and stability

Since the theory as thus far described is canonical, Lagrangian variable equilibria are given by  $\delta H / \delta \mathbf{q} = 0$  and  $\delta H / \delta \boldsymbol{\pi} = 0$ . The second of these conditions clearly implies  $\boldsymbol{\pi} = 0$ , which means Lagrangian equilibria correspond to static configurations in which fluid particles do not move. Thus, Eulerian stationary equilibria, i.e., equilibria with time-independent flows, are not Lagrangian equilibria. To accommodate stationary equilibria, the description developed in Sec. II A in terms of the relabeled Lagrangian variables  $\mathbf{Q}$  and  $\mathbf{\Pi}$  is convenient.

Consider what happens to the Hamiltonian formalism in terms of ( $\mathbf{Q}, \mathbf{\Pi}$ ) when an expansion about a given time-dependent reference trajectory is effected as follows:

$$\mathbf{Q} = \mathbf{Q}_r(\mathbf{b}, t) + \boldsymbol{\eta}(\mathbf{b}, t), \quad \mathbf{\Pi} = \mathbf{\Pi}_r(\mathbf{b}, t) + \boldsymbol{\pi}_\eta(\mathbf{b}, t), \quad (27)$$

where  $\boldsymbol{\eta}$  and  $\boldsymbol{\pi}_\eta$  will eventually be seen to be related to the displacements in the linear energy principles. But, for now,  $\mathbf{Q}_r(\mathbf{b}, t)$  and  $\mathbf{\Pi}_r(\mathbf{b}, t)$  are completely arbitrary. Expanding

$$\partial_t \mathbf{Q} = \frac{\delta \tilde{H}}{\delta \mathbf{\Pi}} \quad \text{and} \quad \partial_t \mathbf{\Pi} = -\frac{\delta \tilde{H}}{\delta \mathbf{Q}} \quad (28)$$

about the reference trajectory gives the leading order equations

$$\begin{aligned} \partial_t \mathbf{Q}_r &= \frac{\mathbf{\Pi}_r}{\tilde{\rho}} - \mathbf{V} \cdot \nabla_b \mathbf{Q}_r, \\ \partial_t \mathbf{\Pi}_r &= -\nabla_b \cdot (\mathbf{V} \otimes \mathbf{\Pi}_r) + \mathbf{F}_r, \end{aligned} \quad (29)$$

where  $\mathbf{F}_r$  comes from the  $W$  part of the Hamiltonian. Now, it is supposed that  $(\mathbf{Q}_r, \mathbf{\Pi}_r)$  is an equilibrium state, meaning  $\partial_t \mathbf{Q}_r = \partial_t \mathbf{\Pi}_r = 0$ , whence from Eq. (29) it follows that

$$\nabla_b \cdot (\tilde{\rho} \mathbf{V} \mathbf{V} \cdot \nabla_b \mathbf{Q}_e) = \mathbf{F}_e, \quad (30)$$

where the subscript  $r$  has been replaced by  $e$  to indicate an equilibrium state. Note, that in this expression  $\mathbf{V}$  and  $\tilde{\rho}$  could depend on time and we could add explicit time dependence to  $W$  to produce a moving state with the balance of Eq. (30). However, only static Lagrangian equilibria where  $\mathbf{V}(\mathbf{b})$ ,  $\tilde{\rho}(\mathbf{b})$ ,  $\mathbf{Q}_e(\mathbf{b})$ , and  $\mathbf{\Pi}_e(\mathbf{b})$  are considered. And, most importantly, it is assumed that  $\mathbf{Q}_e(\mathbf{b}) = \mathbf{b}$ . With this choice, static equilibria in the present context correspond to a moving state in the original  $(\mathbf{q}, \boldsymbol{\pi})$  context, which in turn will be seen to correspond to stationary equilibria in the Eulerian context. At this point, the relabeling is connected to the dynamics: up to now it has been arbitrary. To see this, return to what this all means in terms of the variable  $\mathbf{q}$ :  $\mathbf{b} = \mathbf{Q}_e(\mathbf{b}) = \mathbf{q}_e(\mathfrak{A}_e(\mathbf{b}, t), t) = \mathfrak{B}_e(\mathbf{a}, t)$ . From the definition of  $\mathbf{V}$  of Eq. (9)  $\mathbf{V}(\mathbf{b}, t) = \mathfrak{B}_e(\mathbf{A}_e(\mathbf{b}, t), t) = \mathbf{v}_e(\mathbf{b})$ , where  $\mathbf{v}_e(\mathbf{b})$  denotes an Eulerian equilibrium state. Upon setting  $\mathbf{b} = \mathbf{x}$ , i.e., identifying the Eulerian observation point with the moving label, Eq. (30) becomes the usual stationary equilibrium equation,

$$\nabla \cdot (\rho_e \mathbf{v}_e \mathbf{v}_e) = \mathbf{F}_e, \quad (31)$$

and  $\tilde{\rho}$  becomes the usual equilibrium  $\rho_e(\mathbf{x})$ . It can be shown that  $\mathbf{v}_e \cdot \nabla s_e = 0$ ,  $\nabla \cdot (\rho_e \mathbf{v}_e) = 0$ , and  $\mathbf{v}_e \cdot \nabla \mathbf{B}_e - \mathbf{B}_e \cdot \nabla \mathbf{v}_e + \mathbf{B}_e \nabla \cdot \mathbf{v}_e = 0$  follow from Eqs. (20), (21), and (22), respectively.

Next, an expansion about a stationary Eulerian equilibrium, which in this context is a static Lagrangian equilibrium, can be effected to obtain a quadratic energy functional that has no explicit time dependence. The identification of  $\mathbf{b} = \mathbf{x}$  leads to a usual procedure of measuring perturbed quantities relative to the unperturbed trajectories of a stationary equilibrium, as in Refs. 3 and 9 for fluid models (and Refs. 13, 24, and 25 for kinetic theories). However, here, evidently for the first time, this idea has been incorporated on the nonlinear level before expansion and treated in the purely Hamiltonian framework.

Before considering stationary equilibria, the usual “ $\delta W$ ” energy principle for static equilibrium of Refs. 4–6 will be treated. This principle is an infinite-dimensional version of Lagrange’s necessary and sufficient stability theorem of mechanics (see, e.g., Ref. 26), which is applicable to natural Hamiltonians of the separable form, kinetic plus potential. For static MHD configurations, the relabeling of Sec. II A is not necessary and the variables  $(\mathbf{q}, \boldsymbol{\pi})$  are sufficient; in fact,  $\mathbf{Q} = \mathbf{q}$ ,  $\mathbf{\Pi} = \boldsymbol{\pi}$ ,  $\mathbf{V} \equiv 0$ ; thus, the equilibrium is described by

$$\mathbf{b} = \mathbf{Q}_e(\mathbf{b}) = \mathbf{q}_e(\mathbf{a}) = \mathbf{a} = \mathbf{x}. \quad (32)$$

Since static equilibria are given by time-independent  $\mathbf{q}_e$ , this point can be taken to be the Eulerian observation position, i.e.,  $\mathbf{q}_e = \mathbf{a} = \mathbf{x}$  as given in Eq. (32). Since the Hamiltonian (23) is of separable form, Lagrange’s theorem would imply that the equilibrium is stable if and only if  $\mathbf{q}_e$  is a local minimum of the potential energy,

$$W[\mathbf{q}] = \int d^3a \left[ \rho_0 U(s_0, \rho_0/J) + \frac{\partial q_i}{\partial a^i} \frac{\partial q^i}{\partial a^k} \frac{B_0^j B_0^k}{8\pi J} \right]. \quad (33)$$

There are mathematical subtleties to this theorem, even in the finite-dimensional case, but as is common in plasma physics this formal statement of Lagrange’s theorem will be assumed. Following convention, the infinitesimal displacement from static equilibria  $\mathbf{q}_e$  will be denoted by  $\boldsymbol{\xi}$ , a displacement relative to an inertial frame, instead of  $\boldsymbol{\eta}$ , i.e.,  $\mathbf{q} = \mathbf{q}_e + \boldsymbol{\xi}$ . The second variation of the potential energy (33) (see Refs. 3 and 15 for details) gives

$$\begin{aligned} \delta^2 W[Z_e; \boldsymbol{\xi}] = & \frac{1}{2} \int d^3x \left[ \left( \rho_e \frac{\partial p_e}{\partial \rho_e} + \frac{B_e^2}{4\pi} \right) (\partial_i \xi^i)^2 \right. \\ & + \left( p_e + \frac{B_e^2}{8\pi} \right) ((\partial_i \xi^j)(\partial_j \xi^i) - (\partial_i \xi^i)^2) \\ & \left. + [(\partial_j \xi_i)(\partial_k \xi^i) - 2(\partial_i \xi^i)(\partial_k \xi_j)] \frac{B_e^j B_e^k}{4\pi} \right], \quad (34) \end{aligned}$$

where  $\partial_i \xi^j := \partial \xi^j / \partial q_e^i = \partial \xi^j / \partial x^i$  and where the Eulerian static equilibrium quantities, denoted by  $e$ , are consistent with the Grad-Shafranov equation (see Ref. 2). Particular care should be paid to the treatment of boundary terms as pointed out in Ref. 15 but, as noted above, an in-depth treatment of boundary conditions, including the plasma vacuum interface will be considered elsewhere.

