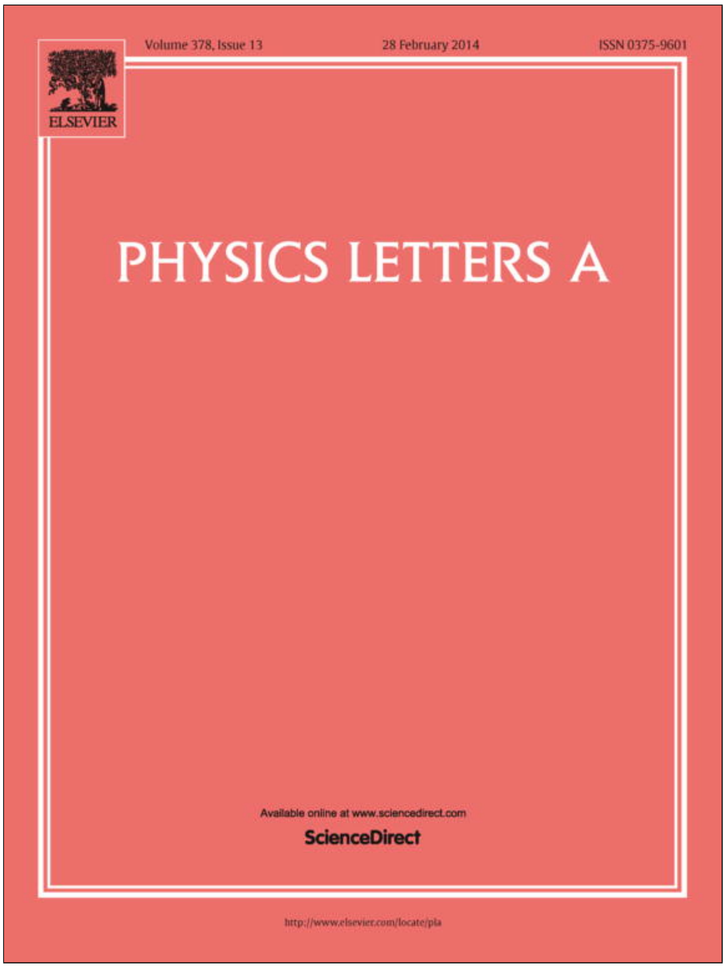
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Hamiltonian formulation of the modified Hasegawa–Mima equation

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ABSTRACT

We derive the Hamiltonian structure of the modified Hasegawa–Mima equation from the ion fluid equations applying Dirac's theory of constraints. We discuss the Casimirs obtained from the corresponding Poisson structure.

Zonal flows are believed to have a dramatic effect on the confinement of magnetized plasmas by suppressing the associated turbulent transport, notably the radial transport. Through the years various reduced models have been developed for describing transport in toroidal plasmas, and zonal flows have been observed in simulations of some of these models. An early example of such a reduced model is the Hasegawa–Mima (HM) equation [1], but more general electrostatic fluid models using both the FLR (e.g., [2]) and gyrofluid (e.g., [3,4]) approaches have been available for many years. Similarly, electromagnetic gyrofluid models (e.g., [5,6]) have been developed. Some of these models have noncanonical Hamiltonian structure (see [7–9]), which has been used to guide construction and led to the identification of new and physically important terms (e.g., [10]), and has also been shown to be important for the consistent calculation of zonal flow dynamics (e.g., [11,12]).

Zonal flows have been associated with drift wave modulational instability subject to the physics contained in the HM equation. However, it was recognized in [13] that the physics is more accurately described by a modified form of the HM equation, in which the electron response is modified and takes into account the geometry of magnetic surfaces. This modification has been shown to enhance the generation of zonal flows (see [14–16]).

The purpose of the present contribution is to demonstrate that the modified Hasegawa–Mima (mHM) equation has a Hamiltonian structure by obtaining it from Dirac's theory of constrained Hamiltonian systems [17–20], a technique used in previous derivations [21–23]. It is known that the modification of the HM equation applies for more general multifield theories (e.g., [24]); consequently, the methods we use and the results we obtain are of general utility and can be adapted to apply to a very large class of reduced fluid models.

