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Action principles for extended magnetohydrodynamic models

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The general, non-dissipative, two-fluid model in plasma physics is Hamiltonian, but this property is sometimes lost or obscured in the process of deriving simplified (or reduced) two-fluid or one-fluid models from the two-fluid equations of motion. To ensure that the reduced models are Hamiltonian, we start with the general two-fluid action functional, and make all the approximations, changes of variables, and expansions directly within the action context. The resulting equations are then mapped to the Eulerian fluid variables using a novel nonlocal Lagrange-Euler map. Using this method, we recover Lüst's general two-fluid model, extended magnetohydrodynamic (MHD), Hall MHD, and electron MHD from a unified framework. The variational formulation allows us to use Noether's theorem to derive conserved quantities for each symmetry of the action. © 2014 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4896336]

I. INTRODUCTION

Fluid models are ubiquitous in the study of plasmas. It is desirable that the non-dissipative limits of fluid models be Hamiltonian, but this property is often lost in the process of deriving them (see, e.g., Refs. 1 and 2). One way to ensure the Hamiltonian property of such models is to derive them from action principles, i.e., to start from a Hamiltonian parent-model action and to make all the approximations and manipulations directly in the action (see, e.g., Refs. 3–6). The equations of motion are then obtained as the stationary points of the action under variation with respect to the dynamical variables. This is known as *Hamilton's principle* in particle mechanics.

Deriving the equations of motion of fluids[7–9] and plasmas[4,5,10–16] from an action principle has a rich history. The reasons that such a formulation is pursued, even after the equations of motion are already known, are numerous. Finding conservation laws using Noether's theorem,[17–20] obtaining variational principles for equilibria,[6,21,22] performing stability analyses,[4,22–27] or imposing constraints on a theory is straight-forward in the action functional, but often not easily done directly in the equations of motion.

Fluids can be described within the Eulerian or the Lagrangian viewpoints. The former describes the fluid in terms of, e.g., the evolution of the fluid velocity field at a position x, whereas the latter tracks the motion of individual fluid elements. The map connecting these two viewpoints is known as the Lagrange-Euler map. Action functionals are naturally expressed in terms of Lagrangian variables, whereas the equations of motion of fluid models are Eulerian.

Here, we are interested in fluid models describing two charged fluids, e.g., an ion and an electron fluid, interacting with an electromagnetic field. The general, non-dissipative two-fluid system is Hamiltonian, and therefore it is desirable that any model attempting to give a reduced description of it, should also be Hamiltonian and, consequently, not possess spurious forms of dissipation.

We will use the Lagrangian viewpoint and construct a general two-fluid action functional. Any subsequent ordering, approximation, and change of variables will be done directly in the action before deriving the equations of motion using Hamilton's principle. To ensure that the final equations of motion are Eulerian, we only construct actions that can be completely expressed in terms of Eulerian variables. This general requirement was elucidated in Refs. 5 and 6, where it was termed the *Eulerian Closure Principle* (ECP).

This paper is organized as follows: In Sec. II, we review the Lagrangian and Eulerian picture of fluid mechanics, including the derivation of the two-fluid equations of motion through Lagrangian variations of a two-fluid action functional. Starting from this action, we derive a new one-fluid action functional using careful approximations, e.g., imposing quasineutrality, and a change of variables in Sec. III. Here, we also introduce a new Lagrange-Euler map and impose locality in order to derive Eulerian equations of motion in the new variables. In Sec. IV, we show in detail how to derive various fluid models, e.g., extended magnetohydrodynamic (MHD) and Hall MHD, from this new one-fluid action functional. Sec. V contains a discussion of Noether's theorem applied to the new action functional, and finally, Sec. VI our conclusions and some discussions of future work.

II. REVIEW: TWO-FLUID MODEL AND ACTION

In this section, we will briefly review the derivation of the non-dissipative two-fluid model equations of motion from the general two-fluid action functional. This action will be the starting point for deriving reduced models further below. In this context we establish our notation and, later on, discuss differences with our new procedure and results.

Non-dissipative fluids can be described in two equivalent ways: The Eulerian (or spatial) point of view, which uses the physical observables of, e.g., fluid velocity $v(x, t)$ and mass density $\rho(x, t)$ as its fundamental variables and describes the fluid at an observation point x in the three-

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dimensional domain as time passes, or the Lagrangian (or material) point of view, which considers individual fluid elements with position q and tracks their time evolution. As described below, both pictures are related through the standard Lagrange-Euler map.

From an action functional/variational point of view, the Lagrangian picture is the more natural one, as it represents the infinite-dimensional generalization of the finite-dimensional Lagrangians of particle mechanics. The equations of motion are then obtained using Hamilton's principle as the stationary points of the action, i.e., the first variation of the action with respect to the variables is equal to zero.

We will use the Lagrangian picture as our starting point and construct a general two-fluid action functional. To ensure physical relevance of the theory, we only construct actions that obey the Eulerian Closure Principle, which states that any action functional of a physical fluid theory must be completely expressible in Eulerian variables after the application of the Lagrange-Euler map.

To simplify our notation (consistent with Ref. 3), we will avoid explicit vector notation and define the following: $q_s = q_s(a, t)$ is the position of a fluid element ($s = (i, e)$ is the species label) in a rectangular coordinate system where $a = (a_1, a_2, a_3)$ is any label identifying the fluid element and $q_s = (q_{s1}, q_{s2}, q_{s3})$. Here we choose a to be the initial position of the fluid particle at $t = 0$, although other choices are possible.[25] The Lagrangian velocity will then be denoted by $\dot{q}_s$.

The Eulerian velocity field will be denoted by $v_s(x, t)$ with $v_s = (v_{s1}, v_{s2}, v_{s3})$ where $x = (x_1, x_2, x_3)$ is the position in the Eulerian picture. Similarly, we define the electric field vector $E(x, t)$, the magnetic field vector $B(x, t)$, and the vector potential $A(x, t)$. If we need to explicitly refer to components of these vectors, we will use subscripts (or superscripts) j and k. To simplify the equations, we will also often suppress the dependence on x, a, and t.

The action functional described below will include integrations over position space $\int d^3x$ and label space $\int d^3a$. We will not explicitly specify the domains of integration, but assume that our functions are well-defined on their respective domains, and that integrating them and taking functional derivatives is allowed. In addition, we will assume that all variations on the boundaries of the domains and any surface terms (due to integration by parts) vanish.

A. Constructing the two-fluid action

The action functional of a general theory of a charged fluid interacting with an electromagnetic field should include the following components: The energy of the electromagnetic field, the fluid-field interaction energy, the kinetic energy of the fluid, and the internal energy of the fluid, which describes the fluid's thermodynamic properties.

We will assume two independent fluids corresponding to two different species (ions and electrons with charge e_s, mass m_s, and initial number density of $n_{s0}(a)$) which interact with the electromagnetic field, but not directly with each other. Therefore, the fluid-dependent parts of the action will naturally split into two parts, one for each species.

The complete action functional is given by

$$S[q_s, A, \phi] = \int_T dt L, \quad (1)$$

where T is a finite time interval and the Lagrangian L is given by

$$L = \frac{1}{8\pi} \int d^3x \left[\left| -\frac{1}{c}\frac{\partial A(x,t)}{\partial t} - \nabla\phi(x,t) \right|^2 - |\nabla \times A(x,t)|^2 \right] \quad (2)$$

$$+ \sum_s \int d^3a\, n_{s0}(a) \int d^3x\, \delta(x - q_s(a,t))$$

$$\times \left[\frac{e_s}{c}\dot{q}_s \cdot A(x,t) - e_s\phi(x,t) \right] \quad (3)$$

$$+ \sum_s \int d^3a\, n_{s0}(a) \left[\frac{m_s}{2}|\dot{q}_s|^2 - m_s U_s\left(m_s n_{s0}(a)/\mathcal{J}_s, s_{s0}\right) \right]. \quad (4)$$

The symbol $\mathcal{J}_s$ is the Jacobian of the map between Lagrangian positions and labels, $q(a, t)$, which we will discuss in more detail below. Here, we have expressed the electric and magnetic fields in terms of the vector and scalar potential, $E = -1/c(\partial A/\partial t) - \nabla\phi$ and $B = \nabla \times A$. The first term (2) is the electromagnetic field energy, while the next expression (3) is the coupling of the fluid to the electromagnetic field, which is achieved here by using the delta function. The last line of the Lagrangian L (4) represents the kinetic and internal energies of the fluid. Note, the specific internal energy (energy per unit mass) of species s, U_s, depends on the Eulerian density as well as a function s_{s0}, an entropy label for each species. Also note that the vector and scalar potentials are Eulerian variables (i.e., functions of x).