Given an equilibrium solution, the functional  $\delta^2 W$  is typically viewed as a quadratic form in  $\boldsymbol{\xi}$ , viz. a functional that, upon variation, defines the linear dynamics of perturbations with respect to the equilibrium. By carrying out some manipulations, the functional (34) can be transformed in the more familiar expression of Ref. 4,

$$\begin{aligned} \delta^2 W[Z_e; \boldsymbol{\xi}] = & \frac{1}{2} \int d^3x \left[ \rho_e \frac{\partial p_e}{\partial \rho_e} (\nabla \cdot \boldsymbol{\xi})^2 + (\nabla \cdot \boldsymbol{\xi})(\nabla p_e \cdot \boldsymbol{\xi}) \right. \\ & \left. + \frac{|\delta \mathbf{B}|^2}{4\pi} + \mathbf{j}_e \times \boldsymbol{\xi} \cdot \delta \mathbf{B} \right], \quad (35) \end{aligned}$$

where  $4\pi \mathbf{j}_e = \nabla \times \mathbf{B}_e$  is the equilibrium current and  $\delta \mathbf{B} := \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_e)$ . The linear Hamiltonian is given by

$$H_{\text{stc}}[\boldsymbol{\xi}, \boldsymbol{\pi}_\xi] = \delta^2 H = \int d^3x \frac{|\boldsymbol{\pi}_\xi|^2}{2\rho_e} + \delta^2 W, \quad (36)$$

which with the linear Poisson bracket

$$\{F, G\}_{(\boldsymbol{\xi}, \boldsymbol{\pi}_\xi)} := \int d^3x \left( \frac{\delta F}{\delta \boldsymbol{\xi}} \cdot \frac{\delta G}{\delta \boldsymbol{\pi}_\xi} - \frac{\delta G}{\delta \boldsymbol{\xi}} \cdot \frac{\delta F}{\delta \boldsymbol{\pi}_\xi} \right), \quad (37)$$

produces the linear Hamiltonian system, obtained by expansion about static equilibria, as

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \{\boldsymbol{\xi}, H_{\text{stc}}\}_{(\boldsymbol{\xi}, \boldsymbol{\pi}_\xi)} = \frac{\delta H_{\text{stc}}}{\delta \boldsymbol{\pi}_\xi} \\ \dot{\boldsymbol{\pi}}_\xi &= \{\boldsymbol{\pi}_\xi, H_{\text{stc}}\}_{(\boldsymbol{\xi}, \boldsymbol{\pi}_\xi)} = -\frac{\delta H_{\text{stc}}}{\delta \boldsymbol{\xi}}. \end{aligned} \quad (38)$$

Thus, this Hamiltonian system is considered to be stable by Lagrange’s theorem if and only if  $\delta^2 W$  is positive for any perturbation  $\boldsymbol{\xi}$ , i.e., if and only if the quadratic form is positive definite.

Now consider Eulerian stationary equilibria in the Lagrangian variable framework, using the relabeling transformation discussed above and in Sec. II A. For such equilibria, Lagrange's theorem in general does not apply: because of the presence of the fictitious term, the Hamiltonian is no longer of separable form and instead one only has Dirichlet's sufficient condition for stability. For stationary equilibria, the analog of Eq. (32) is  $\mathbf{b} = \mathbf{Q}_e(\mathbf{b}) = \mathbf{q}_e(\mathfrak{A}(\mathbf{b}, t), t) = \mathbf{x}$ , the displacement relative to the relabeled fluid elements is given by  $\boldsymbol{\eta}(\mathbf{x}, t) = \boldsymbol{\xi}(\mathbf{a}, t)|_{\mathbf{a}=\mathbf{q}_e^{-1}(\mathbf{x}, t)}$ , and stationary equilibrium quantities are represented in terms of unrelabeled fluid elements by

$$\rho_e(\mathbf{x}) = \frac{\rho_0(\mathbf{a})}{J(\mathbf{a}, t)} \Big|_{\mathbf{a}=\mathbf{q}_e^{-1}(\mathbf{x}, t)}, \quad (39)$$

$$v_{ei}(\mathbf{x}) = \frac{\pi_i(\mathbf{a}, t)}{\rho_0(\mathbf{a})} \Big|_{\mathbf{a}=\mathbf{q}_e^{-1}(\mathbf{x}, t)}, \quad (40)$$

$$s_e(\mathbf{x}) = s_0(\mathbf{a})|_{\mathbf{a}=\mathbf{q}_e^{-1}(\mathbf{x}, t)}, \quad (41)$$

$$B_e^i(\mathbf{x}) = \frac{\partial q^i(\mathbf{a}, t)}{\partial a^i} \frac{B_0^i(\mathbf{a})}{J(\mathbf{a}, t)} \Big|_{\mathbf{a}=\mathbf{q}_e^{-1}(\mathbf{x}, t)}. \quad (42)$$

Following Ref. 3, the second variation of the Hamiltonian in terms of the canonically conjugate variables  $(\boldsymbol{\eta}, \boldsymbol{\pi}_\eta)$  results

$$\delta^2 H_{\text{la}}[Z_e; \boldsymbol{\eta}, \boldsymbol{\pi}_\eta] = \frac{1}{2} \int d^3x \left[ \frac{1}{\rho_e} |\boldsymbol{\pi}_\eta - \rho_e \mathbf{v}_e \cdot \nabla \boldsymbol{\eta}|^2 + \boldsymbol{\eta} \cdot \mathfrak{B}_e \cdot \boldsymbol{\eta} \right], \quad (43)$$

where  $\rho_e$ ,  $\mathbf{v}_e$ , and the operator  $\mathfrak{B}_e$  has no explicit time dependence. The functional

$$\begin{aligned} \delta^2 W_{\text{la}}[\boldsymbol{\eta}] &:= \frac{1}{2} \int d^3x \boldsymbol{\eta} \cdot \mathfrak{B}_e \cdot \boldsymbol{\eta} \\ &= \frac{1}{2} \int d^3x [\rho_e (\mathbf{v}_e \cdot \nabla \mathbf{v}_e) \cdot (\boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta}) \\ &\quad - \rho_e |\mathbf{v}_e \cdot \nabla \boldsymbol{\eta}|^2] + \delta^2 W[\boldsymbol{\eta}], \end{aligned} \quad (44)$$

which is that obtained by Frieman and Rotenberg in Ref. 9 represents a generalization to stationary equilibria of the potential energy of Eq. (35). The linear Hamiltonian about stationary equilibria is  $H_{\text{str}} = \delta^2 H_{\text{la}}$  with Eq. (43), and the linear equations of motion are

$$\frac{\partial \boldsymbol{\eta}}{\partial t} = \{\boldsymbol{\eta}, H_{\text{str}}\}_{(\boldsymbol{\eta}, \boldsymbol{\pi}_\eta)} = \frac{1}{\rho_e} (\boldsymbol{\pi}_\eta - \rho_e \mathbf{v}_e \cdot \nabla \boldsymbol{\eta}), \quad (45)$$

$$\begin{aligned} \frac{\partial \boldsymbol{\pi}_\eta}{\partial t} &= \{\boldsymbol{\pi}_\eta, H_{\text{str}}\}_{(\boldsymbol{\eta}, \boldsymbol{\pi}_\eta)} = -\rho_e \mathbf{v}_e \cdot \nabla \frac{\boldsymbol{\pi}_\eta}{\rho_e} \\ &\quad + \nabla \cdot (\rho_e \boldsymbol{\eta} \mathbf{v}_e \cdot \nabla \mathbf{v}_e) + \nabla \cdot \left( \rho_e \frac{\partial p_e}{\partial \rho_e} \nabla \cdot \boldsymbol{\eta} + \boldsymbol{\eta} \cdot \nabla p_e \right) \\ &\quad + \frac{1}{4\pi} [\mathbf{B}_e \cdot \nabla \delta \mathbf{B} + \delta \mathbf{B} \cdot \nabla \mathbf{B}_e - \nabla (\mathbf{B}_e \cdot \delta \mathbf{B})], \end{aligned} \quad (46)$$

where the equilibrium equations have been used to simplify the functional derivative of  $H_{\text{str}}$  with respect to  $\boldsymbol{\eta}$ . By exploiting the relation

$$\begin{aligned} \frac{\partial \boldsymbol{\pi}_\eta}{\partial t} + \rho_e \mathbf{v} \cdot \nabla \frac{\boldsymbol{\pi}_\eta}{\rho_e} &= \rho_e \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} + 2\rho_e \mathbf{v}_e \cdot \nabla \frac{\partial \boldsymbol{\eta}}{\partial t} \\ &\quad + \nabla \cdot (\rho_e \mathbf{v}_e \mathbf{v}_e \cdot \nabla \boldsymbol{\eta}). \end{aligned} \quad (47)$$

Equations (45) and (46) can be put into the form of Ref. 9, i.e., as

$$\rho_e \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} + 2\rho_e \mathbf{v}_e \cdot \nabla \frac{\partial \boldsymbol{\eta}}{\partial t} - \mathbf{F}_e^v = 0, \quad (48)$$

where the force operator with velocity terms is

$$\begin{aligned} \mathbf{F}_e^v(\boldsymbol{\eta}) &:= \nabla \cdot (\rho_e \boldsymbol{\eta} \mathbf{v}_e \cdot \nabla \mathbf{v}_e - \rho_e \mathbf{v}_e \mathbf{v}_e \cdot \nabla \boldsymbol{\eta}) \\ &\quad + \nabla \cdot \left( \rho_e \frac{\partial p_e}{\partial \rho_e} \nabla \cdot \boldsymbol{\eta} + \boldsymbol{\eta} \cdot \nabla p_e \right) \\ &\quad + \frac{1}{4\pi} [\mathbf{B}_e \cdot \nabla \delta \mathbf{B} + \delta \mathbf{B} \cdot \nabla \mathbf{B}_e - \nabla (\mathbf{B}_e \cdot \delta \mathbf{B})]. \end{aligned}$$

In this case, it is clear from expression (43) that the Hamiltonian is not of separable form and in general Lagrange's theorem does not apply. However, due to the arbitrariness of  $\boldsymbol{\pi}_\eta$ , which does not contribute to  $\delta^2 W_{\text{la}}$ , the quadratic term in the integrand of Eq. (43) can be put equal to zero and a sufficient condition for stability is given by  $\delta^2 W_{\text{la}} > 0$  for any perturbation  $\boldsymbol{\eta}$ . This is an infinite-dimensional version of Dirichlet's theorem.