For simplicity of our argumentation, we consider a slab geometry in which x corresponds to the radial direction and y to the poloidal angle (the magnetic field is in the z-direction). The zonal part of any field $\chi(x, y)$ is given by $\tilde{\chi} = \chi - \bar{\chi}$, where

$$\bar{\chi} = \frac{1}{2\pi}\int_0^{2\pi} dy\, \chi \equiv P\chi.$$

The mHM equation is given by

$$(\Delta - 1 + P)\dot{\phi} = [\Delta\phi + P\phi + \lambda, \phi],$$

where the dot indicates the partial derivative with respect to time and Δ denotes the Laplacian. Here the bracket is defined as $[f, g] = \hat{\mathbf{z}} \cdot \nabla f \times \nabla g$, where $\hat{\mathbf{z}}$ is the unit vector in the z-direction, and $\lambda(x, y)$ is a function related to the electron density at equilibrium. The mHM equation arises under the assumption that only the magnetic flux surface-varying part of ϕ, the electrostatic potential, is considered in the adiabatic electron response. In this way unphysical fluctuations in the averaged density are removed, and it is now understood that such a removal facilitates zonal flow production.

For the derivation of the Hamiltonian structure of the mHM equation, we first consider a parent model with two dynamical equations, from which the mHM equation can be derived: one describing the transverse dynamics of the ion velocity field $\mathbf{v}(x, y, t)$ and the other describing the dynamics of the ion density field $n(x, y, t)$:

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\phi + \mathbf{v} \times \mathbf{B}, \tag{1}$$

$$\dot{n} = -\nabla \cdot (n\mathbf{v}). \tag{2}$$

Here we use units such that the ion mass is $M = 1$ and its charge $e = 1$. The magnetic field is assumed to be constant and uniform. Without restrictions, we assume that the amplitude of

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the magnetic field is unity. The usual HM derivation is performed by a rather straightforward combination of the two equations for the density and the velocity field, assuming that the ion polarization velocity is much smaller than the $\mathbf{E}\times\mathbf{B}$ drift. In a previous work [22] we have shown a different way of deriving this equation from the Hamiltonian structure of the parent model. This method of derivation allows one to derive the reduced equation with the Hamiltonian structure naturally provided.

The total energy of the ions, given by the sum of their kinetic energy plus the potential energy provided by an external electrostatic potential ϕ, is a conserved quantity that is also the Hamiltonian of the system of equations (1)–(2), viz.

$$H[n,\mathbf{v}]=\int d^2x\left[n\frac{v^2}{2}+n\phi\right],$$

where the integration is over the two-dimensional cylinder $\mathbb{R}\times[0,2\pi]$ and $d^2x=dx\,dy$. The dynamics is determined by the Poisson bracket [25,26]

$$\{F,G\}=-\int d^2x\left[F_{\mathbf{v}}\cdot\nabla G_n-\nabla F_n\cdot G_{\mathbf{v}}-\left(\frac{\nabla\times\mathbf{v}+\hat{\mathbf{z}}}{n}\right)\cdot F_{\mathbf{v}}\times G_{\mathbf{v}}\right], \tag{3}$$

where we denote the functional derivatives of a given observable $F[n,\mathbf{v}]$ by subscripts, i.e. $F_{\mathbf{v}}=\delta F/\delta\mathbf{v}$ and $F_n=\delta F/\delta n$. In the present context we assume that the electrostatic potential ϕ is determined by the dynamics of the electrons which leads to a function $\phi(n_e)$, where n_e is the electron density. From the quasineutrality condition, $n=n_e$, the internal energy is changed from $\int d^2x\,n\phi$ to $\int d^2x\,\psi(n)$ such that $\psi'(n)=\phi(n)$, so that the Hamiltonian becomes

$$H[n,\mathbf{v}]=\int d^2x\left[n\frac{v^2}{2}+\psi(n)\right]. \tag{4}$$

Using Hamiltonian (4) and Poisson bracket (3), we obtain the equations of motion (1)–(2). The usual HM equation is obtained by neglecting the inertia of the electrons so that their density obeys a Boltzmann law $n_e=n_0\exp\phi$, where $n_0=n_0(x,y)=1-\lambda(x,y)$ is the electron density at equilibrium. For the mHM equation, this response has to take into account the prescription of Ref. [13], which here reads $n_e=n_0\exp\tilde{\phi}$, where $\tilde{\phi}$ is the zonal part of the potential.