B. Lagrange-Euler map

In accordance with the above-mentioned Eulerian Closure Principle, we need to ensure that the action Eqs. (1)–(4) can be completely expressed in terms of the desired set of Eulerian variables, which in turn ensures that the resulting equations of motion will also be completely Eulerian, hence representing a physically meaningful model.

The connection between the Lagrangian and Eulerian pictures of fluids is the *Lagrange-Euler map*. Before looking at the mathematical implementation of this map, it is instructive to discuss its meaning. As an example, consider the Eulerian velocity field $v(x, t)$ at a particular position x at time t. The velocity of the fluid at that point will be the velocity of the particular fluid element $\dot{q}(a, t)$ which has started out at position a at time $t = 0$ and then arrived at point $x = q(a, t)$ at time t.

To implement this idea, we define the Eulerian number density $n_s(x, t)$ in terms of Lagrangian quantities as follows:

$$n_s(x, t) = \int d^3a\, n_{s0}(a)\delta(x - q_s(a, t)). \quad (5)$$

Using properties of the delta function, this relation can be rewritten as

$$n_s(x, t) = \left.\frac{n_{s0}(a)}{\mathcal{J}_s}\right|_{a=q_s^{-1}(x,t)}, \quad (6)$$

where, $\mathcal{J}_s = \det(\partial q_s / \partial a)$ is the Jacobian determinant. Note that Eq. (6) implies the continuity equation

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s v_s) = 0, \tag{7}$$

which corresponds to local mass conservation if we define the mass density as $\rho_s = m_s n_s$.

The corresponding relation for the Eulerian velocity is

$$v_s(x,t) = \dot{q}_s(a,t)|_{a=q_s^{-1}(x,t)}, \tag{8}$$

where the dot means differentiation with respect to time at fixed particle label a. This relation follows from integrating out the delta function in the definition of the Eulerian momentum density, $M_s := m_s n_s v_s$

$$M_s(x,t) = \int d^3a \, n_{s0}(a,t) \delta(x - q_s(a,t)) m_s \dot{q}_s(a,t). \tag{9}$$

Finally, our Eulerian entropy per unit mass, $s_s(x,t)$, is defined by

$$\rho_s s_s(x,t) = \int d^3a \, n_{s0}(a) s_{s0}(a) m_s \delta(x - q_s(a,t)), \tag{10}$$

completing our set of fluid Eulerian variables for this theory, which is $\{n_s, s_s, M_s\}$. It is easy to check that the closure principle is satisfied by these variables.

For later use, we quote (without proof) some results involving the determinant and its derivative

$$\frac{\partial q^k}{\partial a^i} \frac{A_k^i}{\mathcal{J}} = \delta_j^i, \tag{11}$$

where A_k^i is the cofactor of $\partial q^k / \partial a^i := q_{,i}^k$. A convenient expression for A_k^i is

$$A_k^i = \frac{1}{2} \epsilon_{kjl} \epsilon^{imn} \frac{\partial q^j}{\partial a^m} \frac{\partial q^l}{\partial a^n}, \tag{12}$$

where $\epsilon_{ijk}(= \epsilon^{ijk})$ is the Levi-Civita tensor. Using Eq. (11) one can show that

$$\frac{\partial \mathcal{J}}{\partial q_{,k}^i} = A_i^j \tag{13}$$

and using the chain rule

$$\frac{\partial}{\partial q^k} = \frac{1}{\mathcal{J}} A_k^i \frac{\partial}{\partial a^i}. \tag{14}$$

For further discussions see, e.g., Refs. 3, 5, and 28.

C. Varying the two-fluid action

The action of Eq. (1) depends on four dynamical variables: the scalar and vector potentials, ϕ and A, and the positions of the fluid elements q_s.

Varying with respect to ϕ yields Gauss's law

$$\partial_k \left(-\frac{1}{c} \frac{\partial A_k}{\partial t} - \partial_k \phi \right) = 4\pi e \int d^3a \, n_{i0}(a) \delta(x - q_i)$$
$$- 4\pi e \int d^3a \, n_{e0}(a) \delta(x - q_e),$$

where $\partial_k := \partial / \partial x^k$, or in more familiar form

$$\nabla \cdot E = 4\pi e (n_i(x,t) - n_e(x,t)). \tag{15}$$

Similarly, the variation with respect to A recovers the Maxwell-Ampere law

$$\frac{1}{4\pi} \left[-\nabla \times \nabla \times A + \frac{1}{c} \frac{\partial}{\partial t} \left(-\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \phi \right) \right]$$
$$- \frac{e}{c} \int d^3a \left[\delta(x - q_e) n_{e0} \dot{q}_e - \delta(x - q_i) n_{i0} \dot{q}_i \right] = 0,$$

or in more familiar form

$$\nabla \times B = \frac{4\pi J}{c} + \frac{1}{c} \frac{\partial E}{\partial t}, \tag{16}$$

where the Eulerian current density J is defined as

$$J(x,t) = e(n_i v_i - n_e v_e). \tag{17}$$

Recall that the other two Maxwell equations are contained in the definition of the potentials.

Variation with respect to the q_s's is slightly more complex, and we will show a few intermediate steps. Varying the kinetic energy term is straight forward and yields

$$-n_{s0}(a) m_s \ddot{q}_s(a,t). \tag{18}$$

The jth component of the interaction term results in

$$e_s n_{s0}(a) \left[-\frac{1}{c} \frac{dA^j(q_s,t)}{dt} + \frac{1}{c} \dot{q}_s^k \frac{\partial A^k(q_s,t)}{\partial q_s^j} - \frac{\partial \phi(q_s,t)}{\partial q_s^j} \right]$$
$$= e_s n_{s0}(a) \left[-\frac{1}{c} \frac{\partial A^j(q_s,t)}{\partial t} - \frac{1}{c} \dot{q}_s^k \frac{\partial A^j(q_s,t)}{\partial q_s^k} \right.$$
$$\left. + \frac{1}{c} \dot{q}_s^k \frac{\partial A^k(q_s,t)}{\partial q_s^j} - \frac{\partial \phi(q_s,t)}{\partial q_s^j} \right]$$
$$= e_s n_{s0}(a) \left[E(q_s,t) + \frac{1}{c} \dot{q}_s(a,t) \times (\nabla_{q_s} \times A(q_s,t)) \right]_j. \tag{19}$$

Note that this expression is purely Lagrangian. The fields A and E are evaluated at the positions q_s of the fluid elements and the curl $\nabla_{q_s} \times$ is taken with respect to the Lagrangian position. Also note, since $q_s = q_s(a,t)$, any total time derivative of, e.g., $A(q_s,t)$ will result in two terms.

Variation of the internal energy term yields

$$A_i^j \frac{\partial}{\partial a_j} \left(\frac{\rho_{s0}^2}{\mathcal{J}_s^2} \frac{\partial U \left(\frac{\rho_{s0}}{\mathcal{J}_s}, s_{s0} \right)}{\partial \left(\frac{\rho_{s0}}{\mathcal{J}_s} \right)} \right). \tag{20}$$

Setting the sum of Eqs. (18)–(20) equal to zero and invoking the usual thermodynamic relations between internal energy and pressure and temperature

$$p_s = (m_s n_s)^2 \frac{\partial U_s}{\partial (m_s n_s)} \quad \text{and} \quad T_s = \frac{\partial U_s}{\partial s_s}, \tag{21}$$

results in the well-known (non-dissipative) two-fluid equations of motion

$$m_s n_s \left(\frac{\partial v_s}{\partial t} + v_s \cdot \nabla v_s \right) = e_s n_s \left(E + \frac{1}{c} v_s \times B \right) - \nabla p_s. \quad (22)$$

Further analysis (see, e.g., Refs. 29 and 30) of these equations usually involves the addition and subtraction of the two-fluid equations and a change of variable transformation to

$$
\begin{aligned}
V &= \frac{1}{\rho_m} (m_i n_i v_i + m_e n_e v_e), \\
J &= e(n_i v_i - n_e v_e), \\
\rho_m &= m_i n_i + m_e n_e, \\
\rho_q &= e(n_i - n_e).
\end{aligned}
\quad (23)
$$

The resulting equations are then simplified by, e.g., making certain assumptions (quasineutrality, $v \ll c$, etc.) and ordering to obtain two new *one*-fluid equations—one often referred to as the *one-fluid momentum equation* and the other as the *generalized Ohm's law*.