For completeness, we record the first order Eulerian perturbations that are induced by the Lagrangian variation  $\boldsymbol{\xi}$ , written in terms of the "Eulerianized" displacement  $\boldsymbol{\eta}$ . They are given as follows:

$$\delta \rho_{\text{la}} = -\nabla \cdot (\rho_e \boldsymbol{\eta}), \quad (49)$$

$$\delta \mathbf{M}_{\text{la}} = \boldsymbol{\pi}_\eta - \rho_e \boldsymbol{\eta} \cdot \nabla \mathbf{v}_e - \mathbf{v}_e \nabla \cdot (\rho_e \boldsymbol{\eta}), \quad (50)$$

$$\delta \sigma_{\text{la}} = -\nabla \cdot (\sigma_e \boldsymbol{\eta}), \quad (51)$$

$$\delta \mathbf{B}_{\text{la}} = -\nabla \times (\mathbf{B}_e \times \boldsymbol{\eta}), \quad (52)$$

where the momentum and entropy perturbations  $\delta \mathbf{M}_{\text{la}}$  and  $\delta \sigma_{\text{la}}$  can be replaced by the following velocity and pressure perturbations:

$$\delta p_{\text{la}} = -\gamma p_e \nabla \cdot \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla p_e, \quad (53)$$

$$\delta \mathbf{v}_{\text{la}} = \frac{\partial \boldsymbol{\eta}}{\partial t} + \mathbf{v}_e \cdot \nabla \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla \mathbf{v}_e, \quad (54)$$

as were used in Ref. 9.

### III. EULERIAN STABILITY—THE ENERGY-CASIMIR METHOD

Before proceeding, we briefly list results of Ref. 2 for readability and containedness.

The Hamilton form of the MHD equations in the Eulerian variables  $Z = (\rho, \mathbf{v}, s, \mathbf{B})$  is

$$\frac{\partial Z}{\partial t} = \{Z, H\}, \quad (55)$$

where  $H[Z]$ , the Hamiltonian is the energy expressing of Eq. (24), and  $\{\cdot, \cdot\}$  represents the noncanonical Poisson bracket of Refs. 27 and 28, which is not of canonical form because the Eulerian variables  $Z$  are not canonical variables.

Because Eulerian variables are not canonical variables, the noncanonical Poisson bracket has degeneracy that gives rise to Casimir invariants, special invariants  $C$  that satisfy  $\{C, F\} = 0$  for all functionals  $F$ , and these give rise to the variational principles of Ref. 2 that we can use further for determining stability. This follows from the general form for noncanonical Poisson brackets given by

$$\{F, G\} = \int d^3x \frac{\delta F}{\delta Z} \cdot \mathbb{J} \cdot \frac{\delta G}{\delta Z}, \quad (56)$$

which is defined on two functionals  $F$  and  $G$ . Here  $\mathbb{J}$ , the cosymplectic operator, is formally anti-self-adjoint and must satisfy a strenuous condition for the Jacobi identity.<sup>3,19</sup> In terms of Eq. (56), the equilibrium variational principles of Ref. 2 amount to  $\delta \mathfrak{F} / \delta Z = 0 \Rightarrow \dot{Z} = 0$ , where  $\mathfrak{F} := H + C$ . This follows from  $\dot{Z} = \mathbb{J} \cdot \frac{\delta H}{\delta Z} = \mathbb{J} \cdot \frac{\delta \mathfrak{F}}{\delta Z} = 0$  since  $\mathbb{J} \cdot \delta C / \delta Z = 0$  by definition. In finite-dimensional Hamiltonian systems, Dirichlet<sup>14</sup> showed that the Hamiltonian provides a sufficient condition for nonlinear stability of an equilibrium point if energy surfaces in the vicinity of the equilibrium are ellipsoidal, which is equivalent to definiteness of the second variation of the Hamiltonian. Carrying this idea over to the present infinite-dimensional noncanonical setting, this amounts to definiteness of the second variation of  $\mathfrak{F}$ . This is sufficient for linear stability and points toward nonlinear stability, but a rigorous mathematical proof requires information about the existence of solutions for MHD, which is a famous open problem. It should be noted that by nonlinear stability we mean stability to infinitesimal perturbations under the full nonlinear dynamics of the system. A nonlinearly unstable system is unstable to infinitesimal perturbations as opposed to finite amplitude instability, which requires a sufficiently large perturbation for instability as with a dimpled potential well.

In this section, we consider MHD with symmetry, as described in Ref. 2. All geometric symmetries can be described as a combination of axial symmetry and translational symmetry or, in other terms, as a helical symmetry. Given a cylindrical coordinate system  $(r, \phi, z)$ , we define a helical coordinate  $u = \phi[l] \sin \alpha + z \cos \alpha$ , where  $[l]$  is a scale length and  $\alpha$  defines the helical angle. The unit vector in the direction of the coordinate  $u$  can be written as  $\mathbf{u} = kr \nabla u = (k[l] \sin \alpha) \phi + (kr \cos \alpha) \mathbf{z}$ , where  $k^2 = 1 / ([l]^2 \sin^2 \alpha + r^2 \cos^2 \alpha)$  represents a metric factor. Thus, the second helical direction results,  $\mathbf{h} = kr \nabla r \times \nabla u = -(kr \cos \alpha) \phi + (k[l] \sin \alpha) \mathbf{z}$ , and the helical symmetry is expressed by the fact that  $\mathbf{h} \cdot \nabla f = 0$ , where  $f$  is a generic scalar function. The direction  $\mathbf{h}$ , which is called the symmetry direction, can be chosen to obtain axial ( $\alpha = 0$ ), translational ( $\alpha = \pi/2$ ), or true helical ( $0 < \alpha < \pi/2$ ) symmetry and the metric factor  $k$  changes accordingly. In the following, we use the identities

$$\nabla \cdot \mathbf{h} = 0, \quad \nabla \times (k\mathbf{h}) = -(k^3[l] \sin 2\alpha) \mathbf{h}, \quad (57)$$

which also show that, for  $\sin 2\alpha = 0$ , we can define a coordinate in the symmetry direction as  $\nabla h = k\mathbf{h}$ .

Using the notation described before, the magnetic field can be rewritten as

$$\mathbf{B}(r, u) = B_h(r, u) \mathbf{h} + \nabla \psi \times k\mathbf{h}, \quad (58)$$

where  $\psi = \psi(r, u)$  is the magnetic flux function, while the velocity becomes

$$\mathbf{v}(r, u) = v_h(r, u) \mathbf{h} + \mathbf{v}_\perp(r, u). \quad (59)$$

In terms of the variables  $Z_S = (\rho, \mathbf{v}_\perp, v_h, \psi, B_h)$ , where we assume the entropy is a flux function, i.e.,  $s = S(\psi)$ , the energy-Casimir functional is given by

$$\begin{aligned} \mathfrak{F} = \int d^3x \left( \frac{\rho |\mathbf{v}_\perp|^2}{2} + \frac{\rho v_h^2}{2} + \rho U + \frac{k^2 |\nabla \psi|^2}{8\pi} + \frac{B_h^2}{8\pi} \right. \\ \left. - \rho \mathcal{J} - kB_h \mathcal{H} - (k^4[l] \sin 2\alpha) \mathcal{H}^- \right. \\ \left. - \frac{\rho}{k} v_h \mathcal{G} - \mathbf{v} \cdot \mathbf{B} \mathcal{F} \right), \quad (60) \end{aligned}$$

where  $\mathcal{F}$ ,  $\mathcal{G}$ ,  $\mathcal{H}$ , and  $\mathcal{J}$  are four arbitrary functions of  $\psi$  and  $\mathcal{H}^-(\psi) := \int^\psi \mathcal{H}(\psi') d\psi'$ .

As discussed in Ref. 2, the first variation of Eq. (60) gives the equilibrium equations

$$0 = \rho \mathbf{v} - \mathcal{F} \mathbf{B} - \frac{1}{k} \rho \mathcal{G} \mathbf{h}, \quad (61)$$

$$0 = \frac{|\mathbf{v}|^2}{2} + U + \frac{p}{\rho} - \mathcal{J} - \frac{1}{k} v_h \mathcal{G}, \quad (62)$$

$$0 = \frac{B_h}{4\pi} - k \mathcal{H} - v_h \mathcal{F}, \quad (63)$$

$$\begin{aligned} 0 = -\nabla \cdot \left( \frac{k^2}{4\pi} \nabla \psi \right) + \rho T S' - \rho \mathcal{J}' - kB_h \mathcal{H}' \\ - k^4[l] \sin 2\alpha \mathcal{H}' - \frac{1}{k} \rho v_h \mathcal{G}' - \mathbf{v} \cdot \mathbf{B} \mathcal{F}' \\ + \nabla \cdot (\mathcal{F} k \mathbf{h} \times \mathbf{v}_\perp), \quad (64) \end{aligned}$$

where in the last equation primes indicate derivatives with respect to  $\psi$ . We assume the equilibrium is known and proceed to the second variation.