Next we perform a change of variables $(n,\mathbf{v})\mapsto(n,\phi,D)$ where

$$\Delta\phi=\hat{\mathbf{z}}\cdot\nabla\times\mathbf{v},\qquad \Delta D=\nabla\cdot\mathbf{v}.$$

In terms of the new variables (n,ϕ,D), Hamiltonian (4) becomes

$$H[n,\phi,D]=\int d^2x\left[n\left(\frac{|\nabla\phi|^2+|\nabla D|^2}{2}+[\phi,D]\right)+\psi(n)\right],$$

and the bracket (3) becomes

$$\{F,G\}=\int d^2x\big[F_nG_D-F_DG_n-F_\phi\Delta^{-1}\mathcal{L}\Delta^{-1}G_\phi-F_D\Delta^{-1}\mathcal{L}\Delta^{-1}G_D+F_D\Delta^{-1}\Lambda\Delta^{-1}G_\phi-F_\phi\Delta^{-1}\Lambda\Delta^{-1}G_D\big], \tag{5}$$

where the two linear operators $\mathcal{L}$ and Λ acting on a function $f(\mathbf{x})$ are defined by

$$\mathcal{L}f=\left[\frac{\Delta\phi+1}{n},f\right],\qquad \Lambda f=-\nabla\cdot\left(\frac{\Delta\phi+1}{n}\nabla f\right).$$

In order to reduce this three-field model to a one-field one, we impose constraints on the dynamics. The constraints originate by imposing that the velocity field $\mathbf{v}$ is equal to the $\mathbf{E}\times\mathbf{B}$ drift velocity, with the zonal flow prescription (i.e., ϕ replaced by $\tilde{\phi}$), and that a relation exists between the potential and the electron density fluctuations. Therefore the first constraint is incompressibility, i.e., $D=0$, and the second one is a relation between the streamfunction ϕ and the density n. Near equilibrium and taking into account the linear response of the electrons, this relation is given by $n=N(\phi)\equiv\tilde{\phi}-\lambda$. The reduced dynamics is obtained by applying Dirac's theory of constrained Hamiltonian systems with two local constraints $\Phi_1(\mathbf{x})=D$ and $\Phi_2(\mathbf{x})=n-N(\phi)$, where $\mathbf{x}=(x,y)$. This theory starts with a parent bracket, which is here provided by the Poisson bracket (5). The first step is to compute $C_{ij}(\mathbf{x},\mathbf{x}')=\{\Phi_i(\mathbf{x}),\Phi_j(\mathbf{x}')\}$ for $i,j=1,2$. From the computation of the inverse, i.e., $D_{ij}(\mathbf{x},\mathbf{x}')$ satisfying

$$\int d^2x'\,C_{ij}(\mathbf{x},\mathbf{x}')D_{jk}(\mathbf{x}',\mathbf{x}'')=\delta_{ik}\delta(\mathbf{x}-\mathbf{x}''),$$

we obtain the Dirac bracket $\{F,G\}_*$ from the expression

$$\{F,G\}_*=\{F,G\}-\int d^2x\int d^2x'\,\{F,\Phi_i(\mathbf{x})\}D_{ij}(\mathbf{x},\mathbf{x}')\{\Phi_j(\mathbf{x}'),G\}.$$

The remarkable feature is that the bracket $\{\cdot,\cdot\}_*$ is a Poisson bracket if the original bracket $\{\cdot,\cdot\}$ is a Poisson bracket. The calculation of the Dirac bracket has been done in Ref. [22] for a different function N, corresponding to the Hasegawa–Mima equation. It was shown that the Dirac bracket is given by

$$\{F,G\}_*=\int d^2x\,\big(\Delta\phi-N(\phi)\big)\big[(\Delta-\hat{N})^{-1}F_\phi,(\Delta-\hat{N})^{-1}G_\phi\big], \tag{6}$$

where $\hat{N}$ is the Fréchet derivative of the pseudo-differential function N. The Hamiltonian is written as

$$H=\frac{1}{2}\int d^2x\left(|\nabla\phi|^2+\big(N(\phi)+\lambda\big)^2\right).$$

Compared to the HM equation, where the second constraint was $N(\phi)=\phi-\lambda$, here the second constraint is $N(\phi)=\phi-P\phi-\lambda$. Therefore, $\hat{N}=1-P$. We notice that P is symmetric in the sense that $\int fPg=\int gPf$.