III. THE NEW ONE-FLUID ACTION

The first step in building an action functional for fluid models is to decide on the relevant Eulerian observables of the model. Since we want to derive, e.g., the two-fluid model of Lüst and various reductions, our Eulerian observables are going to be the set $\{n, s, s_e, V, J, E, B\}$, where $s = (m_i s_i + s_e m_e)/m$, with $m = m_e + m_i$, and n is a single number density variable.

Next, we have to define our Lagrangian variables and with them construct the action. Any additional assumption (e.g., quasineutrality) and ordering will be implemented on the action level. Varying the new action will then result in equations of motion that, using properly defined Lagrange-Euler maps, will Eulerianize to, e.g., Lüst's equation of motion and the generalized Ohm's law.

A. New Lagrangian variables

We will start by defining a new set of Lagrangian variables, inspired by Eq. (23),[31]

$$
\begin{aligned}
Q(a,t) &= \frac{1}{\rho_{m0}(a)} \big(m_i n_{i0}(a) q_i(a,t) + m_e n_{e0}(a) q_e(a,t) \big), \\
D(a,t) &= e\big(n_{i0}(a) q_i(a,t) - n_{e0}(a) q_e(a,t) \big), \\
\rho_{m0}(a) &= m_i n_{i0}(a) + m_e n_{e0}(a), \\
\rho_{q0}(a) &= e\big(n_{i0}(a) - n_{e0}(a) \big).
\end{aligned}
\quad (24)
$$

Here, $Q(a,t)$ can be interpreted as a center of mass position variable and $D(a,t)$ as a local dipole moment variable, connecting an ion fluid element to an electron fluid element. It is then straight-forward to take the time-derivative of Q and D which can be interpreted as the center-of-mass velocity $\dot{Q}(a,t)$ and a Lagrangian current $\dot{D}(a,t)$, respectively. Using appropriately defined Lagrange-Euler maps, $\dot{Q}(a,t)$ will map

to the Eulerian velocity $V(x,t)$ and $\dot{D}(a,t)$ to the Eulerian current $J(x,t)$ as defined by Eq. (23).

We will also need the inverse of this transformation

$$
\begin{aligned}
q_i(a,t) &= \frac{\rho_{m0}(a)Q(a,t) + \dfrac{m_e}{e}D(a,t)}{\rho_{m0}(a) + \dfrac{m_e}{e}\rho_{q0}}, \\[2ex]
q_e(a,t) &= \frac{\rho_{m0}(a)Q(a,t) - \dfrac{m_i}{e}D(a,t)}{\rho_{m0}(a) - \dfrac{m_i}{e}\rho_{q0}}, \\[2ex]
n_{i0}(a) &= \frac{\rho_{m0}(a) + \dfrac{m_e}{e}\rho_{q0}(a)}{m}, \\[2ex]
n_{e0}(a) &= \frac{\rho_{m0}(a) - \dfrac{m_e}{e}\rho_{q0}(a)}{m}.
\end{aligned}
\quad (25)
$$

B. Ordering of fields and quasineutrality

Typically, reductions of the full two-fluid model are obtained by imposing an auxiliary ordering on the equations of motion. In order to preserve the variational formulation, we perform an ordering directly in the action.

To construct the action, we will start with the two-fluid action of Eq. (1) and change variables to Q and D, but in light of the fluid models we are interested in, we will first make two simplifying assumptions: We order the fields in the action so that the displacement current in Eq. (16) will vanish, and we assume quasineutrality. In this section, we describe this field ordering in detail and discuss quasineutrality in the Lagrangian variable context, which as far as we know has not been done before.

The omission of the displacement current is allowed, when the time scale of changes in the field configuration is long relative to the time it takes for radiation to "communicate" these changes across the system.[32] We use non-dimensional variables by introducing a characteristic scale B_0 for the magnetic field and a characteristic length scale ℓ for gradients. Times are then normalized by the Alfvén time $t_A = \ell/v_A$ and the $\dot{q}_s$'s by the Alfvén speed $v_A = B_0/\sqrt{4\pi\rho}$, resulting in the following form for the sum of the field and interaction terms of the Lagrangian (4):

$$
\begin{aligned}
\frac{B_0^2}{8\pi} &\int dt \int d^3\hat{x} \left[\left| -\frac{v_A}{c}\frac{\partial \hat{A}}{\partial \hat{t}} - \frac{\phi_0}{B_0\ell}\hat{\nabla}\hat{\phi} \right|^2 - |\hat{\nabla} \times \hat{A}|^2 \right] \\
&+ \sum_s B_0^2 \left[\int dt \int d^3\hat{a}\, n_0 \hat{n}_{s0}(a) e_s \int d^3\hat{x}\, \delta(\hat{x} - \hat{q}_s) \right. \\
&\quad \left. \times \left(\frac{v_A\ell}{B_0 c}\dot{\hat{q}}_s \cdot \hat{A} - \frac{\phi_o}{B_0^2}\hat{\phi} \right) \right],
\end{aligned}
$$

where ϕ_0 and n_0 are yet to be specified scales for the electrostatic potential and the densities of both species, respectively. We also require that the two species' velocities are of the same scale. Requiring the two interaction terms in the Lagrangian to be of the same order results in a scaling for ϕ; viz., $\phi_0 \equiv B_0 \ell v_A/c$. Thus, both parts of the $|E|^2$ term are of order $\mathcal{O}(v_A/c)$. Neglecting this term and varying with respect to $\hat{A}$ results in

$$\hat{\nabla} \times \hat{B} = \frac{4\pi e n_0 v_A}{c} \frac{\ell}{B_0} \left(\int d^3 a \delta(\hat{x} - \hat{q}_i) \hat{n}_{i0}(a) \hat{q}_i \right.$$
$$\left. - \int d^3 a \delta(\hat{x} - \hat{q}_e) \hat{n}_{e0}(a) \hat{q}_e \right),$$

which can be written as

$$\frac{B_0}{\ell} \hat{\nabla} \times \hat{B} = \frac{4\pi j_0}{c} \hat{J}, \tag{26}$$

where $j_0 = e n_0 v_A$ is a scale for the current.

Varying the scaled action with respect to $\hat{\phi}$ yields

$$0 = \int d^3 \hat{a} \delta(\hat{x} - \hat{q}_i) \hat{n}_i - \int d^3 \hat{a} \delta(\hat{x} - \hat{q}_e) \hat{n}_e \equiv \Delta \hat{n}. \tag{27}$$

The above equation states that the difference in the two densities is zero, i.e., the plasma is quasineutral, a property that holds locally, i.e., $n_i(x,t) = n_e(x,t)$. Using Eq. (6), this statement would correspond to the following in the Lagrangian variable picture:

$$\left. \frac{n_{i0}(a)}{\mathcal{J}_i(a,t)} \right|_{a=q_i^{-1}(x,t)} = \left. \frac{n_{e0}(a)}{\mathcal{J}_e(a,t)} \right|_{a=q_e^{-1}(x,t)}. \tag{28}$$

In the Lagrangian picture, we will make the additional assumption of homogeneity: $n_{i0}(a) = n_{e0}(a) =$ constant, which is natural for the plasma we are modeling. It states that at $t = 0$ all fluid elements are identical in the amount of density they carry. Therefore, n_{i0} and n_{e0} can be replaced by a constant n_0. Equation (28) then reduces to a statement about the two Jacobians

$$\mathcal{J}_i(a,t)|_{a=q_i^{-1}(x,t)} = \mathcal{J}_e(a,t)|_{a=q_e^{-1}(x,t)}, \tag{29}$$

which will play a central role in our development below.