The second variation of the energy-Casimir principle results

$$\delta^2 \mathfrak{F} = \int d^3x (\delta \tilde{Z}_S \cdot \mathbb{K} \cdot \delta \tilde{Z}_S), \quad (65)$$

where we rearranged the terms in the vectorial form

$$\delta \tilde{Z}_S = (\delta v_r, \delta v_u, \delta v_h, k \partial_r \delta \psi, k \partial_u \delta \psi, \delta B_h, \delta \rho, \delta \psi)$$

and the matrix quadratic form



$$\mathbb{K} = \begin{bmatrix} \rho & 0 & 0 & 0 & -\mathcal{F} & 0 & \frac{1}{\rho}\mathcal{F}B_r & -\mathcal{F}'B_r \\ " & \rho & 0 & \mathcal{F} & 0 & 0 & \frac{1}{\rho}\mathcal{F}B_u & -\mathcal{F}'B_u \\ " & " & \rho & 0 & 0 & -\mathcal{F} & \frac{1}{\rho}\mathcal{F}B_h & -\mathcal{F}'B_h - \frac{1}{k}\rho\mathcal{G}' \\ " & " & " & \frac{1}{4\pi} & 0 & 0 & 0 & \frac{1}{\rho}\mathcal{F}\mathcal{F}'B_u \\ " & " & " & " & \frac{1}{4\pi} & 0 & 0 & -\frac{1}{\rho}\mathcal{F}\mathcal{F}'B_r \\ " & " & " & " & " & \frac{1}{4\pi} & 0 & -\frac{1}{\rho}\mathcal{F}\mathcal{F}'B_h - k\mathcal{H}' - \frac{1}{k}\mathcal{G}\mathcal{F}' \\ " & " & " & " & " & " & \frac{1}{\rho}c_s^2 & -\frac{1}{k\rho}\left(B_h\mathcal{F} + \frac{1}{k}\rho\mathcal{G}\right)\mathcal{G}' - \mathcal{J}' \\ " & " & " & " & " & " & " & -\Upsilon \end{bmatrix}. \quad (66)$$

Since  $\mathbb{K}$  is symmetric, we only showed the terms above and on the diagonal. The notation  $\delta\tilde{Z}_S$  is used to emphasize that in the rearranged vectorial form we have included, as separated elements, not only the perturbations of the variables  $Z_S$  but also the spatial derivatives of the perturbation  $\delta\psi$ . Moreover, in the quadratic form (66), the notation  $c_s^2 = \partial p/\partial\rho$  represents the square of the plasma sound speed, while  $\Upsilon$  indicates the expression

$$\begin{aligned} \Upsilon = & -\rho T\mathcal{S}'' - \rho(\partial T/\partial s)\mathcal{S}'^2 + \rho\mathcal{J}'' + kB_h\mathcal{H}'' \\ & + (k^4[l]\sin 2\alpha)\mathcal{H}' + \frac{1}{k}B_h\mathcal{F}\mathcal{G}'' + \frac{1}{k^2}\rho\mathcal{G}\mathcal{G}'' \\ & + \frac{B^2}{\rho}\mathcal{F}\mathcal{F}'' + B_h\frac{1}{k}\mathcal{G}\mathcal{F}'' . \end{aligned} \quad (67)$$

Next, we consider Eq. (61) and we define

$$\mathbf{S} := \rho\mathbf{v} - \mathcal{F}\mathbf{B} - \frac{1}{k}\rho\mathcal{G}\mathbf{h}. \quad (68)$$

At the equilibrium  $\mathbf{S} = 0$  and we use this equation to obtain  $\mathbf{v}$  as a function of the other variables. Next, we define

$$Q := \frac{B^2}{2\rho^2}\mathcal{F}^2 + U + \frac{p}{\rho} - \mathcal{J} - \frac{1}{2k^2}\mathcal{G}^2, \quad (69)$$

which follows from Eq. (62) after substitution of Eq. (61). Again,  $Q = 0$  at the equilibrium and we can use this equation to obtain  $\rho = \rho(\psi, \mathbf{B})$ . Last, we define

$$\mathbf{R} := \frac{(1 - M^2)}{4\pi}\mathbf{B} - \left(k\mathcal{H} + \frac{1}{k}\mathcal{F}\mathcal{G}\right)\mathbf{h}, \quad (70)$$

where substituting the previous results, the poloidal Alfvén Mach number

$$M^2 = \frac{4\pi\rho|\mathbf{v}_\perp|^2}{B_\perp^2} = \frac{4\pi\mathcal{F}^2}{\rho(\psi, \mathbf{B})} \quad (71)$$

is considered as a function of  $\psi$  and  $\mathbf{B}$ . The component of  $\mathbf{R}$  in the symmetry direction is related to Eq. (63) (after

substituting of  $v_h$  and  $\rho$ ) and at the equilibrium  $R_h = 0$ , whereas  $\mathbf{R}_\perp$  is related to the derivatives of  $\psi$  and in general is not zero.

By introducing in Eq. (65), the variations

$$\begin{aligned} \delta\mathbf{S} &= \rho\delta\mathbf{v} + \mathbf{v}\delta\rho - \mathcal{F}\delta\mathbf{B} - \mathcal{F}'\mathbf{B}\delta\psi - \frac{\mathbf{h}}{k}(\mathcal{G}\delta\rho + \rho\mathcal{G}'\delta\psi), \\ \delta Q &= \frac{1}{\rho}(c_s^2 - M^2c_a^2)\delta\rho - \mathcal{M}\delta\psi + \frac{M^2\mathbf{B}}{4\pi\rho}\delta\mathbf{B}, \\ \delta\mathbf{R} &= \frac{(c_s^2 + c_a^2)(M_c^2 - M^2)}{4\pi(c_s^2 - M^2c_a^2)}\delta\mathbf{B} \\ &\quad - \left[\frac{2\mathcal{F}\mathcal{F}'}{\rho}\mathbf{B} + \mathcal{N}\mathbf{h} + \frac{M^2\mathcal{M}}{(c_s^2 - M^2c_a^2)}\frac{\mathbf{B}}{4\pi}\right]\delta\psi, \end{aligned} \quad (72)$$

where

$$\begin{aligned} \mathcal{M} &:= \mathcal{G}\mathcal{G}'/k^2 + \mathcal{J}' - (T + \rho\partial T/\partial\rho)\mathcal{S}' - B^2\mathcal{F}\mathcal{F}'/\rho^2, \\ \mathcal{N} &:= k\mathcal{H}' + \mathcal{F}'\mathcal{G}/k + \mathcal{F}\mathcal{G}'/k, \end{aligned} \quad (73)$$

$c_a^2 = B^2/(4\pi\rho)$  is the square of the Alfvén velocity and  $M_c^2 = c_s^2/(c_s^2 + c_a^2)$  represents the square Alfvén Mach number corresponding to the ‘‘cusp velocity,’’ the second variation of the energy-Casimir principle can be rewritten in a form for which the matrix  $\mathbf{K}$  becomes diagonal, i.e.,

$$\delta^2\mathfrak{F} = \int d^3x [a_1|\delta\mathbf{S}|^2 + a_2(\delta Q)^2 + a_3|\delta\mathbf{R}|^2 + a_4(\delta\psi)^2], \quad (74)$$

where

$$a_1 = \frac{1}{\rho}, \quad a_2 = \frac{\rho}{(c_s^2 - M^2c_a^2)},$$

$$a_3 = \frac{4\pi(c_s^2 - M^2c_a^2)}{(c_s^2 + c_a^2)(M_c^2 - M^2)},$$

$$a_4 = -\Upsilon - a_1\left|\frac{\delta\mathbf{S}}{\delta\psi}\right|_{\tilde{Z}_S}^2 - a_2\left|\frac{\delta Q}{\delta\psi}\right|_{\tilde{Z}_S}^2 - a_3\left|\frac{\delta\mathbf{R}}{\delta\psi}\right|_{\tilde{Z}_S}^2. \quad (75)$$

From Eq. (74) it follows that, if  $a_i > 0$  for  $i = 1, 4$ , the equilibrium is a local minimum (i.e.,  $\delta^2 \mathfrak{F} > 0$ ) and stability is thus proved. In particular, the coefficient  $a_1$  is always positive while the positiveness of the coefficients  $a_2$  and  $a_3$  can be reduced to the condition  $M < M_c$ . However, the coefficient  $a_4$  involves in a very complicated way the second derivatives of the flux functions  $\mathcal{F}$ ,  $\mathcal{G}$ ,  $\mathcal{H}$ ,  $\mathcal{J}$ , and  $\mathcal{S}$  and considerations on its positiveness require, for each specific problem, specific investigations. Notice that stability is assured for generic Eulerian perturbations, which in general do not satisfy any of the dynamic constraints of the equilibrium (e.g., Casimir invariants can be modified by the perturbations).

In the equations above, we have considered  $\psi$  and  $\mathbf{B}_\perp$  as two independent variables and in the last of Eqs. (75) the notation  $\delta/\delta\psi|_{\bar{z}_s}$  indicates only the terms on the right hand side of the first of Eqs. (72) that multiply  $\delta\psi$ . Given an equilibrium, the terms  $\delta\mathbf{S}/\delta\psi|_{\bar{z}_s}$ ,  $\delta Q/\delta\psi|_{\bar{z}_s}$ ,  $\delta\mathbf{R}/\delta\psi|_{\bar{z}_s}$  are completely known, since they depend only on equilibrium quantities and not on the perturbations. However, the variation of the poloidal magnetic field is related to the variation of  $\psi$  by the equation

$$\delta\mathbf{B}_\perp = \nabla\delta\psi \times \mathbf{kh}. \quad (76)$$

The stability conditions deduced by considering the positiveness of the coefficients of Eqs. (75) are thus over-estimated and represents only sufficient criteria.