The Poisson bracket for the mHM equation is given by

$$\{F,G\}_*=\int d^2x\,\big(\Delta\phi-(1-P)\phi+\lambda\big)\times\big[(\Delta-1+P)^{-1}F_\phi,(\Delta-1+P)^{-1}G_\phi\big], \tag{7}$$

and its Hamiltonian is

$$H=\frac{1}{2}\int d^2x\left(|\nabla\phi|^2+(\phi-P\phi)^2\right).$$

To see that this defines the correct Hamiltonian structure, note that $H_\phi=-(\Delta-1+P)\phi$ and the mHM equation for ϕ is given by $\dot{\phi}=\{\phi,H\}_*$ which reads as

$$(\Delta-1+P)\dot{\phi}=[\Delta\phi-\phi+P\phi+\lambda,\phi]. \tag{8}$$

In order to separate the two dynamics for the mHM system, we perform the change of variables $\phi\mapsto(\bar{\phi},\tilde{\phi})$ where we recall that $\phi=\bar{\phi}+\tilde{\phi}$ and $\bar{\phi}=P\phi$. Any observable $F(\phi)=\hat{F}(\bar{\phi},\tilde{\phi})$ has the functional derivative chain rule relation

$$F_\phi=(1-P)\hat{F}_{\tilde{\phi}}+P\hat{F}_{\bar{\phi}},$$

where we notice that the part $P\hat{F}_{\bar{\phi}}-P\hat{F}_{\tilde{\phi}}$ is only a function of x and that $P\hat{F}_{\bar{\phi}}=\hat{F}_{\bar{\phi}}$. In what follows we drop the hats on F for simplicity. The Poisson bracket in these new field variables is

$$\{F,G\}_* = \int d^2x\,(\Delta\bar\phi + \Delta\tilde\phi - \tilde\phi + \lambda) \times \big[(\Delta - 1 + P)^{-1}\big((1-P)F_{\tilde\phi} + PF_{\bar\phi}\big), (\Delta - 1 + P)^{-1}\big((1-P)G_{\tilde\phi} + PG_{\bar\phi}\big)\big], \tag{9}$$

and the Hamiltonian is

$$H = \frac{1}{2}\int d^2x\,\big(|\nabla\tilde\phi|^2 + \tilde\phi^2\big) + \frac{1}{2}\int dx\,|\nabla\bar\phi|^2,$$

where we have used $\int d^2x\,\nabla\tilde\phi \cdot \nabla\bar\phi = 0$. The two dynamical equations for the mHM system are

$$\dot{\bar\phi} = \{\bar\phi, H\} = \Delta^{-1}P[\Delta\tilde\phi, \tilde\phi],$$
$$\dot{\tilde\phi} = \{\tilde\phi, H\} = (\Delta - 1)^{-1}\big((1-P)[\Delta\tilde\phi, \tilde\phi] + [\Delta\tilde\phi - \tilde\phi, \bar\phi] + [\Delta\bar\phi + \lambda, \tilde\phi]\big),$$

where we have used the fact that $H_{\bar\phi} - PH_{\tilde\phi} = -\Delta\bar\phi$ and $H_{\tilde\phi} = -\Delta\tilde\phi + \tilde\phi$. We notice that these equations can also be obtained by projecting Eq. (8) using P and $1-P$ as projectors.

Next we determine the Casimir invariants for the Poisson bracket (7) and compare them to the ones of the bracket (9). The Casimir invariants of the Poisson bracket given by Eq. (6) are functionals C of ϕ such that $\{C,F\}_* = 0$ for all functionals F. If we perform the change of variables $\omega = \Delta\phi - N(\phi)$, the bracket (7) becomes $\{F,G\}_* = \int d^2x\,\omega[F_\omega, G_\omega]$ for which it is well known that the Casimir invariants are $C(\omega) = \int d^2x\,\alpha(\omega)$ for any scalar function α. This is found by imposing that $[\omega, C_\omega] = 0$ or in other words that C_ω is a function of ω which we call α'. Therefore the Casimir invariants of the bracket (7) are given by