Note, that the homogeneity assumption ($n_{i0} = n_{e0} = n_0$) does not prohibit us from describing quasineutral plasmas with density gradients. What we would have to do in this case, would be to pick our labeling scheme, and hence the Jacobian, accordingly, as to reflect the initial density gradient of the configuration. Thus, there is freedom in this regard beyond what we are assuming now.

C. Action functional

We are now ready to implement the change of variables discussed in Sec. III A. Because of the homogeneity assumption $n_{i0}(a) = n_{e0}(a) = n_0$, the new variables of Eq. (24) reduce to

$$\begin{aligned} Q(a,t) &= \frac{m_i}{m} q_i(a,t) + \frac{m_e}{m} q_e(a,t), \\ D(a,t) &= e n_0 (q_i(a,t) - q_e(a,t)), \\ \rho_{m0}(a) &= m n_0, \\ \rho_{q0}(a) &= 0, \end{aligned} \tag{30}$$

and the inverse transformation of Eq. (25) to

$$\begin{aligned} q_i(Q,D) &:= q_i(a,t) = Q(a,t) + \frac{m_e}{m e n_0} D(a,t), \\ q_e(Q,D) &:= q_e(a,t) = Q(a,t) - \frac{m_i}{m e n_0} D(a,t), \end{aligned} \tag{31}$$

where we choose the notation $q_s(Q,D)$ to emphasize that the q_s should not be thought of as ion/electron trajectories any more but as specific linear combinations of $Q(a,t)$ and $D(a,t)$. In addition, we will need the ion and electron Jacobians, $\mathcal{J}_i(Q,D)$ and $\mathcal{J}_e(Q,D)$, now expressed in terms of Q and D.

The resulting action functional has the form

$$\begin{aligned} S = &-\frac{1}{8\pi} \int dt \int d^3x |\nabla \times A(x,t)|^2 + \int dt \int d^3x \int d^3a n_0 \left\{ \delta(x - q_i(Q,D)) \left[\frac{e}{c} \left(\dot{Q}(a,t) + \frac{m_e}{m e n_0} \dot{D}(a,t) \right) \cdot A(x,t) - e\phi(x,t) \right] \right\} \\ &+ \int dt \int d^3x \int d^3a n_0 \left\{ \delta(x - q_e(Q,D)) \left[-\frac{e}{c} \left(\dot{Q}(a,t) - \frac{m_i}{m e n_0} \dot{D}(a,t) \right) \cdot A(x,t) + e\phi(x,t) \right] \right\} \\ &+ \frac{1}{2} \int dt \int d^3a n_0 \left[m_i |\dot{Q}|^2(a,t) + \frac{m_i m_e}{m e^2 n_0^2} |\dot{D}|^2(a,t) \right] \\ &- \int dt \int d^3a n_0 \left[m_i U_i \left(\frac{m_i n_0}{\mathcal{J}_i(Q,D)}, (m s_0 - m_e s_{e0})/m_i \right) + m_e U_e \left(\frac{m_e n_0}{\mathcal{J}_e(Q,D)}, s_{e0} \right) \right], \end{aligned} \tag{32}$$

where we recall $s_0 = (m_i s_{i0} + m_e s_{e0})/m$.

D. Nonlocal Lagrange-Euler maps

Now we define the Lagrange-Euler maps that connect the Eulerian observables V and J to the new Lagrangian variables Q and D. Referring to Sec. II B, one can see that a Lagrange-Euler map is a relationship between a Lagrangian quantity and some Eulerian observables, which holds only when it is evaluated on a trajectory $x = q_s(a,t)$. If we apply

the inverse Lagrange-Euler maps from Eqs. (6) and (8) to Eq. (23) and assume quasineutrality, we get

$$V(x,t) = \left. \frac{m_i}{m} \dot{q}_i(a,t) \right|_{a=q_i^{-1}(x,t)} + \left. \frac{m_e}{m} \dot{q}_e(a,t) \right|_{a=q_e^{-1}(x,t)},$$

$$\begin{aligned} J(x,t) = &\left. e \left(\frac{n_0}{\mathcal{J}_i(a,t)} \dot{q}_i(a,t) \right) \right|_{a=q_i^{-1}(x,t)} \\ &\left. - e \left(\frac{n_0}{\mathcal{J}_e(a,t)} \dot{q}_e(a,t) \right) \right|_{a=q_e^{-1}(x,t)}, \end{aligned} \tag{33}$$

$$n(x,t) = \frac{m_i}{m}\left(\frac{n_0}{\mathcal{J}_i(a,t)}\right)\Bigg|_{a=q_i^{-1}(x,t)} + \frac{m_e}{m}\left(\frac{n_0}{\mathcal{J}_e(a,t)}\right)\Bigg|_{a=q_e^{-1}(x,t)},$$

$$s(x,t) = \frac{m_i}{m}\, s_{i0}\Bigg|_{a=q_i^{-1}(x,t)} + \frac{m_e}{m}\, s_{e0}\Bigg|_{a=q_e^{-1}(x,t)},$$

(34)

$$s_e(x,t) = s_{e0}\big|_{a=q_e^{-1}(x,t)}. \tag{35}$$

The definitions of $Q(a,t)$ and $D(a,t)$ in Eq. (30) suggest that their time-derivatives should be associated with V and J, respectively. However, both $\dot{Q}$ and $\dot{D}$ are nonlocal objects, since they relate the velocities of electrons and ions which are located at different points in space. This means that neither $\dot{Q}$ nor $\dot{D}$, when evaluated at the inverse maps for a, can Eulerianize to a local velocity or current, since, in general, $x = q_i(Q,D)$ and $x' = q_e(Q,D)$ with $x \neq x'$ or, they are simultaneously evaluated at different trajectories. Therefore, we have two different inverse functions where the Lagrangian quantities are to be evaluated, namely, $a = q_i^{-1}(x,t)$ and $a = q_e^{-1}(x',t)$ which should be thought of as the inverse functions of $x = q_i(Q,D)$ and $x' = q_e(Q,D)$. To make this work, we *define* our Lagrange-Euler maps with $x = x'$ as

$$V(x,t) = \frac{m_i}{m}\left(\dot{Q}(a,t) + \frac{m_e}{men_0}\dot{D}(a,t)\right)\Bigg|_{a=q_i^{-1}(x,t)}$$
$$+ \frac{m_e}{m}\left(\dot{Q}(a,t) - \frac{m_i}{men_0}\dot{D}(a,t)\right)\Bigg|_{a=q_e^{-1}(x,t)},$$

$$J(x,t) = \frac{en_0}{\mathcal{J}_i(a,t)}\left(\dot{Q}(a,t) + \frac{m_e}{men_0}\dot{D}(a,t)\right)\Bigg|_{a=q_i^{-1}(x,t)}$$
$$- \frac{en_0}{\mathcal{J}_e(a,t)}\left(\dot{Q}(a,t) - \frac{m_i}{men_0}\dot{D}(a,t)\right)\Bigg|_{a=q_e^{-1}(x,t)}.$$

(36)

Due to Eq. (29), the two Jacobian determinants are equal (as long as they are evaluated at the respective inverse functions) and can be replaced by a common Jacobian determinant, $\mathcal{J}$.

The maps just defined are straight-forward to apply for mapping an Eulerian statement to a Lagrangian one, but for our purpose, we have to invert them. To keep careful track of the two inverse functions, we first invert the intermediate relations

$$V(x,t) + \frac{m_e}{men(x,t)}J(x,t)$$
$$= \left(\dot{Q}(a,t) + \frac{m_e}{men_0}\dot{D}(a,t)\right)\Bigg|_{a=q_i^{-1}(x,t)}, \tag{37}$$

$$V(x,t) - \frac{m_i}{men(x,t)}J(x,t)$$
$$= \left(\dot{Q}(a,t) - \frac{m_i}{men_0}\dot{D}(a,t)\right)\Bigg|_{a=q_e^{-1}(x,t)}, \tag{38}$$

where we have used Eq. (6). The inverse Lagrange-Euler maps are now given by

$$\dot{Q}(a,t) = \frac{m_i}{m}\left(V(x,t) + \frac{m_e}{men(x,t)}J(x,t)\right)\Bigg|_{x=q_i(Q,D)}$$
$$+ \frac{m_e}{m}\left(V(x',t) - \frac{m_i}{men(x',t)}J(x',t)\right)\Bigg|_{x'=q_e(Q,D)},$$

$$\dot{D}(a,t) = en_0\left(V(x,t) + \frac{m_e}{men(x,t)}J(x,t)\right)\Bigg|_{x=q_i(Q,D)}$$
$$- en_0\left(V(x',t) - \frac{m_i}{men(x',t)}J(x',t)\right)\Bigg|_{x'=q_e(Q,D)}. \tag{39}$$

Note that the construction of the maps of Eqs. (36) and (39) can be done with any invertible linear combination of the time derivatives of our Lagrangian variables. The only restriction is that the action should comply with the Eulerian Closure Principle, i.e., it should be expressible entirely in terms of the Eulerian observables. It is straightforward to show that this is true in our case.