In order to obtain, for a given equilibrium, a better stability condition, we exploit relation (76) between  $\delta\mathbf{B}_\perp$  and  $\delta\psi$  and we consider  $\delta^2 \mathfrak{F}$  as a function of  $\delta\mathbf{S}$ ,  $\delta Q$ ,  $\delta R_h$ , and  $\delta\psi$ . Upon variation with respect to  $\delta\mathbf{S}$ ,  $\delta Q$ , and  $\delta R_h$ , it is straightforward to show that the minimum of  $\delta^2 \mathfrak{F}$  corresponds to

$$\delta\mathbf{S} = \delta Q = \delta R_h = 0, \quad (77)$$

provided that  $a_2$  and  $a_3$  are positive. Equations (77), which yield  $\mathbf{S} = Q = R_h = \text{const} = 0$ , correspond to the reduced variational principle presented in Ref. 2, where the constraints arising from the equilibrium equations have been used to obtain a variational principle for  $\psi$  alone.

We are reduced to the functional

$$\delta^2 \mathfrak{F}[\delta\psi] = \int d^3x [a_3 |\delta\mathbf{R}_\perp|^2 + a_4 (\delta\psi)^2], \quad (78)$$

which now depends only on the perturbation  $\delta\psi$  and its derivatives. By using the last expression of Eq. (72), which can be rewritten as

$$\delta\mathbf{R}_\perp = \frac{1}{a_3} \delta\mathbf{B}_\perp + \frac{\delta\mathbf{R}_\perp}{\delta\psi} \Big|_{\bar{z}_s} \delta\psi, \quad (79)$$

the second variation of the constrained energy functional becomes

$$\begin{aligned} \delta^2 \mathfrak{F} = & \int d^3x \left[ \frac{1}{a_3} |\delta\mathbf{B}_\perp|^2 + 2 \frac{\delta\mathbf{R}_\perp}{\delta\psi} \Big|_{\bar{z}_s} \cdot \delta\mathbf{B}_\perp \delta\psi \right. \\ & \left. - \left( \Upsilon + a_1 \left| \frac{\delta\mathbf{S}}{\delta\psi} \right|_{\bar{z}_s}^2 + a_2 \left| \frac{\delta Q}{\delta\psi} \right|_{\bar{z}_s}^2 + a_3 \left| \frac{\delta R_h}{\delta\psi} \right|_{\bar{z}_s}^2 \right) (\delta\psi)^2 \right]. \end{aligned}$$

Then, we consider the term

$$\begin{aligned} 2 \frac{\delta\mathbf{R}_\perp}{\delta\psi} \Big|_{\bar{z}_s} \cdot \delta\mathbf{B}_\perp \delta\psi &= 2 \frac{\delta\mathbf{R}_\perp}{\delta\psi} \Big|_{\bar{z}_s} \cdot (\nabla\delta\psi \times \mathbf{kh}) \delta\psi \\ &= -\nabla(\delta\psi)^2 \cdot \left( \frac{\delta\mathbf{R}_\perp}{\delta\psi} \Big|_{\bar{z}_s} \times \mathbf{kh} \right), \end{aligned}$$

and, by integrating by parts (neglecting the surface integral), the second variation becomes

$$\delta^2 \mathfrak{F} = \int d^3x [b_1 |\nabla\delta\psi|^2 + b_2 (\delta\psi)^2], \quad (80)$$

where  $b_1 = k^2/a_3$  and

$$\begin{aligned} b_2 = & -\Upsilon - a_1 \left| \frac{\delta\mathbf{S}}{\delta\psi} \right|_{\bar{z}_s}^2 - a_2 \left| \frac{\delta Q}{\delta\psi} \right|_{\bar{z}_s}^2 - a_3 \left| \frac{\delta R_h}{\delta\psi} \right|_{\bar{z}_s}^2 \\ & + \nabla \cdot \left( \frac{\delta\mathbf{R}_\perp}{\delta\psi} \Big|_{\bar{z}_s} \times \mathbf{kh} \right). \end{aligned}$$

Thus, the Euler-Lagrange equation associated with the extrema of Eq. (78) is

$$\nabla \cdot (b_1 \nabla\delta\psi) - b_2 \delta\psi = 0, \quad (81)$$

which represents a generalized form of the Newcomb equation<sup>29,30</sup> for MHD symmetric equilibria with flow.

Although we have obtained an equation that effectively minimizes the second variation of the energy-Casimir functional, in most cases the solution of Eq. (81) requires significant effort. However, a different way to estimate the minimum of  $\delta^2 \mathfrak{F}$  can be obtained by introducing into Eq. (78) a Poincarè inequality  $\|\nabla\delta\psi\|_{L^2} \geq \Lambda \|\delta\psi\|_{L^2}$ , where  $\Lambda$  is a constant that depends only on the domain and we have assumed that the mean value of  $\delta\psi$  is zero. A sufficient condition for the stability results

$$\delta^2 \mathfrak{F} \geq \int d^3x [(b_{1\min}\Lambda + b_2)(\delta\psi)^2] \geq 0, \quad (82)$$

where  $b_{1\min}$  represents the minimum value of  $b_1$  for the considered equilibrium. Condition (82) requires at each point  $b_{1\min}\Lambda + b_2 \geq 0$ .

#### IV. DYNAMICALLY ACCESSIBLE STABILITY

Stability of MHD equilibria with flow for perturbations that are confined to surfaces of constant Casimirs can be assessed by means of the so-called dynamically accessible variations, which are explicitly constructed in order to satisfy the Casimir constraints. Dynamically accessible variations, which are a restricted class of the Eulerian variations presented in Sec. III, are generated by means of the noncanonical Poisson bracket of the problem as  $\delta Z_{\text{da}} = \{G, Z\}$ , where the functional  $G = \int d^3x Z^i g_i$  plays the role of a generic Hamiltonian and where the generating functions  $g_i$  embody the arbitrariness in the variations. In particular, for the MHD model, in terms of the density variables introduced in Ref. 2, where, as defined in Sec. II,  $\mathbf{M} = \rho\mathbf{v}$ , the momentum density, and  $\sigma = \rho s$ , the entropy per unit volume, the functional that generates the dynamically accessible variations can be written as

$$G = \int d^3x (\mathbf{g}_1 \cdot \mathbf{M} + g_2 \sigma + g_3 \rho + \mathbf{g}_4 \cdot \mathbf{B}), \quad (83)$$

and the Poisson bracket is of Lie-Poisson form<sup>27,28</sup>

$$\begin{aligned} \{F, G\} = & - \int d^3x [\rho(F_{\mathbf{M}} \cdot \nabla G_\rho - G_{\mathbf{M}} \cdot \nabla F_\rho) \\ & + \mathbf{M} \cdot [(F_{\mathbf{M}} \cdot \nabla) G_{\mathbf{M}} - (G_{\mathbf{M}} \cdot \nabla) F_{\mathbf{M}}] \\ & + \sigma(F_{\mathbf{M}} \cdot \nabla G_\sigma - G_{\mathbf{M}} \cdot \nabla F_\sigma) \\ & + F_{\mathbf{M}} \cdot (\mathbf{B} \times (\nabla \times G_{\mathbf{B}})) + F_{\mathbf{B}} \cdot \nabla \times (\mathbf{B} \times G_{\mathbf{M}})], \end{aligned}$$

i.e., linear with respect to each variable. (Note, for compactness we have written the bracket of Ref. 27; for equilibria with  $\nabla \cdot \mathbf{B}_e = 0$ , results are identical to those with the more general bracket of Ref. 28.)

Stability is thus given by the positiveness of the second dynamically accessible variation of the Hamiltonian

$$H = \int d^3x \left( \frac{M^2}{2\rho} + \rho U + \frac{B^2}{8\pi} \right). \quad (84)$$

The first order dynamically accessible variations result

$$\delta\rho_{\text{da}} = \nabla \cdot (\rho \mathbf{g}_1), \quad (85)$$

$$\begin{aligned} \delta\mathbf{M}_{\text{da}} = & \rho \nabla g_3 + (\nabla \times \mathbf{M}) \times \mathbf{g}_1 + \mathbf{M} \nabla \cdot \mathbf{g}_1 \\ & + \nabla (\mathbf{M} \cdot \mathbf{g}_1) + \sigma \nabla g_2 + \mathbf{B} \times (\nabla \times \mathbf{g}_4), \end{aligned} \quad (86)$$

$$\delta\sigma_{\text{da}} = \nabla \cdot (\sigma \mathbf{g}_1), \quad (87)$$

$$\delta\mathbf{B}_{\text{da}} = \nabla \times (\mathbf{B} \times \mathbf{g}_1), \quad (88)$$

and the first variation of the Hamiltonian can be written as

$$\begin{aligned} \delta H_{\text{da}} = & \int d^3x \left[ \frac{\mathbf{M}}{\rho} \cdot \delta\mathbf{M}_{\text{da}} + T \delta\sigma_{\text{da}} + \frac{1}{4\pi} \mathbf{B} \cdot \delta\mathbf{B}_{\text{da}} \right. \\ & \left. + \left( -\frac{M^2}{2\rho^2} + U + \frac{p}{\rho} - \frac{\sigma}{\rho} T \right) \delta\rho_{\text{da}} \right]. \end{aligned} \quad (89)$$

By inserting into Eq. (89) the expressions obtained for the dynamically accessible variations, Eqs. (85)–(88), we get the set of MHD equilibrium equations.