$$C(\phi) = \int d^2x\,\alpha\big(\Delta\phi - N(\phi)\big).$$

This form of Casimir invariant for the bracket (7) is ubiquitous, although its specific from for mHM is reported here for the first time. Such Casimir invariants appear in all two-dimensional advection models where the advecting velocity is divergence-free, e.g., the HM, Euler, and Vlasov–Poisson systems, systems that are distinguished by how the advected quantity is related to the advecting velocity (see e.g., [8]). This Casimir invariant can be viewed as arising from Liouville's theorem for the characteristic equations of the system – indeed, the conservation of the area between any two nested ω-contours is a consequence of these Casimir invariants (see [27]). Also, it should be noted that the generalized enstrophy, the conservation of which plays an important role in the dynamics (see, e.g., [28]), is a Casimir invariant obtained in the particular case where $\alpha(\omega) = \omega^2/2$.

In a similar way, for the Poisson bracket (9), the condition that determines the Casimir invariants is

$$C_{\tilde\phi} - PC_{\tilde\phi} + C_{\bar\phi} = (\Delta - 1 + P)\alpha'(\Delta\tilde\phi - \tilde\phi + \Delta\bar\phi + \lambda),$$

whence we obtain

$$C(\tilde\phi, \bar\phi) = \int d^2x\,\alpha(\Delta\tilde\phi - \tilde\phi + \Delta\bar\phi + \lambda) + \int dx\,\beta(x)P\tilde\phi,$$

where α and β are two arbitrary scalar functions of one variable. Notice in particular that $P\tilde\phi(x,y)$ is a local Casimir invariant, obtained for $\alpha = 0$ and $\beta(x) = \delta(x - x')$. We have chosen here to restrict to $P\tilde\phi = 0$.

It should be pointed out that Dirac's theory ensures the Jacobi identity for the bracket (7) before the change of variables. Therefore if the change of variables is invertible, i.e., under the condition $P\tilde\phi = 0$, then the Jacobi identity is ensured too. However we show explicitly that the Poisson bracket (9) satisfies the Jacobi identity unconditionally, i.e., for all field variables $(\bar\phi, \tilde\phi)$, not necessarily restricted to $P\tilde\phi = 0$. In order to demonstrate this, we write the bracket (9) as

$$\{F,G\}_* = \int d^2x\,\big((A-P)\tilde\phi + \Delta\bar\phi + \lambda\big) \times \big[A^{-1}\big((1-P)F_{\tilde\phi} + PF_{\bar\phi}\big), A^{-1}\big((1-P)G_{\tilde\phi} + PG_{\bar\phi}\big)\big],$$

where $A = \Delta - 1 + P$. In the following, we let $f := (1-P)F_{\tilde\phi} + PF_{\bar\phi}$, and adopt analogous relations between g and G and h and H. As was shown in [7], only the functional derivatives of $\{F,G\}$ that take into account the explicit dependence of the bracket on the variables are needed. These are

$$\{F,G\}_{\tilde\phi} = (A-P)\big[A^{-1}f, A^{-1}g\big],$$
$$\{F,G\}_{\bar\phi} = P\Delta\big[A^{-1}f, A^{-1}g\big].$$

The computation of $\{\{F,G\},H\}$ leads to

$$\big\{\{F,G\},H\big\} = \int d^2x\big((A-P)\tilde\phi + \Delta\bar\phi + \lambda\big) \times \big[A^{-1}\big((1-P)(A-P)\big[A^{-1}f, A^{-1}g\big] + P\Delta\big[A^{-1}f, A^{-1}g\big]\big), A^{-1}h\big]. \tag{10}$$

Since $(1-P)(A-P) + P\Delta = A$, Eq. (10) becomes

$$\big\{\{F,G\},H\big\} = \int d^2x\big((A-P)\tilde\phi + \Delta\bar\phi + \lambda\big) \times \big[\big[A^{-1}f, A^{-1}g\big], A^{-1}h\big].$$

Therefore, the Jacobi identity for the bracket $\{\cdot,\cdot\}$ follows from the Jacobi identity for the bracket $[\cdot,\cdot]$.

Given the Hamiltonian structure we are now set to use Hamiltonian techniques, e.g., for investigating equilibrium and stability (e.g., [29,8,30,31]) or studies of absolute equilibria and cascading in turbulence theory (e.g., [32,11,12]). In addition, the proposed construction is sufficiently general that it could be used beyond mHM, in more general Hamiltonian zonal flow models.

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