E. Lagrange-Euler maps without quasineutrality

Had we not assumed quasineutrality, we would have to proceed differently: Eq. (23) implies that the proper Lagrangian variables that would Eulerianize to velocity and current would be

$$V(x,t) = \frac{m_i\left(\dfrac{n_{i0}}{\mathcal{J}_i}\dot{q}_i(a,t)\right)\Big|_{a=q_i^{-1}(x,t)}}{m_i\left(\dfrac{n_{i0}}{\mathcal{J}_i}\right)\Big|_{a=q_i^{-1}(x,t)} + m_e\left(\dfrac{n_{e0}}{\mathcal{J}_e}\right)\Big|_{a=q_e^{-1}(x,t)}}$$
$$+ \frac{m_e\left(\dfrac{n_{e0}}{\mathcal{J}_e}\dot{q}_e(a,t)\right)\Big|_{a=q_e^{-1}(x,t)}}{m_i\left(\dfrac{n_{i0}}{\mathcal{J}_i}\right)\Big|_{a=q_i^{-1}(x,t)} + m_e\left(\dfrac{n_{e0}}{\mathcal{J}_e}\right)\Big|_{a=q_e^{-1}(x,t)}},$$

$$J(x,t) = e\left(\frac{n_{i0}}{\mathcal{J}_i}\dot{q}_i(a,t)\right)\Bigg|_{a=q_i^{-1}(x,t)}$$
$$- e\left(\frac{n_{e0}}{\mathcal{J}_e}\dot{q}(a,t)\right)\Bigg|_{a=q_e^{-1}(x,t)}.$$

The above equations suggest that without quasineutrality, the definitions for $\dot{Q}$, $\dot{D}$, etc., should be modified to the following:

$$\dot{Q}(a,t) = \frac{1}{\rho_{m0}(a)}\left(m_i\mathcal{J}_e n_{i0}(a)\dot{q}_i(a,t) + m_e\mathcal{J}_i n_{e0}(a)\dot{q}_e(a,t)\right),$$

$$\dot{D}(a,t) = e\left(\mathcal{J}_e n_{i0}(a)\dot{q}_i(a,t) - \mathcal{J}_i n_{e0}(a)\dot{q}_e(a,t)\right),$$

$$\rho_{m0}(a) = m_i\mathcal{J}_e n_{i0}(a) + m_e\mathcal{J}_i n_{e0}(a),$$

where $\dot{Q}/(\mathcal{J}_i\mathcal{J}_e)$ maps to $V(x,t)$ and $\dot{D}/(\mathcal{J}_i\mathcal{J}_e)$ to $J(x,t)$. In this case, however, both $\dot{Q}$ and $\dot{D}$ are implicitly defined, since $\mathcal{J}_i$ and $\mathcal{J}_e$ depend on them. This problem is absent when only manipulating the Eulerian equations of motion. It might suggest though that when quasineutrality does not hold, the one-fluid description might not be appropriate. This can also be seen in the most general case derived by Lüst in Ref. 29. The resulting equations of motion in V and J still contain terms explicitly referring to ion/electron quantities, e.g., n_i and n_e. From a variational point of view, it is not obvious how to apply the Eulerian Closure Principle without quasineutrality. It seems

that in order to preserve it, one would need to distinguish between integrations over ion and electron labels, so that the d^3a could be related to the proper $\mathcal{J}_s$.

F. Derivation of the continuity and entropy equations

Before we derive the equations of motion for several different models in the next section, we derive here the continuity equation, which all of the models below have in common, and the entropy equations.

Due to the identity of the Jacobians from Eq. (29), the equation for n (Eq. (33)) reduces to

$$n(x,t) = \left(\frac{n_0}{\mathcal{J}_i(a,t)}\right)\bigg|_{a=q_i^{-1}(x,t)} = \left(\frac{n_0}{\mathcal{J}_e(a,t)}\right)\bigg|_{a=q_e^{-1}(x,t)}, \quad (40)$$

where q_s^{-1} are still the inverse functions of $q_s(Q,D)$. Inverting the equation for the ions and taking the time derivative yields

$$\frac{dn}{dt}\bigg|_{x=q_i(Q,D)} = \frac{d}{dt}\frac{n_0}{\mathcal{J}_i(a,t)} = -\frac{n_0}{\mathcal{J}_i^2(a,t)}\frac{\partial \mathcal{J}_i}{\partial t}.$$

To Eulerianize the equation above, we use the well-known relations $d/dt = \partial/\partial t + v \cdot \nabla$ and $\partial \mathcal{J}/\partial t = \mathcal{J}\nabla \cdot v$. The key here is to use the correct Eulerian velocity, in this case the ion velocity in terms of V and J. The result is

$$\frac{\partial n}{\partial t} + \left(V + \frac{m_e}{m_e n}J\right) \cdot \nabla n = -n\nabla \cdot \left(V + \frac{m_e}{m_e n}J\right),$$

which can be further reduced to

$$\frac{\partial n}{\partial t} + \nabla \cdot (nV) + \frac{m_e}{m_e}\nabla \cdot J = 0.$$

However, we already know from Eq. (26) that $\nabla \cdot J = 0$. Therefore, no matter which equality we choose in Eq. (40), the continuity equation will be the same

$$\frac{\partial n}{\partial t} + \nabla \cdot (nV) = 0. \quad (41)$$

Similarly, from Eq. (34) we obtain

$$\frac{\partial s}{\partial t} + V \cdot \nabla s = 0,$$

and from Eq. (35)

$$\frac{\partial s_e}{\partial t} + \left(V - \frac{m_i}{m_e n}J\right) \cdot \nabla s_e = 0,$$

or to leading order in m_e/m_i

$$\frac{\partial s_e}{\partial t} + \left(V - \frac{1}{en}J\right) \cdot \nabla s_e = 0.$$

IV. DERIVATION OF REDUCED MODELS

If we vary the action functional (32) with respect to Q and D and subsequently apply the Lagrange-Euler map we

recover the momentum equation and generalized Ohm's law of Lüst[33] (in the non-dissipative limit)

$$nm\left(\frac{\partial V}{\partial t} + (V \cdot \nabla)V\right) = -\nabla p + \frac{J \times B}{c}$$
$$- \frac{m_i m_e}{m_e^2}(J \cdot \nabla)\left(\frac{J}{n}\right), \quad (42)$$

$$E + \frac{V \times B}{c} = \frac{m_i m_e}{m_e^2 n}\left(\frac{\partial J}{\partial t} + (J \cdot \nabla)V - (J \cdot \nabla)\left(\frac{J}{n}\right) + (\nabla \cdot V)J\right)$$
$$+ \frac{m_i m_e}{m_e^2}(V \cdot \nabla)\left(\frac{J}{n}\right) + \frac{(m_i - m_e)}{m_e nc}(J \times B)$$
$$+ \frac{m_i m_e}{m_e n^2 e^2}J(V \cdot \nabla)n - \frac{m_i}{m_e n}\nabla p_e + \frac{m_e}{m_e n}\nabla p_i. \quad (43)$$

We will not show this lengthy, although straightforward, calculation here, but instead show the detailed derivation of extended MHD in Sec. IV A which requires one more ordering in the action of Eq. (32).