The second variation of the Hamiltonian results

$$\begin{aligned} \delta^2 H_{\text{da}} = & \int d^3x \left[ \frac{1}{\rho} |\delta\mathbf{M}_{\text{da}}|^2 - 2 \frac{\mathbf{M}}{\rho^2} \cdot \delta\mathbf{M}_{\text{da}} \delta\rho_{\text{da}} \right. \\ & + \left( \frac{M^2}{\rho^3} + \rho \frac{\partial^2 U}{\partial \rho^2} + 2 \frac{\partial U}{\partial \rho} - 2 \frac{\sigma}{\rho} \frac{\partial^2 U}{\partial \rho \partial s} + \frac{\sigma^2}{\rho^3} \frac{\partial^2 U}{\partial s^2} \right) (\delta\rho_{\text{da}})^2 \\ & + 2 \left( \frac{\partial^2 U}{\partial \rho \partial s} - \frac{\sigma}{\rho^2} \frac{\partial^2 U}{\partial s^2} \right) \delta\rho_{\text{da}} \delta\sigma_{\text{da}} + \frac{1}{\rho} \frac{\partial^2 U}{\partial s^2} (\delta\sigma_{\text{da}})^2 \\ & + \frac{1}{4\pi} |\delta\mathbf{B}_{\text{da}}|^2 + \frac{2\mathbf{M}}{\rho} \cdot \delta\mathbf{M}_{\text{da}} + 2 \frac{\partial U}{\partial s} \delta^2 \sigma_{\text{da}} \\ & \left. + 2 \left( \rho \frac{\partial U}{\partial \rho} + U - \frac{\sigma}{\rho} \frac{\partial U}{\partial s} - \frac{M^2}{2\rho^2} \right) \delta^2 \rho_{\text{da}} + \frac{\mathbf{B}}{2\pi} \cdot \delta^2 \mathbf{B}_{\text{da}} \right], \end{aligned} \quad (90)$$

where  $\delta^2 Z_{\text{da}}$  are the second order variations obtained as

$$\delta^2 Z_{\text{da}} = \{G^{(2)}, Z\} + \frac{1}{2} \{G^{(1)}, \{G^{(1)}, Z\}\}. \quad (91)$$

Notice that  $G^{(1)}$  represents the first order generating functional, i.e., functional (83), whereas  $G^{(2)}$  is a second order functional. However, it is easy to show that altogether the terms corresponding to this second functional become null at the equilibrium points.

The second order variations result

$$\delta^2 \rho_{\text{da}} = \frac{1}{2} \nabla \cdot (\delta\rho_{\text{da}} \mathbf{g}_1),$$

$$\begin{aligned} \delta^2 \mathbf{M}_{\text{da}} = & \frac{1}{2} [\delta\rho_{\text{da}} \nabla g_3 + (\nabla \times \delta\mathbf{M}_{\text{da}}) \times \mathbf{g}_1 + \delta\mathbf{M}_{\text{da}} \nabla \cdot \mathbf{g}_1 \\ & + \nabla (\delta\mathbf{M}_{\text{da}} \cdot \mathbf{g}_1) + \delta\sigma_{\text{da}} \nabla g_2 + \delta\mathbf{B}_{\text{da}} \times (\nabla \times \mathbf{g}_4)], \end{aligned}$$

$$\delta^2 \sigma_{\text{da}} = \frac{1}{2} \nabla \cdot (\delta\sigma_{\text{da}} \mathbf{g}_1),$$

$$\delta^2 \mathbf{B}_{\text{da}} = \frac{1}{2} \nabla \times (\delta\mathbf{B}_{\text{da}} \times \mathbf{g}_1).$$

In order to compare the results of dynamical accessible variations with those of the Lagrangian approach (see Sec. II), we introduce a ‘‘Lagrangian’’ velocity variation defined as  $\delta\mathbf{v}_{\text{la}} = \partial\boldsymbol{\eta}/\partial t - \boldsymbol{\eta} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \boldsymbol{\eta}$ . After some manipulations, we obtain

$$\begin{aligned} \delta^2 H_{\text{da}}[\mathbf{g}] = & \int d^3x \rho |\delta\mathbf{v}_{\text{da}} - \mathbf{g}_1 \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{g}_1|^2 \\ & + \delta W_{\text{la}}[\mathbf{g}_1], \end{aligned} \quad (92)$$

where  $\mathbf{g} := (\mathbf{g}_1, g_2, g_3, \mathbf{g}_4)$ ,

$$\begin{aligned} \delta\mathbf{v}_{\text{da}} = & \nabla(g_3 + \mathbf{g}_1 \cdot \mathbf{v}) - \mathbf{g}_1 \times (\nabla \times \mathbf{v}) \\ & + \frac{\sigma}{\rho} \nabla g_2 + \frac{1}{\rho} \mathbf{B} \times (\nabla \times \mathbf{g}_4), \end{aligned}$$

and the quadratic form  $\delta W_{\text{la}}$  is now expressed in terms of  $\mathbf{g}_1$ . Notice that the variation  $\delta^2 H_{\text{da}}$  is formally identical to the variation  $\delta^2 H$  obtained in the Lagrangian description, where  $\boldsymbol{\eta} = -\mathbf{g}_1$  and  $\delta\mathbf{v}_{\text{da}}$  replaces  $\delta\mathbf{v}_{\text{la}}$ . As shown in Sec. II C in the Lagrangian description, the integrand in the first term on the r.h.s. of Eq. (92) can be re-expressed in terms of  $(\boldsymbol{\pi}_\eta - \rho_e \mathbf{v}_e \cdot \nabla \boldsymbol{\eta}) = \partial\boldsymbol{\eta}/\partial t$  and an arbitrary variation  $\boldsymbol{\pi}_\eta$  can make this term null. On the contrary, in the case of dynamically accessible perturbations the arbitrariness of the variation  $\delta\mathbf{v}_{\text{da}}$  is described by the functions  $\mathbf{g}_1$ ,  $g_2$ ,  $g_3$ , and  $\mathbf{g}_4$  and the first term of  $\delta^2 H_{\text{da}}$  can be written as

$$\begin{aligned} \Delta = & \int d^3x \rho |\delta\mathbf{v}_{\text{da}} - \mathbf{g}_1 \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{g}_1|^2 \\ = & \int d^3x \rho |\nabla g_3 + \mathbf{v} \times (\nabla \times \mathbf{g}_1) + 2(\mathbf{v} \cdot \nabla) \mathbf{g}_1 \\ & + \frac{\sigma}{\rho} \nabla g_2 + \frac{1}{\rho} \mathbf{B} \times (\nabla \times \mathbf{g}_4)|^2. \end{aligned} \quad (93)$$

The functions  $g_2$ ,  $g_3$ , and  $\mathbf{g}_4$  inside  $\Delta$ , which is non-negative, do not appear inside the last term in Eq. (92). Thus, one might think that a suitable choice of these functions exists

that yields  $\Delta = 0$ . However, such a choice can be made only if specific solvability conditions (analogous to those of the magnetic differential equations discussed by Newcomb in Ref. 31) are satisfied. Thus, in general, we can only try to minimize  $\Delta$  with respect to the functions  $g_2$ ,  $g_3$ , and  $\mathbf{g}_4$ . This kind of minimization was first suggested in the dynamically accessible context applied to Vlasov theory in Ref. 25, but a similar procedure was adopted for MHD equilibrium configurations with nested flux surfaces in Ref. 32 and without flux surfaces in Ref. 33. In the following, we are going to analyze the symmetric case, which represents a good benchmark for the procedure and permits a direct comparison with the results obtained in Sec. III.

The first variation of  $\Delta$  with respect to  $g_{2,3}$ , and  $\mathbf{g}_4$  yields  $\delta\Delta/\delta g_2 = \nabla \cdot (\sigma \mathbf{X})$ ,  $\delta\Delta/\delta g_3 = \nabla \cdot (\rho \mathbf{X})$ , and  $\delta\Delta/\delta \mathbf{g}_4 = \nabla \times (\mathbf{X} \times \mathbf{B})$ , where we defined the vector field  $\mathbf{X}$  as

$$\begin{aligned} \mathbf{X} := & \nabla g_3 + \mathbf{v} \times (\nabla \times \mathbf{g}_1) + 2(\mathbf{v} \cdot \nabla) \mathbf{g}_1 \\ & + \frac{\sigma}{\rho} \nabla g_2 + \frac{1}{\rho} \mathbf{B} \times (\nabla \times \mathbf{g}_4). \end{aligned} \quad (94)$$

The minimum of term (93) satisfies

$$\nabla \cdot (\rho \mathbf{X}) = 0 \iff \nabla \cdot (\delta \mathbf{M}_{\text{da}}) = 0, \quad (95)$$

$$\mathbf{X} \cdot \nabla \frac{\sigma}{\rho} = 0 \iff \nabla \cdot (\delta (\mathbf{M} \sigma / \rho)_{\text{da}}) = 0, \quad (96)$$

$$\nabla \times (\mathbf{X} \times \mathbf{B}) = 0 \iff \nabla \times (\delta \mathbf{E}_{\text{da}}) = 0, \quad (97)$$

where  $\mathbf{E} = -\mathbf{v} \times \mathbf{B}/c$  and the equivalencies above, which are new and give insight, can be ascertained by a straightforward calculation. By considering symmetric configurations, where  $\mathbf{B} = B_h \mathbf{h} + \nabla \psi \times k \mathbf{h}$  and  $\frac{\sigma}{\rho} = s(\psi)$ , we obtain  $\mathbf{X} = X_h \mathbf{h} + \frac{1}{\rho} \nabla \chi \times k \mathbf{h}$  from Eq. (95),  $\nabla \chi \times \nabla \psi \cdot k \mathbf{h} = 0 \rightarrow \chi(\psi)$  from Eq. (96), and

$$\nabla \times \left[ k \left( X_h - \frac{\chi'}{\rho} B_h \right) \nabla \psi \right] = 0 \rightarrow$$

$X_h = \frac{\chi'}{\rho} B_h + \frac{1}{k} G(\psi)$  from Eq. (97).

Thus, for symmetric configurations, the vector field  $\mathbf{X}$  that minimizes the term  $\Delta$  can be written as

$$\mathbf{X}_{\text{min}} = \frac{F}{\rho} \mathbf{B} + G \frac{\mathbf{h}}{k}, \quad (98)$$

where  $F = \chi'$  and  $G$  are two generic functions of  $\psi$ .

Then, we consider the symmetric version of Eq. (94), which yields

$$\begin{aligned} \mathbf{X} := & \nabla \left( g_3 + \frac{\sigma}{\rho} g_2 \right) - \left( k \mathbf{h} \cdot \frac{1}{\rho} \nabla \times \mathbf{g}_4 + s' g_2 \right) \nabla \psi \\ & + \frac{k B_h}{\rho} \nabla \frac{g_{4h}}{k} - k \mathbf{h} \left( \frac{1}{\rho} \mathbf{B}_{\perp} \cdot \nabla \frac{g_{4h}}{k} \right) + \frac{v_h}{k} \nabla (k g_{1h}) \\ & + \frac{v_h}{k} \nabla \times (k \mathbf{h}) \times \mathbf{g}_{1\perp} - 2 v_h \mathbf{h} \left( \mathbf{g}_1 \cdot \frac{1}{k} \nabla k \right) \\ & + \mathbf{v}_{\perp} \times (\nabla \times \mathbf{g}_1) + 2(\mathbf{v}_{\perp} \cdot \nabla) \mathbf{g}_1 \end{aligned} \quad (99)$$

and, by combining Eqs. (98) and (99), we obtain

$$\begin{aligned} \frac{F}{\rho} B_h + G \frac{1}{k} = & -k \mathbf{B}_{\perp} \cdot \frac{1}{\rho} \nabla \frac{g_{4h}}{k} - 2 \frac{v_h}{k} \mathbf{g}_1 \cdot \nabla k \\ & + \mathbf{v}_{\perp} \cdot \left[ \frac{1}{k} \nabla (k g_{1h}) + \frac{1}{k} \nabla \times (k \mathbf{h}) \times \mathbf{g}_{1\perp} \right] \end{aligned}$$

and

$$\begin{aligned} \frac{F}{\rho} \nabla \psi \times k \mathbf{h} = & \nabla \left( g_3 + \frac{\sigma}{\rho} g_2 \right) + \frac{k B_h}{\rho} \nabla \frac{g_{4h}}{k} + \frac{v_h}{k} \nabla (k g_{1h}) \\ & - \left( k \mathbf{h} \cdot \frac{1}{\rho} \nabla \times \mathbf{g}_4 + s' g_2 \right) \nabla \psi + \frac{v_h}{k} \nabla \times (k \mathbf{h}) \times \mathbf{g}_{1\perp} \\ & + \mathbf{v}_{\perp} \times (\nabla \times \mathbf{g}_{1\perp}) + 2(\mathbf{v}_{\perp} \cdot \nabla) \mathbf{g}_{1\perp}. \end{aligned}$$

Now, in order to determine  $F$  and  $G$ , we multiply Eq. (99) by  $\rho \mathbf{h}/k$  and we integrate this expression in a domain  $\Psi$  bounded by two magnetic flux surfaces  $\psi$  and  $\psi + d\psi$

$$\int_{\Psi} d^3x F \frac{B_h}{k} + \int_{\psi} d^3x \frac{G}{k^2} = \int_{\Psi} d^3x \rho \frac{X_h}{k},$$

and we obtain

$$\begin{aligned} F \left\langle \frac{B_h}{k} \right\rangle + G \left\langle \frac{1}{k^2} \right\rangle \\ = \left\langle -2 \frac{\rho v_h}{k^2} \mathbf{g}_1 \cdot \nabla k + \frac{\rho \mathbf{v}_{\perp}}{k^2} \cdot [\nabla (k g_{1h}) + \nabla \times (k \mathbf{h}) \times \mathbf{g}_{1\perp}] \right\rangle, \end{aligned} \quad (100)$$

where  $\langle f \rangle = \int_{\psi} \frac{d^2x}{|\nabla \psi|} f$  indicates the surface integral on a flux surface. In order to obtain Eq. (100), we used the fact that  $F = F(\psi)$  and  $G = G(\psi)$  and the equation

$$\int_{\Psi} d^3x \left[ \mathbf{B}_{\perp} \cdot \nabla \frac{g_{4h}}{k} \right] = \int_{\Psi} d^3x \nabla \cdot \left[ \mathbf{B}_{\perp} \frac{g_{4h}}{k} \right] = 0,$$

where the last equality follows from the fact that the boundaries are flux surfaces.

Next, we multiply Eq. (99) by  $\mathbf{B}$  and again we integrate in  $\Psi$  to obtain

$$F \left\langle \frac{|\mathbf{B}|^2}{\rho} \right\rangle + G \left\langle \frac{B_h}{k} \right\rangle = \langle 2\mathbf{B} \cdot (\mathbf{v} \cdot \nabla \mathbf{g}_1) \rangle, \quad (101)$$

where we use the expressions  $\mathbf{B} \cdot \nabla \psi = 0$ ,  $\int_{\Psi} d^3x [\mathbf{B} \cdot \mathbf{v} \times (\nabla \times \mathbf{g}_1)] = \int_{\Psi} d^3x \nabla \cdot [\mathbf{g}_1 \times (\mathbf{B} \times \mathbf{v})]$ , and  $\int_{\Psi} d^3x \mathbf{B} \cdot \nabla g = \int_{\Psi} d^3x \nabla \cdot (\mathbf{B} g)$  for  $g = g_3 + \sigma g_2 / \rho$ .

Equations (100) and (101) can be rewritten as  $\mathbb{A} \cdot \Xi = \Gamma$ , where  $\Xi = (F, G)^t$ , the  $2 \times 2$  matrix  $\mathbb{A}$  is

$$\mathbb{A} = \begin{bmatrix} \left\langle \frac{|\mathbf{B}|^2}{\rho} \right\rangle & \left\langle \frac{B_h}{k} \right\rangle \\ \left\langle \frac{B_h}{k} \right\rangle & \left\langle \frac{1}{k^2} \right\rangle \end{bmatrix} \quad (102)$$

and the vector  $\Gamma$  is



$$\Gamma = \left[ \left\langle \frac{2\mathbf{B} \cdot (\mathbf{v} \cdot \nabla \mathbf{g}_1)}{k^2} \mathbf{v}_\perp \cdot \mathcal{L}_k \mathbf{g}_1 - \frac{2\rho}{k^2} v_h \mathbf{g}_1 \cdot \nabla k \right\rangle \right]^t, \quad (103)$$

where  $\mathcal{L}_k \mathbf{g}_1 := \nabla(kg_{1h}) + \nabla \times (k\mathbf{h}) \times \mathbf{g}_{1\perp}$ . Moreover, we notice that the coefficients of  $\mathbb{A}$  depend only on the equilibrium fields, while the vector  $\Gamma$  depends also on  $\mathbf{g}_1$ . Finally, we solve the linear system  $\mathbb{A} \cdot \Xi = \Gamma$  as  $\Xi = \mathbb{A}^{-1} \cdot \Gamma$ , and then substitute the solution  $\mathbf{X}_{\min} = \Xi_1 \mathbf{B}/\rho + \Xi_2 \mathbf{h}/k$ , into Eq. (93) to obtain

$$\begin{aligned} \Delta_{\min} &= \int d^3x \rho |\mathbf{X}_{\min}|^2 \\ &= \int d^3x \left( \Xi_1^2 \frac{|\mathbf{B}|^2}{\rho} + 2\Xi_1 \Xi_2 \frac{B_h}{k} + \Xi_2^2 \frac{1}{k^2} \right) \\ &= \int_{\psi} d\psi \Xi^t \cdot \mathbb{A} \cdot \Xi = \int_{\psi} d\psi \Gamma^t \cdot \mathbb{A}^{-1} \cdot \Gamma. \end{aligned} \quad (104)$$

It can be shown that the solvability conditions of Eq. (93) correspond to  $\Gamma = 0$ . In this case, the condition  $\delta^2 H_{\text{da}} > 0$  corresponds to  $\delta^2 W_{\text{la}} > 0$ , i.e., dynamically accessible stability conditions are equivalent to those obtained in Ref. 9.

## V. SUMMARY AND CONCLUSIONS

In this paper, we have described various forms of stability for MHD within the Hamiltonian framework, which is an efficacious stability framework because the Hamiltonian can serve as a Lyapunov functional.

We first described the Hamiltonian structure in terms of the Lagrangian variables, which being particle-like naturally has the canonical Hamiltonian form. We then described how time-dependent relabeling is a canonical transformation that amounts to a local frame change that can be used to remove the time dependence of fluid element trajectories that occur on the Lagrangian variable level for stationary (time-independent) Eulerian equilibria. For MHD in Ref. 9 and also in other works in the kinetic theory context (e.g., Ref. 13), time dependence was removed by measuring the displacement relative to the equilibrium trajectory. This can be viewed as a linear ramification of our fully nonlinear relabeling development, which to our knowledge is new. We also discussed the Hamiltonian in the relabeled frame and compared it to that for global transformations such as that occur for the frame shift corresponding to the total momentum. Then, the interrelationship between relabeling and the Euler-Lagrange map was described for equilibrium states. With these tools at hand, we were able to arrive at an energy expression that was compared to that of Ref. 9.

Next, we described Hamiltonian stability on the Eulerian variable level. This was done within the confines of a formulation that represents general symmetry, which affords a rich Casimir structure for ascertaining stability within various symmetry classes. General sufficient conditions for stability were obtained by incisive analysis of the energy-Casimir functional.