A. Extended MHD

At this point we will make one more simplification: We define the mass ratio $\mu = m_e/m_i$ and order the action functional keeping terms up to first order in μ. Up to first order, the change of variables is

$$q_i(Q,D) = Q(a,t) + \frac{\mu}{en_0}D(a,t),$$
$$q_e(Q,D) = Q(a,t) - \frac{1-\mu}{en_0}D(a,t), \quad (44)$$

and the action takes on the form

$$S = -\frac{1}{8\pi}\int dt \int d^3x |\nabla \times A(x,t)|^2$$
$$+ \int dt \int d^3x \int d^3a n_0 \left\{\delta\left(x - q_i(Q,D)\right)\right.$$
$$\times \left[\frac{e}{c}\dot{Q}(a,t) + \frac{\mu}{cn_0}\dot{D}(a,t) \cdot A(x,t) - e\phi(x,t)\right]\right\}$$
$$+ \int dt \int d^3x \int d^3a n_0 \left\{\delta\left(x - q_e(Q,D)\right)\right.$$
$$\times \left[-\frac{e}{c}\dot{Q}(a,t) + \frac{(1-\mu)}{cn_0}\dot{D}(a,t) \cdot A(x,t) + e\phi(x,t)\right]\right\}$$
$$+ \frac{1}{2}\int dt \int d^3a\, n_0 m_i\left((1+\mu)|\dot{Q}|^2(a,t) + \frac{\mu}{e^2 n_0^2}|\dot{D}|^2(a,t)\right)$$
$$- \int dt \int d^3a\, n_0\left(\mathfrak{U}_e\left(\frac{n_0}{\mathcal{J}_e(Q,D)}, s_{e0}\right)\right.$$
$$+ \mathfrak{U}_i\left(\frac{n_0}{\mathcal{J}_i(Q,D)}, s_{i0}\right)\right), \quad (45)$$

where for convenience we have replaced the U_s, the internal energy per unit mass, by $\mathfrak{U}_s$, the internal energy per particle. The pressure is obtained from the latter according to $p_s = n^2\partial\mathfrak{U}_s/\partial n$.

Varying the action with respect to Q_k yields

$$
\begin{aligned}
0 = & -n_0 m_i (1+\mu)\ddot{Q}_k(a,t) - \partial_k p \\
& + n_0 \left[\frac{e}{c}\left(\dot{Q}_j(a,t) + \frac{\mu}{en_0}\dot{D}_j(a,t) \right) \frac{\partial A_j(x,t)}{\partial x^k} \right. \\
& \left. \left. - e\partial_k \phi(x,t) - \frac{e}{c}\frac{d}{dt}A_k(x,t) \right] \right|_{x=q_i(Q,D)} \\
& + n_0 \left[-\frac{e}{c}\left(\dot{Q}_j(a,t) - \frac{(1-\mu)}{en_0}\dot{D}_j(a,t) \right) \frac{\partial A_j(x,t)}{\partial x^k} \right. \\
& \left. \left. + e\partial_k \phi(x,t) + \frac{e}{c}\frac{d}{dt}A_k(x,t) \right] \right|_{x=q_e(Q,D)}.
\end{aligned}
\tag{46}
$$

The variation of the internal energy term proceeds by varying q_s through Eqs. (44), giving $\delta q_s = \delta Q$ and using these expressions in the variation of the Jacobians $\mathcal{J}_s$. We have given the Eulerian result since the Lagrangian one has two terms of the form of Eq. (20), and it is cumbersome to carry this throughout the rest of the calculation. (See Ref. 1 for a treatment that orders the Eulerian equations directly.) Consistent with Dalton's law, the total single fluid pressure is $p = p_i + p_e$ and both these pressures come in entirely at the zeroth order of μ. Note that the two time derivatives of A do not cancel, because they are advected by different flows, or, since we are still in the Lagrangian framework, they are evaluated at different arguments.

To find the Eulerian equations of motion, we start with Eq. (39) (up to first order in μ) and impose locality, i.e., $x = x'$, such that $\dot{Q}$ maps to $V(x,t)$ and $\dot{D}$ to $J(x,t)$. However, the time derivatives of $\dot{Q}$ and $\dot{D}$ have to be treated with care as they each consists of two terms that are advected with different velocities. We will show how to Eulerianize the equations of motion in detail.

The $\ddot{Q}$ in the first term of Eq. (46) can be re-written as

$$
\begin{aligned}
\ddot{Q}(a,t) = & \left. \mu \frac{d}{dt}\left(V(x,t) - \frac{(1-\mu)}{en(x)}J(x,t) \right) \right|_{x=q_e(Q,D)} \\
& + \left. (1-\mu)\frac{d}{dt}\left(V(x,t) + \frac{\mu}{en(x)}J(x,t) \right) \right|_{x=q_i(Q,D)}
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
= & (1-\mu)\left(\frac{\partial V}{\partial t} + \frac{\partial q_i}{\partial t}\cdot \nabla V + \frac{\mu}{en}\frac{\partial}{\partial t}\left(\frac{J}{n}\right) \right. \\
& \left. + \frac{\mu}{e}\frac{\partial q_i}{\partial t}\cdot \nabla\left(\frac{J}{n}\right) \right) + \mu\left(\frac{\partial V}{\partial t} + \frac{\partial q_e}{\partial t}\cdot \nabla V \right. \\
& \left. - \frac{(1-\mu)}{en}\frac{\partial}{\partial t}\left(\frac{J}{n}\right) - \frac{(1-\mu)}{e}\frac{\partial q_e}{\partial t}\cdot \nabla\left(\frac{J}{n}\right) \right).
\end{aligned}
\tag{48}
$$

From Eqs. (44), we can find explicit expressions for the time derivatives of the $q_s(Q,D)$

$$
\frac{\partial q_i}{\partial t} = \dot{Q} + \frac{\mu}{en_0}\dot{D} \rightarrow V + \frac{\mu}{en}J,
\tag{49}
$$

$$
\frac{\partial q_e}{\partial t} = \dot{Q} - \frac{1-\mu}{en_0}\dot{D} \rightarrow V - \frac{1-\mu}{en}J.
\tag{50}
$$

Inserting these expression into Eq. (48), we find after some algebra that

$$
\ddot{Q}(a,t) \rightarrow \frac{\partial V}{\partial t} + (V\cdot\nabla)V + \frac{\mu(1-\mu)}{ne^2}(J\cdot\nabla)\left(\frac{J}{n}\right).
\tag{51}
$$

Next we Eulerianize the interaction terms of Eq. (46) using Eq. (39) (up to first order in μ) and Eq. (44). The result is

$$
\begin{aligned}
& \frac{ne}{c}\left[\left(V_j + \frac{\mu}{en}J_j \right)\frac{\partial A_j}{\partial x^k} - c\partial_k\phi - \frac{\partial A_k}{\partial t} - \frac{\partial q_i}{\partial t}\cdot\nabla A_k \right] \\
& + \frac{ne}{c}\left[\left(-V_j + \frac{(1-\mu)}{en}J_j \right)\frac{\partial A_j}{\partial x^k} + c\partial_k\phi + \frac{\partial A_k}{\partial t} \right. \\
& \left. + \frac{\partial q_e}{\partial t}\cdot\nabla A_k \right],
\end{aligned}
\tag{52}
$$

which, after substitution using Eqs. (49) and (50), yields

$$
\frac{1}{c}\left(J_j\frac{\partial A_j}{\partial x^k} - J_j\frac{\partial A_k}{\partial x^j} \right) = \frac{(J\times(\nabla\times A))_k}{c}.
\tag{53}
$$

The full Eulerian version of the equation of motion for the velocity of Eq. (46), also referred to as the *momentum equation* is

$$
\begin{aligned}
nm\left(\frac{\partial V}{\partial t} + (V\cdot\nabla)V \right) = & -\nabla p + \frac{J\times B}{c} \\
& - \frac{m_e}{e^2}(J\cdot\nabla)\left(\frac{J}{n}\right).
\end{aligned}
\tag{54}
$$

Note, it was shown in Ref. 1 that the last term of Eq. (54) is necessary for energy conservation.