Finally, the dynamically accessible variations, based on the theory introduced in Refs. 13 and 25 and developed in

generality in Refs. 3 and 34, were employed. This allowed the investigation of arbitrary equilibria without the imposition of symmetry. Extremization of the energy functional was done as in Refs. 13, 25, and 33 and stability under this kind of constraint was determined.

As pointed out in Ref. 3, differences in the various stability conditions arise because different representations of a theory can incorporate different constraints. In closing we make a few comments on the comparisons between the various stability results, leaving more in-depth comparisons to the companion paper,<sup>8</sup> where specific examples will be treated in detail.

First consider the development of Sec. II in terms of Lagrangian variables. Although our sufficient conditions for stability are the same as those of Ref. 9, the manner of derivation and meaning are different. In Ref. 9, the stability conditions are obtained by manipulation of the linear equations of motion and subsequent analysis based on the insertion of exponential time dependence. However, our development is purely Hamiltonian: it proceeds by expansion of the fully nonlinear invariant energy, in the manner of Lagrange and Dirichlet of usual Hamilton theory, and no assumption is made about the temporal behavior of the solution. It is important to realize that linear equations of motion can have more than one quadratic invariant, and such invariants need not be the expansion of an invariant of the nonlinear system. For finite-dimensional systems, definiteness of the expansion of the Hamiltonian to second order actually implies nonlinear stability, i.e., stability under the full nonlinear dynamics. However, stability based on the definiteness of an invariant obtained by manipulation of a linear equation of motion is significantly weaker. In fact, it is possible that systems shown to be stable by such a procedure can in fact be unstable to arbitrarily small perturbations. (See Sec. VI of Ref. 3 for discussion.) For infinite-dimensional systems, definiteness of  $\delta^2 H$  of Eq. (43) is a step toward a proof of nonlinear stability. However, rigorous proofs of stability can be quite subtle and difficult; since stability is norm dependent, functional analysis is unavoidable (see, e.g., Refs. 35 and 36).

Direct comparison of the stability conditions of Secs. II, III, and IV is complicated by the fact that not all apply to the same equilibria. Although the Lagrangian and dynamically accessible methods apply to general equilibria, the energy-Casimir results as developed only apply to symmetric equilibria. Consequently, our comparisons below will implicitly assume equivalent equilibria.

Let us denote by  $\mathfrak{P} := \{\delta\rho, \delta\mathbf{M}, \delta\sigma, \delta\mathbf{B}\}$  the set of first order unconstrained perturbations of the Eulerian variables, i.e., the perturbed variables  $\delta\rho, \delta\mathbf{v}$ , etc., are arbitrary and completely independent of each other. This is the largest set of perturbations. The set  $\mathfrak{P}_{\text{ec}}$  used in Sec. III is similarly unconstrained, except within our symmetry class we have built in  $\nabla \cdot \delta\mathbf{B}_{\text{ec}} = 0$ . The set  $\mathfrak{P}_{\text{ec}}$  is the largest of this paper. Upon comparing Eqs. (49)–(52) with Eqs. (85)–(88), we see that with the identification  $\mathbf{g}_1 \equiv -\boldsymbol{\eta}$ , the sets  $\mathfrak{P}_{\text{la}}$  and  $\mathfrak{P}_{\text{da}}$  have all equivalent elements except for the momentum perturbations, where  $\delta\mathbf{M}_{\text{la}}$  is given by Eq. (50) and  $\delta\mathbf{M}_{\text{da}}$  is given by Eq. (86). It is easy to see that dynamically accessible variations are less general than Lagrangian variations.

Because of the freedom to choose  $\pi_\eta$  in Eq. (50) at will,  $\delta\mathbf{M}_{\text{la}}$  is completely arbitrary. However, to see that this is not the case for  $\delta\mathbf{M}_{\text{da}}$ , consider the special case of static, uniform, hydrodynamic equilibria, where  $\rho_e = \text{constant}$ ,  $\mathbf{M}_e \equiv 0$ ,  $\sigma_e = \text{constant}$ , and  $\mathbf{B}_e \equiv 0$ , in which case

$$\delta\mathbf{v}_{\text{da}} = \nabla g_3 + \frac{\sigma_e}{\rho_e} \nabla g_2 = \nabla \left( g_3 + \frac{\sigma_e}{\rho_e} g_2 \right). \quad (105)$$

Thus, because  $\nabla \times \delta\mathbf{v}_{\text{da}} \equiv 0$ , this kind of perturbation is incapable of introducing vorticity into such a static fluid, in contrast to  $\delta\mathbf{v}_{\text{la}}$ . For more general equilibria, the constraints implied by dynamical accessibility are more subtle and these will be considered on a case by case basis in Ref. 8. However, in general the following is true:

$$\mathfrak{P}_{\text{da}} \subset \mathfrak{P}_{\text{la}} \subset \mathfrak{P}_{\text{ec}}.$$

As a side note, we observe that the expression  $\delta\mathbf{v}_{\text{da}}$ , with  $\mathbf{v} \equiv 0$  and  $\sigma = \text{constant}$ , is identical to the Clebsch representation introduced in Ref. 19. Thus, this Clebsch representation is not capable of expressing all vector fields.

Given that dynamically accessible perturbations are constrained, one must make a decision based on the physics of the situation to determine which kinds of perturbations are relevant, an idea that was emphasized in Ref. 13, where the notion of dynamical accessible stability was introduced, and also in subsequent work. For example, if one is interested in ideal perturbations of a normal, fluid, i.e., the case where viscosity is not important, and it is assumed that the walls containing the fluid do not move normal to themselves, i.e., there is no stirring mechanism, then there is no physical mechanism by which vorticity can be introduced into the fluid, and we have a situation consistent with the case described above. However, if nondynamically accessible perturbations are important, then one might want to reassess the completeness of the dynamical system governing the phenomena. If nondynamically accessible perturbations are allowed, then one might want a dynamical system that reflects their evolution in time.

It is not enough to just consider the first order perturbations: one must consider the energy expressions to which they correspond. The perturbations  $\mathfrak{P}_{\text{la}}$  are to be introduced into Eq. (43), while the perturbations  $\mathfrak{P}_{\text{da}}$  go into Eq. (92). Since  $\pi_\eta$  is arbitrary the first term of Eq. (43) was made to vanish in Sec. II, leaving only  $\delta^2 W_{\text{la}}$ . While this was not the case when we analyzed Eq. (92) in Sec. IV. If one replaces  $\pi_\eta$  in Eq. (43) by its dynamically accessible counterpart,

$$\delta\pi_{\text{da}} = -\nabla \mathbf{g}_1 \cdot \mathbf{M}_e - \sigma_e \nabla g_2 - \rho_e \nabla g_3$$

(see Eq. (462) of Ref. 3), then one obtains Eq. (92). Thus, the same energy expression applies to both, but in the dynamically accessible case one is constrained away from the minimum available in the Lagrangian case. Therefore, to the extent that these expressions determine stability, Lagrangian stability implies dynamically accessible stability.

A comparison between the energy expressions used for Lagrangian and energy-Casimir stability is also possible, if the former is restricted to symmetrical perturbations. If one

inserts for  $\delta Z_S$  in  $\delta^2 \mathcal{F}$  the Lagrangian induced symmetric variations of Eqs. (49)–(52), adapted for symmetry, then  $\delta^2 \mathcal{F}$  becomes identical to  $\delta^2 H_{\text{la}}$ . This calculation was done for a reduced system (compressible reduced MHD of Ref. 37) in Ref. 38, but the calculation here for general symmetry is more complicated. To effect this calculation, Eqs. (52) and (58) are used to obtain  $\delta\psi_{\text{la}} = -\boldsymbol{\eta} \cdot \nabla\psi_e$ . Beginning with the first term of Eq. (74), with  $\delta\mathbf{S}$  given by the first of Eqs. (72), and the perturbations  $\delta\mathbf{B}$ ,  $\delta\psi$ , etc., replaced by their Lagrangian induced versions, we obtain

$$\int d^3x a_1 |\delta\mathbf{S}|^2 = \int d^3x \rho |\delta\mathbf{v} + \boldsymbol{\eta} \cdot \nabla\mathbf{v}_e - \mathbf{v}_e \cdot \nabla\boldsymbol{\eta}|^2.$$

This calculation requires the removal of the functions  $\mathcal{F}$  and  $\mathcal{G}$  in lieu of  $\mathbf{v}_e$  by making use of Eq. (68) and the use of metric identities such as  $\mathbf{h} \cdot \nabla\mathbf{h} \propto \mathbf{r}$ . The identity of the remaining portions of these energies follows similarly. Given that Lagrangian perturbations are a subset of energy-Casimir variations, we conclude that energy-Casimir stability implies Lagrangian stability, as implied by these energy expressions. Similarly, it was shown in Sec. VI C 2 of Ref. 3 that insertion of the perturbations  $\mathfrak{P}_{\text{da}}$  into Eq. (43) produces Eq. (92), i.e.,  $\delta^2 \mathfrak{F}_{\text{da}} \equiv \delta^2 H_{\text{da}}$  when the former is evaluated on first order dynamically accessible variations. Thus, we are led to two conclusions

$$\text{stab}_{ec} \Rightarrow \text{stab}_{\text{la}} \Rightarrow \text{stab}_{\text{da}},$$

to the extent that each of these quadratic forms implies stability, and that all the quadratic forms are in fact the identical physical energy contained in a perturbation away from an equilibrium state, but how much of that energy can be tapped depends on the constraints embodied in the forms of the perturbations  $\delta Z$ .

*Postscript:* We wish to point out two references that were brought to our attention after the completion of this work. In Ref. 39, the authors have used a form of Noether's theorem in an action principle setting of MHD to compare Lagrangian and dynamically accessible perturbations, while in Ref. 40 the author considers a case of energy-Casimir stability that is in the same vein as that of our Sec. III.

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