Next, varying the action with respect to D_k yields

$$
\begin{aligned}
0 = & \frac{m_i\mu}{n_0 e^2}\ddot{D}_k(a,t) + \frac{(1-\mu)}{en_0}\partial_k p_e - \frac{\mu}{en_0}\partial_k p_i \\
& + \mu\left[\left(-\frac{1}{c}\frac{d}{dt}A_k(x,t) - \partial_k\phi(x,t) + \frac{1}{c}\left(\dot{Q}_j(a,t) \right. \right. \right. \\
& \left. \left. \left. + \frac{\mu}{en_0}\dot{D}_j(a,t) \right) \frac{\partial A_j(x,t)}{\partial x^k} \right) \right]\Bigg|_{x=q_i(Q,D)} \\
& + (1-\mu)\left[\left(-\frac{1}{c}\frac{d}{dt}A_k(x,t) - \partial_k\phi(x,t) + \frac{1}{c}\left(\dot{Q}_j(a,t) \right. \right. \right. \\
& \left. \left. \left. - \frac{(1-\mu)}{en_0}\dot{D}_j(a,t) \right) \frac{\partial A_j(x,t)}{\partial x^k} \right) \right]\Bigg|_{x=q_e(Q,D)}.
\end{aligned}
\tag{55}
$$

This time the Jacobians of the internal energies are varied using $\delta q_e = -(1-\mu)\delta D/(en_0)$ and $\delta q_i = \mu\delta D/(en_0)$, which again follow from Eqs. (44). Note, it is for this reason that only the electron pressure appears to leading order in Ohm's law for extended MHD.

Eulerianizing the $\ddot{D}$ term in Eq. (55) yields

$$
\begin{aligned}
\frac{m_i\mu}{n_0 e^2}\ddot{D}(a,t) = & \frac{m_i\mu}{e^2 n}\left(\frac{\partial J}{\partial t} + (J\cdot\nabla)V - (J\cdot\nabla)\left(\frac{J}{n}\right) + (\nabla\cdot V)J \right) \\
& + \frac{m_i\mu}{e^2}(V\cdot\nabla)\left(\frac{J}{n}\right) + \frac{m_i\mu}{e^2 n^2}J(V\cdot\nabla)n,
\end{aligned}
\tag{56}
$$

where we have used the continuity equation Eq. (41) to eliminate the time derivative of n and kept leading order in μ terms. The interaction terms in Eq. (55) reduce to

$$E + \frac{V \times (\nabla \times A)}{c} - \frac{(1 - 2\mu)}{enc} J \times (\nabla \times A). \qquad (57)$$

In Eqs. (56) and (57), we see the presence of some terms involving μ, in front of $\ddot{D}$ and $J \times B$, respectively. However, in the latter case it occurs in the factor $(1 - 2\mu)$ and since our ordering is $\mu \ll 1$, we can drop the μ-dependence in Eq. (57), to lowest order. However, in Eq. (56), we cannot throw out all the terms that depend on μ since the factor $\mu m_i/(ne^2)$ cannot be cast into a dimensionless form, and hence one cannot invoke the ordering $\mu \ll 1$ here. Post-variation, the discrepancy in the order of the derived terms, i.e., the existence of these anomalous terms, has also been observed elsewhere.[34]

The Eulerian version of the equation of motion of the current Eq. (55) (after keeping zeroth order in μ), also known as *generalized Ohm's law*, is then

$$E + \frac{V \times B}{c} = \frac{m_e}{e^2 n} \left(\frac{\partial J}{\partial t} + (J \cdot \nabla)V - (J \cdot \nabla)\left(\frac{J}{n}\right) + (\nabla \cdot V)J \right)$$
$$+ \frac{(J \times B)}{enc} - \frac{\nabla p_e}{en} + \frac{m_e}{e^2}(V \cdot \nabla)\left(\frac{J}{n}\right)$$
$$+ \frac{m_e}{e^2 n^2} J(V \cdot \nabla)n. \qquad (58)$$

The last two terms on the right hand side of Eq. (58) can be combined to give $(m_e/(e^2 n))(V \cdot \nabla)J$ and since $\nabla \cdot J = 0$, we can add a $V(\nabla \cdot J)$ term without changing the result, and combine most terms in the divergence of the tensor $VJ + JV$ to obtain the following equation:

$$E + \frac{V \times B}{c} = \frac{m_e}{e^2 n} \left(\frac{\partial J}{\partial t} + \nabla \cdot (VJ + JV) \right)$$
$$- \frac{m_e}{e^2 n}(J \cdot \nabla)\left(\frac{J}{n}\right) + \frac{(J \times B)}{enc} - \frac{\nabla p_e}{en}. \qquad (59)$$

Equations (54) and (59) constitute the extended MHD model.

B. Hall MHD

Hall MHD is a limiting case, for which previous work of an action functional nature exists.[35,36] Here, we obtain the action functional by expanding and retaining only terms up to zeroth order in μ, i.e., if we neglect the electron inertia ($m_e \to 0$), the action of Eq. (32) reduces to

$$S = -\frac{1}{8\pi} \int dt \int d^3x |\nabla \times A(x,t)|^2 + \int dt \int d^3a\, n_0$$
$$\times \left[\frac{1}{2} m |\dot{Q}|^2(a,t) - \mathfrak{U}_i\left(\frac{n_0}{\mathcal{J}_i(Q)}, s_{i0}\right) - \mathfrak{U}_e\left(\frac{n_0}{\mathcal{J}_e(Q,D)}, s_{e0}\right) \right]$$
$$+ \int dt \int d^3x \int d^3a\, n_0 \left\{ \delta\left(x - Q(a,t) - \frac{1}{en_0}D(a,t)\right) \right.$$
$$\times \left[-\frac{e}{c}\left(\dot{Q}(a,t) - \frac{1}{en_0}\dot{D}(a,t)\right) \cdot A(x,t) + e\phi(x,t) \right] \right\}$$
$$+ \int dt \int d^3x \int d^3a\, n_0 \left\{ \delta(x - Q(a,t)) \right.$$
$$\times \left[\frac{e}{c}\dot{Q}(a,t) \cdot A(x,t) - e\phi(x,t) \right] \right\}, \qquad (60)$$

and Eqs. (44) become

$$q_i(Q,D) = Q(a,t),$$
$$q_e(Q,D) = Q(a,t) - D(a,t)/(en_0). \qquad (61)$$

Observe we have also replaced m_i by m in the kinetic energy term, which is correct to leading order in μ.

The inverse maps required for Eulerianizing the equations of motion reduce to

$$\dot{Q}(a,t) = V(x,t)|_{x = q_i = Q},$$
$$\dot{D}(a,t) = en_0 V(x,t)|_{x = q_i = Q}$$
$$- en_0 \left(V(x',t) - \frac{J(x',t)}{en(x',t)} \right)\bigg|_{x' = q_e(Q,D)}. \qquad (62)$$

Following the procedure outlined in the previous section for extended MHD, we arrive at what is commonly referred to as Hall MHD

$$nm\left(\frac{\partial V}{\partial t} + (V \cdot \nabla)V \right) = -\nabla p + \frac{J \times B}{c}, \qquad (63)$$

$$E + \frac{V \times B}{c} = \frac{J \times B}{nec} - \frac{1}{ne}\nabla p_e, \qquad (64)$$

which are the usual forms of the momentum equation and Ohm's law for Hall MHD.

C. Electron MHD

Electron MHD,[16,37–39] is another limiting case where we neglect the ion motion completely. This theory is used to describe the short time scale motion of the electrons in a neutralizing ion background. Since the ions are immobile, we require $\dot{q}_i = 0$ and $q_i = q_i(a)$. Also, we require that there be no electric field and, consequently, we neglect ϕ from the action. In this case, using the Q, D formulation of Sec. III A is redundant since there is only a single fluid. From $\dot{q}_i = 0$ we find $\dot{D} = -(en_0 m/m_e)\dot{Q}$. (The same relation holds between Q and D up to an additive constant which represents the constant position of the ion). In addition, the Lagrange-Euler map takes on the simple form

$$v_e(x,t) = \left(1 + \frac{1}{\mu}\right)\dot{Q}(a,t)\bigg|_{a = q_e^{-1}(x,t)}, \qquad (65)$$

where

$$q_e(a,t) = \left(1 + \frac{1}{\mu}\right)Q(a,t). \qquad (66)$$

The remaining terms in the action are

$$S = -\frac{1}{8\pi} \int dt \int d^3x |\nabla \times A(x,t)|^2$$
$$+ \int dt \int d^3a\, n_0 \left[\frac{1}{2} m_e |\dot{q}_e|^2(a,t) - \mathfrak{U}_e\left(\frac{n_0}{\mathcal{J}_e(Q)}\right) \right]$$
$$- \int dt \int d^3x \int d^3a\, \delta(x - q_e)\frac{en_0}{c}\dot{q}_e \cdot A(x,t), \qquad (67)$$

which is essentially the same action as that of Ref. 16. It is also straight-forward to express this action in terms of Q using Eqs. (65) and (66).

Upon varying the action (either in terms of q_e or Q) and Eulerianizing the following equation of motion and constraint are obtained:

$$m_e \left(\frac{\partial v_e}{\partial t} + v_e \cdot \nabla v_e \right) + \frac{e}{c} \frac{\partial A}{\partial t} = \frac{e}{c}(v_e \times B) - \frac{\nabla p_e}{n},$$

$$\nabla \times B = -\frac{4\pi}{c} e n v_e,$$

which are the usual equations of electron MHD.

V. NOETHER'S THEOREM

In this section, we will investigate the invariants of the action functional for the quasineutral Lüst equations of Eq. (32) and the extended MHD system of Eq. (45) using Noether's theorem.

Note that both actions can be expressed either in terms of (Q, D) or in terms of (q_i, q_e), which are related through a simple linear transformation, e.g., Eq. (30). Furthermore, both sets of variables obey the Eulerian Closure Principle. Hence, it is equivalent to work with an action expressed in terms of either set of variables. For convenience, we shall work with the latter set, as the Euler-Lagrange maps are easier to apply. Noether's theorem states that if an action is invariant under the transformations

$$q'_s = q_s + K_s(q_s, t); \quad t' = t + \tau(t), \tag{68}$$

i.e.,

$$S = \int_{t_1}^{t_2} dt \int d^3z \mathcal{L}(q_s, \dot{q}_s, z, t),$$
$$= \int_{t'_1}^{t'_2} dt' \int d^3z' \mathcal{L}(q'_s, \dot{q}'_s, z', t'),$$

then there exist constants of motion given by

$$C = \int d^3z \left[\tau \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_s} \cdot \dot{q}_s - \mathcal{L} \right) - K_s \cdot \frac{\partial \mathcal{L}}{\partial \dot{q}_s} \right], \tag{69}$$

where the index s represents the number of independent variables q in the system. Our actions are mixed Lagrangian and Eulerian, so the variable z can denote a or x.

We note that invariants for general reduced fluid models have also been derived in Ref. 40 via the Noether approach. The primary difference between the two stems from the fact that our approach is purely in terms of Lagrangian variables, i.e., we obtain invariants by investigating symmetries of the Lagrangian through suitable transformations of (Q, D), which serve as our fields. The approach employed in Ref. 40 is complementary as it introduces variations induced by space-time translations, and investigates the ensuing symmetries.

A. Time translation

It is straight-forward to see that the action is invariant under time translation with

$$K_s = 0; \quad \tau = 1.$$

The corresponding constant of motion, the energy, is found to be

$$\mathcal{E} = \int d^3x \left[\frac{|\nabla \times A|^2}{8\pi} + \sum_s \int d^3a \left(\frac{1}{2} n_0 m_s |\dot{q}_s|^2 \right. \right.$$
$$\left. \left. + n_0 \, \mathfrak{U}_s \left(\frac{n_0}{\mathcal{J}_s}, s_{s0} \right) \right) \right].$$

Using suitable Lagrange-Euler maps to express our answer in terms of the Eulerian variables $\{n, V, J\}$, we obtain

$$\mathcal{E} = \int d^3x \left[\frac{|B|^2}{8\pi} + n \mathfrak{U}_i + n \mathfrak{U}_e + mn \frac{|V|^2}{2} + \frac{m_e m_i}{mne^2} \frac{|J|^2}{2} \right], \tag{70}$$

for the quasineutral Lüst model and

$$\mathcal{E} = \int d^3x \left[\frac{|B|^2}{8\pi} + n \mathfrak{U}_i + n \mathfrak{U}_e + mn \frac{|V|^2}{2} + \frac{m_e}{ne^2} \frac{|J|^2}{2} \right], \tag{71}$$

for the extended MHD model. Note that the two energies are different since the extended MHD model includes the mass ratio ordering.

B. Space translation

Space translations correspond to

$$K_s = k; \quad \tau = 0,$$

where k is an arbitrary constant vector. Under space translations, the constant of motion is the momentum, which is found to be

$$P = k \cdot \int d^3a (n_0 m_i \dot{q}_i + n_0 m_e \dot{q}_e)$$
$$+ k \cdot \int d^3x \frac{e}{c} A \left\{ \int d^3a n_0 \left[\delta(x - q_i) - \delta(x - q_e) \right] \right\}.$$

Using the Lagrange-Euler maps one can show that

$$P = k \cdot \int d^3x nmV,$$

is the conserved quantity. Note that k is entirely arbitrary, and hence we see that the total momentum

$$P = \int d^3x \rho V, \tag{72}$$

is conserved. This is also evident from the corresponding dynamical equation for V.

C. Rotations

The actions are also invariant under rotations which correspond to

$$K_s = k \times q_s; \quad \tau = 0.$$

Following the same procedure as before, we have

$$\mathfrak{L} = k \cdot \int d^3 x n m \, r \times V,$$

and since we know that k is arbitrary, we conclude that the angular momentum given by

$$\mathfrak{L} = \int d^3 x \, \rho \, r \times V, \tag{73}$$

is a constant of motion.

D. Galilean boosts

When discussing boosts, we have to consider that the action may remain invariant even when the following holds:

$$
\begin{aligned}
S &= \int_{t_1}^{t_2} dt \int d^3 z \mathcal{L}(q_s, \dot{q}_s, z, t) \\
&= \int_{t'_1}^{t'_2} dt' \int d^3 z' (\mathcal{L}(q'_s, \dot{q}'_s, z', t') + \partial_\mu \lambda^\mu),
\end{aligned}
$$

because the second term vanishes identically. In all the previous derivations of the constants of motion, the infinitesimal transformations did not involve time explicitly. A boost, though, corresponds to

$$K_s = ut; \quad \tau = 0,$$

where u is an arbitrary constant velocity. For a Galilean boost in a one-fluid model, the corresponding invariant quantity is given by

$$\mathcal{B} = \int d^3 a m n_0 (q - \dot{q}t),$$

and since we have two different species, this generalizes to

$$\mathcal{B} = \sum_s \int d^3 a m_s n_{s0} (q_s - \dot{q}_s t).$$

Using the corresponding Lagrange-Euler maps, the Eulerianized expression is given by

$$\mathcal{B} = \int d^3 x \rho (x - Vt). \tag{74}$$

VI. CONCLUSIONS

In this paper, we derived several fluid models from a general two-fluid action functional. All approximations, ordering schemes, and changes of variables were done in the action functional before Hamilton's principle was invoked. We defined a new set of Lagrangian variables, and under the assumption of quasineutrality, we constructed a new set of nonlocal Lagrange-Euler maps assuring that our Lagrangian equations of motion can be Eulerianized. Last, we derived several conservation laws for these models using Noether's theorem.

The novel nonlocal Lagrange-Euler map of this paper is of particular general importance. Usual Lagrange-Euler maps (also known as momentum maps) entail the advection

of various quantities by a single velocity field and this can be traced to the algebraic structure of the Poisson bracket written in terms of Eulerian variables (see, e.g., Ref. 3). For single fluid models like MHD the Poisson bracket[41] has semi-direct product structure, which occurs in a variety of fluid contexts (e.g. Refs. 42–44). However, many systems do not posses this semi-direct product structure (e.g. Refs. 45–48) and indeed a general theory of algebraic extensions was given in Ref. 49. It is the selection of the set of observables and the ECP that give rise to the general algebras underlying Poisson brackets. Detailed construction of general algebras of Ref. 49 will be reported in future publications, along with derivations of other single fluid models including gyroviscosity.[6,50]

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